

## LIMIT THEOREMS FOR A POSITIVELY DRIFTING PROCESS AND ITS RELATED FIRST PASSAGE TIMES

BY

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To the memory of Hsien-Chung Wang

**Abstract.** Let  $S_n$  be a random walk with a positive drift,  $R_n$  be random variables with  $R_n = o(1)$  and  $a_n$  be positive constants with  $n^{-\alpha} a_n = 1 + o(1)$  for some  $\alpha > 0$ . Let  $U_n = R_n + n^{-1} S_n$ ,  $U^*$  be any one of  $\max_{j \leq n} a_j U_j$ ,  $\inf_{j \geq n} a_j U_j$ ,  $\min_{j \leq n} U_j/a_j$  and  $\sup_{j \geq n} U_j/a_j$ . For  $\lambda > 0$  and  $n_\lambda > 0$ , let  $N^*$  be any one of  $\inf \{n \geq n_\lambda : U_n \geq (\lambda a_n)^{-1}\}$ ,  $\inf \{n \geq n_\lambda : 0 < U_n \leq \lambda a_n\}$ ,  $\sup \{n \geq n_\lambda : U_n < (\lambda a_n)^{-1}\}$ ,  $\sup \{n \geq n_\lambda : U_n > \lambda a_n\}$ ,  $\sum_{n_\lambda}^\infty I_{[U_n < (\lambda a_n)^{-1}]}$  and  $\sum_{n_\lambda}^\infty I_{[U_n > \lambda a_n]}$ . The central limit theorem and the law of the iterated logarithm for  $U^*$  and  $N^*$  are given.

The results of Bhattacharya and Mallik [2] and Robbins and Siegmund [12] on sequential estimation, and those of Siegmund [13], Vervaat [17], Gut [9] and Chow and Hsiung [5] on renewal theory are united and extended.

**1. Introduction.** Let  $X, X_1, X_2, \dots$  be independent and identically distributed random variables with  $EX = \mu \in (0, \infty)$ , and  $R_1, R_2, \dots$  be random variables with  $R_n = o(1)$  a.s. as  $n \rightarrow \infty$ . Assume that  $a_1, a_2, \dots$  are positive constants with  $n^{-\alpha} a_n \rightarrow 1$  as  $n \rightarrow \infty$  for some  $\alpha > 0$ . Put

$$(1.1) \quad S_n = \sum_1^n X_i, \quad U_n = n^{-1} S_n + R_n.$$

We shall investigate the asymptotic properties of the first passage times defined in terms of this perturbed random walk with a positive drift. More specifically, for  $\lambda > 0$ , put

$$(1.2) \quad \begin{aligned} N &= N_\lambda = \inf \{n \geq n_\lambda : U_n \geq (\lambda a_n)^{-1}\}, \\ T &= T_\lambda = \inf \{n \geq n_\lambda : 0 < U_n \leq \lambda a_n\}, \end{aligned}$$

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where  $n_\lambda$  is some positive integer which may vary with  $\lambda$ . Closely related to these entities are the last times and the number of boundary crossings:

$$(1.3) \quad N' = N'_\lambda = \sup \{n \geq n_\lambda : U_n < (\lambda a_n)^{-1}\}, \\ T' = T'_\lambda = \sup \{n \geq n_\lambda : U_n > \lambda a_n\},$$

$$(1.4) \quad N'' = N''_\lambda = \sum_{n_\lambda}^{\infty} I_{\{U_n < (\lambda a_n)^{-1}\}}, \quad T'' = T''_\lambda = \sum_{n_\lambda}^{\infty} I_{\{U_n > \lambda a_n\}}.$$

The limiting distributions of these random variables can be derived directly from their definitions and an application of a theorem of Anscombe's. However their laws of the iterated logarithm requires a more intricate inversion from those for

$$(1.5) \quad \max_{j \leq n} a_j U_j, \quad \inf_{j \geq n} a_j U_j,$$

$$(1.6) \quad \min_{j \leq n} a_j^{-1} U_j, \quad \text{and} \quad \sup_{j \geq n} a_j^{-1} U_j.$$

In §2 we shall study the asymptotic behaviour of (1.5) and (1.6) by a sufficiently detailed comparison with  $a_n U_n$  and  $a_n^{-1} U_n$  respectively under various moment conditions on  $X$ . In particular when the second moment is finite, we have been able to obtain the asymptotic distributions and the law of the iterated logarithm for these random variables. In §3 we shall prove the theorems about (1.2)–(1.4) under the finite second moment condition. In §4 we shall assume that  $E|X|^p < \infty$  for  $1 \leq p < 2$  and prove the Marcinkiewicz-Zygmund type of the strong law of large numbers for (1.5) and (1.6), and obtain the corresponding results for (1.2)–(1.4). In §5 we shall remove the assumption of identical distribution for the  $X_n$ 's, and study various asymptotic properties of (1.2)–(1.6).

When  $R_n = 0$ ,  $n_\lambda = 1$  and  $a_n = n^\alpha$  with  $0 < \alpha \leq 1$ , the asymptotic normality of (1.5) has been established in Teicher [16] and Chow and Hsiung [5] where the law of the iterated logarithm has also been proved. In the latter paper such results are applied to obtain the asymptotic distributions and the law of the iterated logarithm for the first passage times  $N$ , the last times  $N'$  and the numbers of boundary crossings  $N''$ . The asymptotic normality for the first passage times  $N$  has been first established by Siegmund [13] and

Gut [9], using different methods. Vervaat [17] has obtained the law of the iterated logarithm for  $N''$  when  $\alpha = 1$ . In the context of sequential estimation, for special cases of  $U_n$ ,  $n_1$  and  $a_n$ , Bhattacharya and Mallik [2] has obtained the asymptotic distributions for  $T$ , and Robbins and Siegmund [12] for  $N$ .

It is partly motivated by the consideration of the problem of sequential estimation that we study the random process (1.1). To see this, let  $Z_1, Z_2, \dots$  be independent observations from a population with mean  $\theta$ , and positive and finite variance  $\mu$ . Let the sample mean and the sample variance be respectively

$$(1.7) \quad \bar{Z}_n = n^{-1} \sum_1^n Z_i, \quad V_n = n^{-1} \sum_1^n (Z_i - \bar{Z}_n)^2 + b_n,$$

where  $b_1, b_2, \dots$  are constants such that  $b_n = o(1)$  as  $n \rightarrow \infty$ . Put  $X_n = (Z_n - \theta)^2$  and  $R_n = -(\bar{Z}_n - \theta)^2 + b_n$ . Then

$$(1.8) \quad V_n = n^{-1} \sum_1^n X_i + R_n,$$

which assumes the form of (1.1) with the side conditions satisfied. The stopping times associated with (1.8) arise very frequently in sequential estimation; for example in the problem considered by Robbins [11]. Subsequently many authors in the literature have considered different aspects of this problem and its ramifications. The asymptotic distributions for these stopping times have been discussed briefly above. Our work here constitutes a natural generalization and extension of those results. For the investigation in the realm of uniform integrability and the performance for these stopping times, see Starr [14], Cabilio and Robbins [4], Woodroffe [18] and Chow, et al. [6]. We shall make a more thorough study of this aspect of the problem and present the results in another paper.

**2. Limiting behaviour of  $\max_{j \leq n} a_j U_j$ ,  $\inf_{j \geq n} a_j U_j$ ,  $\min_{j \leq n} a_j^{-1} U_j$  and  $\sup_{j \geq n} a_j^{-1} U_j$ .** We shall make the following assumptions throughout this section. Let  $X, X_1, X_2, \dots$  be independent and identically distributed random variables with  $EX = \mu > 0$  and  $E|X|^p < \infty$  for some  $p \geq 1$ . Let  $R_1, R_2, \dots$  be random variables with  $R_n = o(1)$  a.s. as  $n \rightarrow \infty$ , and put

$$(2.1) \quad S_n = \sum_1^n X_i, \quad U_n = n^{-1} S_n + R_n.$$

LEMMA 1. Let (2.1) hold. Let  $\alpha > 0$ ,  $1 > \beta > 0$  and  $f_n$  be positive constants such that as  $n \rightarrow \infty$

$$(2.2) \quad \max \{f_j/f_k : n\beta \leq j, k \leq n/\beta\} = O(1).$$

(i) If  $1 \leq p < 4$ ,  $n^\alpha f_n R_n = o(1)$  a.s. and  $f_n = O(n^{1-\alpha-1/p})$ , then as  $n \rightarrow \infty$ , a.s.

$$(2.3) \quad \max \{(j^\alpha U_j - n^\alpha U_n) f_n : n\beta \leq j \leq n\} = o(1),$$

$$(2.4) \quad \max \{(n^\alpha U_n - j^\alpha U_j) f_n : n \leq j \leq n/\beta\} = o(1),$$

$$(2.5) \quad \max \{(j^\alpha U_n - n^\alpha U_j) f_n : n\beta \leq j \leq n\} = o(1),$$

$$(2.6) \quad \max \{(n^\alpha U_j - j^\alpha U_n) f_n : n \leq j \leq n/\beta\} = o(1).$$

(ii) Let  $Y, Y_1, Y_2, \dots$  be independent and identically distributed random variables,  $r \geq 2$ , and

$$(2.7) \quad X = \mu \text{ a.s.}, \quad EY = 0, \quad E|Y|^r < \infty,$$

$$R_n = R'_n - \left( n^{-1} \sum_1^n Y_i \right)^2,$$

with  $n^\alpha f_n R'_n = o(1)$  a.s.. If  $f_n = O(n^{3/2-1/r-\alpha} (\log \log n)^{-1/2})$ , then (2.3)–(2.6) hold a.s., and if  $f_n = O(n^{3/2-1/r-\alpha})$ , then (2.3)–(2.6) hold in probability.

**Proof.** We can assume that  $\mu = 1$ .

(i) Since  $n^\alpha f_n R_n = o(1)$  a.s., by (2.2) we can assume that  $R_n = 0$  a.s.. Then  $U_n = n^{-1} S_n = 1 + n^{-1} W_n$ , where  $W_n = S_n - n$ . First, let  $1 \leq p < 2$ . By Kolmogorov and Marcinkiewicz-Zygmund strong law of large numbers (Loéve [10, p. 242]),

$$\max_{j \leq n} |W_j| = o(n^{1/p}) \text{ a.s.}$$

Since  $\alpha > 0$ , a.s.

$$\begin{aligned} \max_{n\beta \leq j \leq n} (j^\alpha U_n - n^\alpha U_j) &\leq \max_{n\beta \leq j \leq n} (j^\alpha n^{-1} W_n - n^\alpha j^{-1} W_j) \\ &= o(n^{\alpha-1+1/p}) = o(f_n^{-1}), \end{aligned}$$

$$\begin{aligned} \max_{n \leq j \leq n/\beta} (n^\alpha U_j - j^\alpha U_n) &\leq \max_{n \leq j \leq n/\beta} (n^\alpha j^{-1} W_j - j^\alpha n^{-1} W_n) \\ &= o(n^{\alpha-1+1/p}) = o(f_n^{-1}), \end{aligned}$$

yielding (2.5) and (2.6) respectively. Similarly for (2.3) and (2.4). Next, let  $2 \leq p < 4$ . Choose  $\frac{1}{2} < \eta < 2/p$  and put

$$m = [n^\eta], \quad W_{n,j} = W_{n+j} - W_n.$$

By Hartman-Wintner law of the iterated logarithm (Chow and Teicher [7, p. 352]), and the strong law of large numbers for delayed sums (Chow and Hsiung [5, Lemma 2.4]), for  $\xi \in (-\infty, \infty)$ , a.s.

$$\begin{aligned} (2.8) \quad & \max_{n-m \leq j \leq n} (j^\xi W_j - n^\xi W_n) \\ &= \max_{n-m \leq j \leq n} ((j^\xi - n^\xi) W_n - j^\xi W_{j,n-j}) \\ &= O(n^{\xi-1+\eta+1/2} (\log \log n)^{1/2}) + o(n^{\xi+1/p}) \\ &= o(n^{\xi+1/p}), \end{aligned}$$

since  $p \geq 2$  and  $\frac{1}{2} < \eta < 2/p$ , and similarly a.s.

$$\begin{aligned} (2.9) \quad & \max_{n \leq j \leq n+m} (n^\xi W_n - j^\xi W_j) \\ &= \max_{n \leq j \leq n+m} ((n^\xi - j^\xi) W_j - n^\xi W_{n,j-n}) \\ &= o(n^{\xi+1/p}). \end{aligned}$$

By (2.8), (2.9) and  $\alpha > 0$ , a.s.

$$\begin{aligned} (2.10) \quad & \max_{n-m \leq j \leq n} (j^\alpha U_j - n^\alpha U_n) \\ &\leq \max_{n-m \leq j \leq n} (j^{\alpha-1} W_j - n^{\alpha-1} W_n) \\ &= o(n^{\alpha-1+1/p}) = o(f_n^{-1}), \end{aligned}$$

$$\begin{aligned} (2.11) \quad & \max_{n \leq j \leq n+m} (n^\alpha U_n - j^\alpha U_j) \\ &\leq \max_{n \leq j \leq n+m} (n^{\alpha-1} W_n - j^{\alpha-1} W_j) \\ &= o(n^{\alpha-1+1/p}) = o(f_n^{-1}). \end{aligned}$$

Since  $\alpha > 0$  and  $\eta > \frac{1}{2}$ , by LIL again, a.s.

$$\begin{aligned} & \max_{n-p \leq j \leq n-m} (j^\alpha U_j - n^\alpha U_n) \\ &= \max_{n-p \leq j \leq n-m} (j^\alpha - n^\alpha + j^{\alpha-1} W_j - n^{\alpha-1} W_n) \\ &\leq (n-m)^\alpha - n^\alpha + O(n^{\alpha-1/2} (\log \log n)^{1/2}) \\ &= -\alpha n^{\alpha+\eta-1/2} (1 + o(1)). \end{aligned}$$

Hence a.s.

$$(2.12) \quad \max_{n\beta \leq j \leq n-m} (j^\alpha U_j - n^\alpha U_n) f_n \leq o(1),$$

yielding (2.3) by (2.10), and similarly a.s.

$$(2.13) \quad \max_{n+m \leq j \leq n/\beta} (n^\alpha U_n - j^\alpha U_j) f_n \leq o(1),$$

yielding (2.4) by (2.11). Similarly for (2.5) and (2.6).

(ii) Since  $n^\alpha f_n R'_n = o(1)$  a.s., by (2.2) we can assume that  $R'_n = 0$  a.s.. Then  $U_n = 1 - (n^{-1} W_n)^2$ , where  $W_n = \sum_1^n Y_i$ . Let  $0 < \eta < 2/r$ , and

$$m = [n^\eta], \quad W_{n,j} = W_{n+j} - W_n.$$

Then

$$(2.14) \quad j^\alpha U_j - n^\alpha U_n = j^\alpha - n^\alpha + n^{\alpha-2} W_n^2 - j^{\alpha-2} W_j^2,$$

$$(2.15) \quad n^\alpha U_j - j^\alpha U_n = n^\alpha - j^\alpha + n^\alpha j^{-2} W_j^2 - j^\alpha n^{-2} W_n^2.$$

By LIL, (2.14) and (2.15), a.s.

$$(2.16) \quad \begin{aligned} & \max_{n\beta \leq j \leq n-m} (j^\alpha U_j - n^\alpha U_n) \\ & \leq (n-m)^\alpha - n^\alpha + O(n^{\alpha-1} \log \log n) \\ & = -\alpha n^{\alpha-1+\eta} (1 + o(1)), \end{aligned}$$

$$(2.17) \quad \begin{aligned} & \max_{n+m \leq j \leq n/\beta} (n^\alpha U_j - j^\alpha U_n) \\ & \leq n^\alpha - (n+m)^\alpha + O(n^{\alpha-1} \log \log n) \\ & = -\alpha n^{\alpha-1+\eta} (1 + o(1)). \end{aligned}$$

By LIL and SLLN for delayed sums again, a.s.

$$\begin{aligned} & \max_{n-m \leq j \leq n} (j^\alpha U_j - n^\alpha U_n) \\ & \leq \max_{n-m \leq j \leq n} (n^{\alpha-2} W_n^2 - j^{\alpha-2} W_j^2) \\ & = \max_{n-m \leq j \leq n} ((n^{\alpha-2} - j^{\alpha-2}) W_j^2 \\ & \quad + n^{\alpha-2} W_{j,n-j}^2 + 2n^{\alpha-2} W_j W_{j,n-j}) \\ & = O(n^{\alpha-2+\eta} \log \log n) + o(n^{\alpha-2+2/r}) \\ & \quad + 2 \max_{n-m \leq j \leq n} n^{\alpha-2} |W_j W_{j,n-j}|. \end{aligned}$$

Since  $r \geq 2$ , a.s.

$$(2.18) \quad \begin{aligned} & \max_{n-m \leq j \leq n} (j^\alpha U_j - n^\alpha U_n) \\ & = o(n^{\alpha-3/2+1/r}) + 2 \max_{n-m \leq j \leq n} n^{\alpha-2} |W_j W_{j,n-j}|, \end{aligned}$$

and similarly a.s.

$$(2.19) \quad \begin{aligned} & \max_{n \leq j \leq n+m} (n^\alpha U_j - j^\alpha U_n) \\ &= o(n^{\alpha-3/2+1/r}) + 2 \max_{n \leq j \leq n+m} n^\alpha j^{-2} |W_n W_{n,j-n}|. \end{aligned}$$

Assume that  $f_n = O(n^{3/2-1/r-\alpha})$ . Since  $\eta < 2/r \leq 1$ , by SLLN for delayed sums, a.s.

$$\max_{n-m \leq j \leq n} |W_{j,n-j}| = o(n^{1/r}),$$

and by a theorem of Erdős-Kac (Chung [8, p. 204]),  $\max_{j \leq n} n^{-1/2} |W_j|$  converges in distribution. Therefore

$$\max_{n-m \leq j \leq n} n^{-1/2-1/r} |W_j W_{j,n-j}| \xrightarrow{\mathcal{D}} 0,$$

as  $n \rightarrow \infty$ , and by (2.18)

$$(2.20) \quad n^{3/2-1/r-\alpha} \max_{n-m \leq j \leq n} (j^\alpha U_j - n^\alpha U_n) \xrightarrow{\mathcal{D}} 0$$

as  $n \rightarrow \infty$ , yielding (2.3) in probability by (2.16). Similarly for (2.4)–(2.6). Next, assume that

$$f_n = O(n^{3/2-1/r-\alpha} (\log \log n)^{-1/2}).$$

By LIL and SLLN for delayed sums, a.s.

$$\max_{n-m \leq j \leq n} n^{-1/2-1/r} W_n W_{n,j-n} = o((\log \log n)^{1/2}),$$

and by (2.19), a.s.

$$\max_{n \leq j \leq n+m} (n^\alpha U_j - j^\alpha U_n) = O(n^{\alpha-3/2+1/r} (\log \log n)^{1/2}),$$

yielding (2.6) by (2.17). Similarly for (2.3)–(2.5).

**THEOREM 1.** Let (2.1) hold, and  $n_0 = n_0(w)$  be a random variable such that

$$(2.21) \quad U_n > 0 \quad \text{for } n \geq n_0,$$

and let  $\alpha$ ,  $a_n$  and  $f_n$  be positive constants satisfying  $a_n = n^\alpha (1 + o(1))$ ,  $(a_n - n^\alpha) f_n = a + o(1)$  for some  $a \in (-\infty, \infty)$  and

$$(2.22) \quad \lim_{p \rightarrow 1} \limsup_{n \rightarrow \infty} \max \{f_j/f_k : n^\beta \leq j, k \leq n/\beta\} = 1.$$

(i) If  $1 \leq p < 4$ ,  $n^\alpha f_n R_n = c\mu + o(1)$  a.s., and  $f_n = O(n^{1-\alpha-1/p})$ , then as  $n \rightarrow \infty$ , a.s.

$$(2.23) \quad \max_{j \leq n} (a_j U_j - a_n U_n) f_n = o(1),$$

$$(2.24) \quad \sup_{j \geq n} (a_n U_n - a_j U_j) f_n = o(1),$$

$$(2.25) \quad \max_{n_0 \leq j \leq n} (a_j U_n - a_n U_j) f_n = o(1),$$

$$(2.26) \quad \sup_{j \geq n} (a_n U_j - a_j U_n) f_n = o(1).$$

(ii) Let (2.7) hold, and  $n^\alpha f_n R'_n = c\mu + o(1)$  a.s.. If  $f_n = O(n^{3/2-1/r-\alpha})$ , then (2.23)–(2.26) hold in probability and if

$$f_n = O(n^{3/2-1/r-\alpha} (\log \log n)^{-1/2})$$

then (2.23)–(2.26) hold a.s..

(iii) If  $f_n$  in (2.23)–(2.26) is replaced by  $n^\alpha f_n/a_j$  or  $n^\alpha f_n/a_n$ , then (2.23)–(2.26) still hold under the conditions of (i) or (ii).

**Proof.** We can assume that  $\mu = 1$ . Let  $0 < \beta < \delta^{2/\alpha} < 1$ . Since  $a_n = n^\alpha(1 + o(1))$  and  $U_n = 1 + o(1)$  a.s., there exists a random variable  $K = K(w) \geq n_0$  such that if  $j \geq K$  and  $n \geq K$ ,

$$(2.27) \quad \delta < n^{-\alpha} a_n U_j < \delta^{-1}.$$

Since  $a_n \rightarrow \infty$  and  $U_n = 1 + o(1)$  a.s.,

$$\max_{1 \leq j \leq K} (a_j U_j - a_n U_n) < 0 \text{ a.s.}$$

for all large (depending on  $w$ )  $n$ , and since  $U_j > 0$  for  $j \geq n_0$ ,

$$\max_{n_0 \leq j \leq K} (a_j U_n - a_n U_j) < 0 \text{ a.s.}$$

for all large (depending on  $w$ )  $n$ . For  $K < j < n\beta$ ,

$$a_j U_j - a_n U_n < \delta^{-1} j^\alpha - \delta n^\alpha \leq \delta n^\alpha (\beta^\alpha \delta^{-2} - 1) < 0,$$

$$a_j U_n - a_n U_j < \delta^{-1} j^\alpha - \delta n^\alpha < 0.$$

Hence there exists a random variable  $m_0 \geq K$  such that for  $n \geq m_0$ ,

$$(2.28) \quad \max_{j \leq n} (a_j U_j - a_n U_n) = \max_{n\beta \leq j \leq n} (a_j U_j - a_n U_n) \text{ a.s.},$$

$$(2.29) \quad \max_{n_0 \leq j \leq n} (a_j U_n - a_n U_j) = \max_{n\beta \leq j \leq n} (a_j U_n - a_n U_j) \text{ a.s.}.$$

For  $K \leq n \leq n/\beta \leq j$ , by (2.27)

$$a_n U_n - a_j U_j \leq \delta^{-1} n^\alpha - \delta j^\alpha \leq \delta j^\alpha (\beta^\alpha \delta^{-2} - 1) < 0,$$

$$a_n U_j - a_j U_n \leq \delta^{-1} n^\alpha - \delta j^\alpha < 0.$$

Hence if  $n \geq K$

$$(2.30) \quad \sup_{j \geq n} (a_n U_n - a_j U_j) = \max_{n \leq j \leq n/\beta} (a_n U_n - a_j U_j) \text{ a.s.},$$

$$(2.31) \quad \sup_{j \geq n} (a_n U_j - a_j U_n) = \max_{n \leq j \leq n/\beta} (a_n U_j - a_j U_n) \text{ a.s.}.$$

(i) Put  $U_n = V_n + c n^{-\alpha} f_n^{-1}$ . Then  $(V_n - n^{-1} S_n) n^\alpha f_n = n^\alpha f_n R_n - c = o(1)$  a.s.. By Lemma 1 (i), as  $n \rightarrow \infty$ , a.s.

$$(2.32) \quad \max_{n\beta \leq j \leq n} (j^\alpha V_j - n^\alpha V_n) f_n = o(1),$$

$$(2.33) \quad \max_{n \leq j \leq n/\beta} (n^\alpha V_j - j^\alpha V_n) f_n = o(1).$$

By (2.22)

$$(2.34) \quad \lim_{\beta \rightarrow 1} \limsup_{n \rightarrow \infty} \min \{f_j/f_k : n\beta \leq j, k \leq n/\beta\} = 1.$$

From (2.22) and (2.34), for any real constant  $d$ ,

$$(2.35) \quad \begin{aligned} & \lim_{\beta \rightarrow 1} \limsup_{n \rightarrow \infty} \max_{n\beta \leq j \leq n} (df_j^{-1} - df_n^{-1}) f_n \\ &= \lim_{\beta \rightarrow 1} \limsup_{n \rightarrow \infty} \max_{n\beta \leq j \leq n} d \left( \frac{f_n}{f_j} - 1 \right) = 0, \end{aligned}$$

$$(2.36) \quad \begin{aligned} & \lim_{\beta \rightarrow 1} \limsup_{n \rightarrow \infty} \max_{n \leq j \leq n/\beta} (dn^\alpha j^{-\alpha} f_j^{-1} - dj^\alpha n^{-\alpha} f_n^{-1}) f_n \\ &= \lim_{\beta \rightarrow 1} \limsup_{n \rightarrow \infty} \max_{n \leq j \leq n/\beta} d \left( \left( \frac{n}{j} \right)^\alpha \frac{f_n}{f_j} - \left( \frac{j}{n} \right)^\alpha \right) = 0. \end{aligned}$$

By (2.32), (2.35) and  $U_n = V_n + c n^{-\alpha} f_n^{-1}$ , we have

$$(2.37) \quad \lim_{\beta \rightarrow 1} \limsup_{n \rightarrow \infty} \max_{n\beta \leq j \leq n} (j^\alpha U_j - n^\alpha U_n) f_n = 0 \text{ a.s.},$$

and similarly by (2.33) and (2.36),

$$(2.38) \quad \lim_{\beta \rightarrow 1} \limsup_{n \rightarrow \infty} \max_{n \leq j \leq n/\beta} (n^\alpha U_j - j^\alpha U_n) f_n = 0 \text{ a.s.}.$$

Now put  $b_n = a_n - n^\alpha$ . Then by hypothesis,

$$(2.39) \quad b_n f_n = (a_n - n^\alpha) f_n = a + o(1),$$

and by (2.22) and (2.34), as  $n \rightarrow \infty$ ,

$$(2.40) \quad \begin{aligned} & \lim_{\beta \rightarrow 1} \limsup_{n \rightarrow \infty} \max_{n\beta \leq j \leq n} (b_j - b_n) f_n \\ &= \lim_{\beta \rightarrow 1} \limsup_{n \rightarrow \infty} \max_{n\beta \leq j \leq n} \left( b_j f_j \frac{f_n}{f_j} - b_n f_n \right) = 0. \end{aligned}$$

Since  $U_n = 1 + o(1)$  a.s., by (2.22) a.s.

$$\begin{aligned}
 & \max_{n\beta \leq j \leq n} (b_j U_j - b_n U_n) f_n \\
 & \leq \max_{n\beta \leq j \leq n} ((b_j - b_n) + |b_j(U_j - 1)| + |b_n(U_n - 1)|) f_n \\
 & = \max_{n\beta \leq j \leq n} (b_j - b_n) f_n + o(1).
 \end{aligned}$$

and by (2.40)

$$(2.41) \quad \lim_{\beta \rightarrow 1} \limsup_{n \rightarrow \infty} \max_{n\beta \leq j \leq n} (b_j U_j - b_n U_n) f_n = 0 \text{ a.s.},$$

and similarly

$$(2.42) \quad \lim_{\beta \rightarrow 1} \limsup_{n \rightarrow \infty} \max_{n \leq j \leq n/\beta} (b_n U_j - b_j U_n) f_n = 0 \text{ a.s.}.$$

From (2.37), (2.39) and (2.41),

$$(2.43) \quad \lim_{\beta \rightarrow 1} \limsup_{n \rightarrow \infty} \max_{n\beta \leq j \leq n} (a_j U_j - a_n U_n) f_n = 0 \text{ a.s.}$$

yielding (2.23) by (2.28); and from (2.38), (2.39) and (2.42),

$$(2.44) \quad \lim_{\beta \rightarrow 1} \limsup_{n \rightarrow \infty} \max_{n \leq j \leq n/\beta} (a_n U_j - a_j U_n) f_n = 0 \text{ a.s.}$$

yielding (2.26) by (2.31). The proofs of (2.24) and (2.25) are similar.

(ii) Since  $X = 1$  a.s.,  $U_n = 1 + R'_n - (n^{-1} W_n)^2$ , where  $W_n = \sum_i Y_i$ . For (2.23)–(2.26), we only give a proof of (2.25); the proofs for others are similar. For (2.25), by (2.29) it suffices to show that a.s. (or in probability)

$$(2.45) \quad \lim_{\beta \rightarrow 1} \limsup_{n \rightarrow \infty} \max_{n\beta \leq j \leq n} (a_j U_n - a_n U_j) f_n = o(1).$$

Put  $U_n = V_n + c n^{-\alpha} f_n^{-1}$ , Then  $(V_n - 1 + (n^{-1} W_n)^2) n^\alpha f_n = n^\alpha f_n R'_n - c = o(1)$  a.s.. By Lemma 1 (ii), (2.45) holds a.s. (or in probability) for  $a_n = n^\alpha$  and  $U_n = V_n$  if  $f_n = O(n^{3/2-\alpha-1/r} (\log \log n)^{-1/2})$ , (or if  $f_n = O(n^{3/2-\alpha-1/r})$ ). From (2.22) and (2.34),

$$\begin{aligned}
 (2.46) \quad & \lim_{\beta \rightarrow 1} \limsup_{n \rightarrow \infty} \max_{n\beta \leq j \leq n} (c j^\alpha n^{-\alpha} f_n^{-1} - c n^\alpha j^{-\alpha} f_j^{-1}) f_n \\
 & = \lim_{\beta \rightarrow 1} \limsup_{n \rightarrow \infty} \max_{n\beta \leq j \leq n} c \left( \left(\frac{j}{n}\right)^\alpha - \left(\frac{n}{j}\right)^\alpha \frac{f_n}{f_j} \right) = 0,
 \end{aligned}$$

yielding (2.45) for  $a_n = n^\alpha$  and  $U_n = V_n + c n^{-\alpha} f_n^{-1}$ . Now put  $a_n = n^\alpha = b_n$ . By hypothesis, (2.39) holds. By (2.22), (2.34) and  $U_n = 1 + o(1)$  a.s., we have a.s.

$$\begin{aligned}
 & \max_{n\beta \leq j \leq n} (b_j U_n - b_n U_j) f_n \\
 (2.47) \quad & \leq \max_{n\beta \leq j \leq n} ((b_j - b_n) + |b_j(U_n - 1)| + |b_n(U_j - 1)|) f_n \\
 & = \max_{n\beta \leq j \leq n} (b_j - b_n) f_n + o(1),
 \end{aligned}$$

and then by (2.40), (2.45) holds for  $a_n = b_n$ . Therefore (2.45) holds.

(iii) The proof of (2.28) yields that for  $0 < \beta < 1$ , there exists a random variable  $m_0$  such that for  $n \geq m_0$ ,

$$\begin{aligned}
 & \max_{j \leq n} (U_j - a_n U_n / a_j) n^\alpha f_n \\
 (2.48) \quad & = \max_{n\beta \leq j \leq n} (U_j - a_n U_n / a_j) n^\alpha f_n \\
 & = \max_{n\beta \leq j \leq n} (a_j U_j - a_n U_n) n^\alpha f_n / a_j.
 \end{aligned}$$

Hence to prove

$$(2.49) \quad \max_{j \leq n} (U_j - a_n U_n / a_j) n^\alpha f_n = o(1) \text{ a.s.},$$

as  $n \rightarrow \infty$ , it suffices to show

$$(2.50) \quad \lim_{\beta \rightarrow 1} \limsup_{n \rightarrow \infty} \max_{n\beta \leq j \leq n} (a_j U_j - a_n U_n) \left(\frac{n}{j}\right)^\alpha \frac{j^\alpha}{a_j} f_n = 0 \text{ a.s.},$$

which holds by (2.23). The proofs for the other cases are similar.

**THEOREM 2.** Let (2.1) and (2.21) hold and  $\text{Var } X = \mu^2 \sigma^2 < \infty$  and  $\alpha > 0$ .

(i) If  $n^{1/2} R_n = c\mu + o(1)$  a.s., and  $0 < a_n = n^\alpha (1 + o(n^{-1/2}) + o(n^{-1/2}))$ , then as  $n \rightarrow \infty$

$$(2.51) \quad n^{1/2} (1 - \max_{j \leq n} a_j U_j / \mu a_n) + c \xrightarrow{\mathcal{D}} N(0, \sigma^2),$$

$$(2.52) \quad n^{1/2} (1 - \inf_{j \geq n} a_j U_j / \mu a_n) + c \xrightarrow{\mathcal{D}} N(0, \sigma^2),$$

$$(2.53) \quad n^{1/2} (1 - \min_{n_0 \leq j \leq n} a_n U_j / \mu a_j) + c \xrightarrow{\mathcal{D}} N(0, \sigma^2),$$

$$(2.54) \quad n^{1/2} (1 - \sup_{j \geq n} a_n U_j / \mu a_j) + c \xrightarrow{\mathcal{D}} N(0, \sigma^2),$$

and if  $n^{1/2} (2 \log \log n)^{-1/2} R_n = c\mu + o(1)$  a.s., then as  $n \rightarrow \infty$

$$(2.55) \quad \overline{\lim} n^{1/2} (2 \log \log n)^{-1/2} (1 - \max_{j \leq n} a_j U_j / \mu a_n) + c = \pm \sigma \text{ a.s.},$$

$$(2.56) \quad \underline{\lim} n^{1/2} (2 \log \log n)^{-1/2} (1 - \inf_{j \geq n} a_j U_j / \mu a_n) + c = \pm \sigma \text{ a.s.},$$

$$(2.57) \quad \underline{\lim} n^{1/2} (2 \log \log n)^{-1/2} \left( 1 - \min_{n_0 \leq j \leq n} a_n U_j / \mu a_j \right) + c = \pm \sigma \text{ a.s.},$$

$$(2.58) \quad \underline{\lim} n^{1/2} (2 \log \log n)^{-1/2} \left( 1 - \sup_{j \geq n} a_n U_j / \mu a_j \right) + c = \pm \sigma \text{ a.s.}.$$

(ii) Let (2.7) hold and  $\text{Var } Y = \mu^2 \hat{\sigma}^2 < \infty$ .

(a) If  $n R'_n = c\mu + o(1)$  a.s., and  $0 < a_n = n^\alpha (1 + a n^{-1} + o(n^{-1}))$ , then as  $n \rightarrow \infty$

$$(2.59) \quad n \left( 1 - \max_{j \leq n} a_j U_j / \mu a_n \right) + c \xrightarrow{\mathcal{D}} N^2(0, \hat{\sigma}^2),$$

$$(2.60) \quad n \left( 1 - \inf_{j \geq n} a_j U_j / \mu a_n \right) + c \xrightarrow{\mathcal{D}} N^2(0, \hat{\sigma}^2),$$

$$(2.61) \quad n \left( 1 - \min_{n_0 \leq j \leq n} a_n U_j / \mu a_j \right) + c \xrightarrow{\mathcal{D}} N^2(0, \hat{\sigma}^2),$$

$$(2.62) \quad n \left( 1 - \sup_{j \geq n} a_n U_j / \mu a_j \right) + c \xrightarrow{\mathcal{D}} N^2(0, \hat{\sigma}^2).$$

(b) If  $n(2 \log \log n)^{-1} R'_n = c\mu + o(1)$  a.s. and  $0 < a_n = n^\alpha (1 + a n^{-1} (2 \log \log n) + o(n^{-1} \log \log n))$ , then as  $n \rightarrow \infty$

$$(2.63) \quad \underline{\lim} n(2 \log \log n)^{-1} \left( 1 - \max_{j \leq n} a_j U_j / \mu a_n \right) + c = \begin{cases} \hat{\sigma}^2 \\ 0 \end{cases} \text{ a.s.},$$

$$(2.64) \quad \underline{\lim} n(2 \log \log n)^{-1} \left( 1 - \inf_{j \geq n} a_j U_j / \mu a_n \right) + c = \begin{cases} \hat{\sigma}^2 \\ 0 \end{cases} \text{ a.s.},$$

$$(2.65) \quad \underline{\lim} n(2 \log \log n)^{-1} \left( 1 - \min_{n_0 \leq j \leq n} a_n U_j / \mu a_j \right) + c = \begin{cases} \hat{\sigma}^2 \\ 0 \end{cases} \text{ a.s.},$$

$$(2.66) \quad \underline{\lim} n(2 \log \log n)^{-1} \left( 1 - \sup_{j \geq n} a_n U_j / \mu a_j \right) + c = \begin{cases} \hat{\sigma}^2 \\ 0 \end{cases} \text{ a.s..}$$

**Proof.** We can assume that  $\mu = 1$ .

(i) We shall only prove (2.52) and (2.57); the proofs of the rest are similar. Put  $f_n = n^{1/2-\alpha}$ . Then  $n^\alpha f_n R_n = c\mu + o(1)$  a.s. and  $(a_n - n^\alpha) f_n = a + o(1)$ . By (2.24) and (2.25) in Theorem 1 (i) and (iii), a.s.

$$(2.67) \quad n^{1/2} \sup_{j \geq n} (U_n - a_j U_j / a_n) = o(1),$$

$$(2.68) \quad n^{1/2} \max_{n_0 \leq j \leq n} (U_n - a_n U_j / a_j) = o(1).$$

By (2.67) and the central limit theorem for sums of i.i.d. random variables,

$$\begin{aligned}
& n^{1/2} \left( 1 - \inf_{j \geq n} a_j U_j / a_n \right) \\
&= n^{1/2} (1 - U_n) + n^{1/2} \left( U_n - \inf_{j \geq n} a_j U_j / a_n \right) \\
&= n^{1/2} (1 - U_n) + o(1) \\
&= n^{-1/2} (n - S_n) - n^{1/2} R_n + o(1) \quad \text{a.s.} \\
&\xrightarrow{\mathcal{D}} N(-c, \sigma^2), \quad \text{as } n \rightarrow \infty,
\end{aligned}$$

yielding (2.52). By (2.68), a.s.

$$\begin{aligned}
1 - \min_{n_0 \leq j \leq n} a_n U_j / a_j &= (1 - U_n) + o(n^{-1/2}) \\
&= n^{-1} (n - S_n) - R_n + o(n^{-1/2})
\end{aligned}$$

yielding (2.57) by the law of the iterated logarithm.

(ii) We shall only prove (2.61) and (2.66); the proofs of the rest are similar. To prove (2.61), put  $f_n = n^{1-\alpha}$ . Then  $n^\alpha f_n R'_n = c\mu + o(1)$  a.s.,  $(a_n - n^\alpha) f_n = a + o(1)$  and  $f_n = O(n^{3/2-1/r-\alpha})$ . By (2.25) in Theorem 1 (ii) and (iii), as  $n \rightarrow \infty$ ,

$$(2.69) \quad n \max_{n_0 \leq j \leq n} (U_n - a_n U_j / a_j) \xrightarrow{\mathcal{D}} 0.$$

By (2.69) and the central limit theorem,

$$\begin{aligned}
& n \left( 1 - \min_{n_0 \leq j \leq n} a_n U_j / a_j \right) \\
&= n(1 - U_n) + n \max_{n_0 \leq j \leq n} (U_n - a_n U_j / a_j) \\
&= n^{-1} \left( \sum_1^n Y_i \right)^2 - n R'_n + n \max_{n_0 \leq j \leq n} (U_n - a_n U_j / a_j) \\
&\xrightarrow{\mathcal{D}} N^2(0, \hat{\sigma}^2) - c, \quad \text{as } n \rightarrow \infty,
\end{aligned}$$

yielding (2.61). To prove (2.66), put  $f_n = n^{1-\alpha}(2 \log \log n)^{-1}$ . Then  $n^\alpha f_n R'_n = c\mu + o(1)$  a.s.,  $(a_n - n^\alpha) f_n = a + o(1)$  and  $f_n = O(n^{3/2-1/r-\alpha}(\log \log n)^{-1/2})$ . By (2.26) in Theorem 1 (ii) and (iii), a.s.

$$(2.70) \quad n(2 \log \log n)^{-1} \sup_{j \geq n} (a_n U_j / a_j - U_n) = o(1).$$

By (2.70), a.s.

$$\begin{aligned}
 & n(2 \log \log n)^{-1} \left( 1 - \sup_{j \geq n} a_n U_j / a_j \right) \\
 (2.71) \quad & = n(2 \log \log n)^{-1} \left( \left( n^{-1} \sum_1^n Y_i \right)^2 - R'_n \right) + o(1) \\
 & = (2n \log \log n)^{-1} \left( \sum_1^n Y_i \right)^2 - c + o(1).
 \end{aligned}$$

Therefore by the law of the iterated logarithm, as  $n \rightarrow \infty$

$$\overline{\lim} n(2 \log \log n)^{-1} \left( 1 - \sup_{j \geq n} a_n U_j / a_j \right) + c = \sigma^2 \text{ a.s.},$$

and by Strassen's law of the iterated logarithm (Strassen [15]), as  $n \rightarrow \infty$

$$\underline{\lim} n(2 \log \log n)^{-1} \left( 1 - \sup_{j \geq n} a_n U_j / a_j \right) + c = 0 \text{ a.s.}.$$

**REMARK 1.** If  $R_n = 0$ ,  $a_n = n^\alpha$ , and  $f_n = n^{1-\alpha-1/p}$  with  $0 < \alpha \leq 1$  and  $1 \leq p \leq 2$ , (2.23) and (2.24) in Part (i) of Theorem 1 have been given in Chow and Hsiung [5]. For  $p = 2$ , the same paper has also obtained (2.51), (2.52), (2.55) and (2.56) in Part (i) of Theorem 2.

**REMARK 2.** Let  $Z, Z_1, Z_2, \dots$  be independent and identically distributed random variables with  $EZ = 0$ ,  $\text{Var } Z = \mu > 0$ , and  $EZ^4 < \infty$ . Put  $\bar{Z}_n = n^{-1} \sum_1^n Z_i$ ,  $V_n = n^{-1} \sum_1^n (Z_i - \bar{Z}_n)^2$ . If we identify  $(Z_n - \theta)^2$  with  $X_n$  and  $(\bar{Z}_n - \theta)^2$  with  $R_n$ , then  $V_n = U_n$  in Theorems 1 and 2. Thus Part (i) of both theorems holds with  $V$  in place of  $U$ . In particular if  $E((Z - \theta)^2 - \mu)^2 > 0$ , we have the central limit theorem and the law of the iterated logarithm for (2.51)–(2.54) and (2.55)–(2.58) respectively. But if  $E((Z - \theta)^2 - \mu)^2 = 0$ , then  $|Z - \theta| = \mu^{1/2}$  a.s.. Since  $EZ = \theta$ ,  $P[Z - \theta = \mu^{1/2}] = \frac{1}{2} = P[Z - \theta = -\mu^{1/2}]$  and  $Z - \theta$  is a symmetric Bernoulli random variable. Hence

$$V_n = \mu - \left( n^{-1} \sum_1^n (Z_i - \theta) \right)^2$$

satisfies (2.7) with  $R'_n = 0$ . Part (ii) of both Theorem 1 and Theorem 2 is devoted to include this case.

**3. Limiting behaviour of the first passage times, the last times and the numbers of boundary crossings.** Let  $X, X_1, X_2, \dots$  be independent and identically distributed random variables with

$EX = \mu > 0$  and  $\text{Var } X = \mu^2 \sigma^2 < \infty$ , and  $R_1, R_2, \dots$  be random variables with  $R_n = o(1)$  a.s. as  $n \rightarrow \infty$ . For  $\alpha > 0$ , let

$$(3.1) \quad S_n = \sum_i^n X_i, \quad U_n = n^{-1} S_n + R_n,$$

$$0 < a_n = n^\alpha (1 + o(1)) \quad \text{as } n \rightarrow \infty.$$

For  $\lambda > 0$ , let  $0 < n_\lambda = o(\lambda^{-1/\alpha})$  and

$$(3.2) \quad N = N_\lambda = \inf \{n \geq n_\lambda : U_n \geq (\lambda a_n)^{-1}\},$$

$$T = T_\lambda = \inf \{n \geq n_\lambda : 0 < U_n \leq \lambda a_n\},$$

$$(3.3) \quad N' = N'_\lambda = \sup \{n \geq n_\lambda : U_n < (\lambda a_n)^{-1}\},$$

$$T' = T'_\lambda = \sup \{n \geq n_\lambda : U_n > \lambda a_n\},$$

$$(3.4) \quad N'' = N''_\lambda = \sum_{n_\lambda}^\infty I_{[U_n < (\lambda a_n)^{-1}]},$$

$$T'' = T''_\lambda = \sum_{n_\lambda}^\infty I_{[U_n > \lambda a_n]}.$$

THEOREM 3. Let (3.1)–(3.4) hold and for some  $q > 0$ ,

$$(3.5) \quad 0 < a_n = n^\alpha (1 + an^{-q} + o(n^{-q})).$$

Then as  $\lambda \rightarrow 0$ ,

$$(3.6) \quad \infty > N \rightarrow \infty, \quad \infty > T \rightarrow \infty,$$

$$\lambda N^\alpha \rightarrow \mu^{-1}, \quad \lambda T^\alpha \rightarrow \mu \quad \text{a.s.},$$

and if  $N$  and  $T$  are replaced by  $N'$  or  $N''$  and  $T'$  or  $T''$  respectively, (3.6) still holds.

(i) If  $q = \frac{1}{2}$ , and  $n^q R_n = c\mu + o(1)$  a.s., then as  $\lambda \rightarrow 0$

$$(3.7) \quad \alpha(\lambda\mu)^{1/2\alpha} (N - (\lambda\mu)^{-1/\alpha}) + a + c \xrightarrow{\mathcal{D}} N(0, \sigma^2),$$

$$(3.8) \quad \alpha\left(\frac{\lambda}{\mu}\right)^{1/2\alpha} \left(T - \left(\frac{\lambda}{\mu}\right)^{-1/\alpha} + a - c\right) \xrightarrow{\mathcal{D}} N(0, \sigma^2),$$

and if  $N$  and  $T$  are replaced by  $N'$  or  $N''$  and  $T'$  or  $T''$  respectively, (3.7)–(3.8) still hold.

(ii) Let

$$(3.9) \quad X = \mu \quad \text{a.s.}, \quad R_n = R'_n - \left(n^{-1} \sum_i^n Y_i\right)^2,$$

where  $Y, Y_1, Y_2, \dots$  are i.i.d. random variables with

$$(3.10) \quad EY = 0, \quad 0 < EY^2 = \mu\hat{\sigma}^2 < \infty.$$

If  $q = 1$  and  $nR'_n = c\mu + o(1)$  a.s., and  $G(x)$  is the distribution function of a chi-squared distribution with one degree of freedom, then for  $-\infty < x, y < \infty$ , as  $\lambda \rightarrow 0$

$$(3.11) \quad \sup_x |P[\alpha(N - (\lambda\mu)^{-1/\alpha}) + a + c \leq x] - G(x, \hat{\sigma}^{-2})| \rightarrow 0$$

where

$$(3.12) \quad \begin{aligned} & (\lambda\mu)^{-1/\alpha} + (x_\lambda - a - c) \alpha^{-1} \\ &= [(\lambda\mu)^{-1/\alpha} + (x - a - c) \alpha^{-1}], \end{aligned}$$

and

$$(3.13) \quad \sup_y |P\left[\alpha\left(\left(\frac{\lambda}{\mu}\right)^{-1/\alpha} - T\right) - a + c \leq y\right] - G(y, \hat{\sigma}^{-2})| \rightarrow 0$$

where

$$(3.14) \quad \left(\frac{\lambda}{\mu}\right)^{-1/\alpha} - \frac{y_\lambda + a - c}{\alpha} = \left[\left(\frac{\lambda}{\mu}\right)^{-1/\alpha} - \frac{y + a - c}{\alpha}\right],$$

and if  $N$  is replaced by  $N'$  and  $T$  by  $T'$ , then (3.11) and (3.13) still hold.

**Proof.** By SLLN, we can easily see that  $\infty > T \rightarrow \infty$  a.s. as  $\lambda \rightarrow 0$ . By (3.1) and (3.5),

$$U_T = \mu + o(1) \text{ a.s.}, \quad a_n = n^\alpha(1 + o(1)).$$

Since  $U_T \leq \lambda a_T$ ,  $\liminf \lambda T^\alpha \geq \mu$  a.s., and  $T_\lambda > n_\lambda + 1$  a.s. for all sufficiently small  $\lambda$ . Hence a.s. as  $\lambda \rightarrow 0$

$$\begin{aligned} \limsup \lambda T^\alpha &= \limsup \lambda(T - 1)^\alpha \\ &= \limsup \lambda a_{T-1} \leq \lim U_{T-1} = \mu \end{aligned}$$

and  $\lambda T^\alpha = \mu + o(1)$  a.s.. Similarly for  $N$ . Hence (3.6) holds. The proofs of  $N'$  and  $T'$  satisfying (3.6) are similar. Since  $N - 1 \leq N'' \leq N'$ , (3.6) holds for  $N''$  also. Similarly for  $T''$ . For the rest of the proof, we can assume that  $\mu = 1$ .

(i) Put  $z = N - \lambda^{-1/\alpha}$ . By (3.6),  $\lambda^{1/\alpha} z = o(1)$  a.s. and by (3.5) a.s.

$$\begin{aligned} \lambda a_N &= (1 + \lambda^{1/\alpha} z)^\alpha (1 + (a + o(1)) N^{-q}) \\ &= (1 + (\alpha + o(1)) \lambda^{1/\alpha} z)(1 + (a + o(1)) N^{-q}) \\ &= 1 + (\alpha + o(1)) \lambda^{1/\alpha} z + (a + o(1)) N^{-q} \end{aligned}$$

$$(3.15) \quad \begin{aligned} N^q(\lambda a_N - 1) &= (\alpha + o(1)) \lambda^{1/\alpha} z N^q + a + o(1) \\ &= (\alpha + o(1)) \lambda^{(1-q)/\alpha} z + a + o(1). \end{aligned}$$

Similarly a.s.

$$(3.16) \quad \begin{aligned} N^q(\lambda a_{N-1} - 1) &= (\alpha + o(1)) \lambda^{(1-q)/\alpha} (z - 1) + a + o(1). \end{aligned}$$

By the definition of  $N$  and (3.6), for  $N_\lambda > n_\lambda + 1$ , a.s.

$$U_N - 1 \geq (\lambda a_N)^{-1} - 1 = (1 - \lambda a_N)(1 + o(1))$$

$$U_{N-1} - 1 \leq (\lambda a_{N-1})^{-1} - 1 = (1 - \lambda a_{N-1})(1 + o(1)).$$

By (3.15) and (3.16), a.s.

$$N^q(U_N - 1) \geq -(\alpha + o(1)) \lambda^{(1-q)/\alpha} z - a + o(1)$$

$$N^q(U_{N-1} - 1) \leq -(\alpha + o(1)) \lambda^{(1-q)/\alpha} (z - 1) - a + o(1).$$

Hence a.s.

$$(3.17) \quad \alpha \lambda^{(1-q)/\alpha} z \geq N^q(1 - U_N)(1 + o(1)) - a + o(1) \equiv u,$$

say;

$$(3.18) \quad \alpha \lambda^{(1-q)/\alpha} (z - 1) \leq N^q(1 - U_{N-1})(1 + o(1)) - a + o(1) \equiv v,$$

say. Since  $q = \frac{1}{2}$ , and  $n^q R_n = c + o(1)$  a.s., by Anscombe's theorem ([Anscombe, 1952]),

$$(3.19) \quad N^q(U_N - 1) = N^q \left( N^{-1} \sum_1^N (X_i - 1) + R_N \right) \xrightarrow{\mathcal{D}} N(c, \sigma^2),$$

and similarly,

$$(3.20) \quad N^q(U_{N-1} - 1) \xrightarrow{\mathcal{D}} N(c, \sigma^2).$$

From (3.17)–(3.20), we have (3.7) immediately. The proof of (3.8) is identical. The proofs for  $N'$  satisfying (3.7) and for  $T'$  satisfying (3.8) are similar. Since  $N - 1 \leq N'' \leq N'$ , (3.7) holds for  $N''$ . Similarly for  $T''$ .

(ii) Since  $n R'_n = c + o(1)$  a.s., by (3.9) as  $\lambda \rightarrow 0$

$$N(1 - U_N) = -NR'_N + N^{-1} \left( \sum_1^N Y_i \right)^2 \xrightarrow{\mathcal{D}} \hat{\sigma}^2 \chi_1^2 - c,$$

$$N(1 - U_{N-1}) \xrightarrow{\mathcal{D}} \hat{\sigma}^2 \chi_1^2 - c.$$

Since  $q = 1$ , as  $\lambda \rightarrow 0$ ,

$$u + a + c \xrightarrow{\mathcal{D}} \hat{\sigma}^2 \chi_1^2,$$

$$v + a + c \xrightarrow{\mathcal{D}} \hat{\sigma}^2 \chi_1^2,$$

and since  $G$  is continuous, as  $\lambda \rightarrow 0$

$$(3.21) \quad \sup_x |P[u + a + c \leq x] - G(x \hat{\sigma}^{-2})| \rightarrow 0,$$

$$(3.22) \quad \sup_x |P[v + a + c < x] - G(x \hat{\sigma}^{-2})| \rightarrow 0.$$

For  $-\infty < x < \infty$ , by (3.12)

$$\begin{aligned} P[\alpha z + a + c \leq x] &= P[N \leq \lambda^{-1/\alpha} + (x - a - c) \alpha^{-1}] \\ &= P[N \leq \lambda^{-1/\alpha} + (x_\lambda - a - c) \alpha^{-1}] \\ &= P[\alpha z + a + c \leq x_\lambda] \\ &= P[N - 1 < \lambda^{-1/\alpha} + (x_\lambda - a - c) \alpha^{-1}] \\ &= P[\alpha(z - 1) + a + c < x_\lambda]. \end{aligned}$$

By (3.17) and (3.18),

$$(3.23) \quad \begin{aligned} P[v + a + c < x_\lambda] &\leq P[\alpha z + a + c \leq x] \\ &\leq P[u + a + c \leq x_\lambda] \end{aligned}$$

yielding (3.11) by (3.21) and (3.22). Similarly for (3.13). The proofs of  $N'$  and  $T'$  are similar.

**THEOREM 4.** Let (3.1)–(3.4) hold.

(i) Let (3.5) hold with  $q = \frac{1}{2}$ . If  $n^{1/2}(2 \log \log n)^{-1/2} R_n = c\mu + o(1)$  a.s., then as  $\lambda \rightarrow 0$

$$(3.24) \quad \begin{aligned} \overline{\lim} \alpha(\lambda\mu)^{1/2\alpha} (2 \log \log (\lambda\mu)^{-1})^{-1/2} (N - (\lambda\mu)^{-1/\alpha}) + c \\ = \pm \sigma \text{ a.s.}, \end{aligned}$$

$$(3.25) \quad \begin{aligned} \overline{\lim} \alpha \left( \frac{\lambda}{\mu} \right)^{1/2\alpha} \left( 2 \log \log \left( \frac{\lambda}{\mu} \right)^{-1} \right)^{-1/2} \left( T - \left( \frac{\lambda}{\mu} \right)^{-1/\alpha} \right) + c \\ = \pm \sigma \text{ a.s.}, \end{aligned}$$

and if  $N$  and  $T$  are replaced by  $N'$  or  $N''$  and  $T'$  or  $T''$  respectively, (3.24)–(3.25) still hold.

(ii) Let (3.9) and (3.10) hold. If  $n(2 \log \log n)^{-1} R'_n = c\mu + o(1)$  a.s., and  $a_n = n^\alpha(1 + an^{-1}(2 \log \log n) + o(n^{-1} \log \log n))$ , then as  $\lambda \rightarrow 0$ ,

$$(3.26) \quad \limsup \alpha(2 \log \log (\lambda\mu)^{-1})^{-1} (N - (\lambda\mu)^{-1/\alpha}) + c + \alpha = \hat{\sigma}^2 \text{ a.s.},$$

$$(3.27) \quad \liminf \alpha(2 \log \log (\lambda\mu)^{-1})^{-1} (N - (\lambda\mu)^{-1/\alpha}) + c + \alpha = 0 \text{ a.s.},$$

$$(3.28) \quad \limsup \alpha \left(2 \log \log \left(\frac{\lambda}{\mu}\right)^{-1}\right)^{-1} \left(T - \left(\frac{\lambda}{\mu}\right)^{-1/\alpha}\right) + c + \alpha = \hat{\sigma}^2 \text{ a.s.},$$

$$(3.29) \quad \liminf \alpha \left(2 \log \log \left(\frac{\lambda}{\mu}\right)^{-1}\right)^{-1} \left(T - \left(\frac{\lambda}{\mu}\right)^{-1/\alpha}\right) + c + \alpha = 0 \text{ a.s.},$$

and if  $N$  and  $T$  are replaced by  $N'$  or  $N''$  and  $T'$  or  $T''$  respectively, (3.26)–(3.29) still hold.

**Proof.** We can assume that  $\mu = 1$

(i) For  $0 < \eta < 1$ ,  $\eta \neq c/\sigma$ , by (2.55) of Theorem 2,

$$\begin{aligned} 1 &= P \left[ \max_{j \leq n} a_j U_j < a_n - a_n(\eta\sigma - c) n^{-1/2} (2 \log \log n)^{1/2} \text{ i.o.} \right] \\ &= P \left[ \max_{j \leq n} a_j U_j < n^\alpha - (\eta\sigma - c) n^{\alpha-1/2} (2 \log \log n)^{1/2} \text{ i.o.} \right]. \end{aligned}$$

For all large  $n$ , put

$$\lambda_n^{-1} = n^\alpha - (\eta\sigma - c) n^{\alpha-1/2} (2 \log \log n)^{1/2}.$$

Then  $n^\alpha \lambda_n = 1 + o(1)$ ,  $n = \lambda_n^{-1/\alpha} (1 + (\eta\sigma - c) \alpha^{-1} n^{-1/2} (2 \log \log n)^{1/2} \cdot (1 + o(1)))$ , and

$$\begin{aligned} 1 &= P \left[ \max_{j \leq n} a_j U_j < \lambda_n^{-1} \text{ i.o.} \right] \\ (3.30) \quad &= P[N_{\lambda_n} > n \text{ i.o.}] \\ &= P[N_{\lambda_n} > \lambda_n^{-1/\alpha} (1 + (\eta\sigma - c) \alpha^{-1} n^{-1/2} (2 \log \log n)^{1/2} \cdot (1 + o(1)))) \text{ i.o.}]. \end{aligned}$$

Hence as  $n \rightarrow \infty$

$$\limsup \alpha \lambda_n^{1/2\alpha} (2 \log \log \lambda_n^{-1})^{-1/2} (N_{\lambda_n} - \lambda_n^{-1/\alpha}) \geq \eta\sigma - c \text{ a.s.}.$$

Letting  $\eta \rightarrow 1$ ,

$$\limsup \alpha \lambda_n^{1/2\alpha} (2 \log \log \lambda_n^{-1})^{-1/2} (N_{\lambda_n} - \lambda_n^{-1/\alpha}) + c \geq \sigma \text{ a.s..}$$

And a fortiori

$$(3.31) \quad \limsup_{\lambda \rightarrow 0} \alpha \lambda^{1/2\alpha} (2 \log \log \lambda^{-1})^{-1/2} (N_\lambda - \lambda^{-1/\alpha}) + c \geq \sigma \text{ a.s..}$$

For  $1 < \eta \neq c/\sigma$ , and small  $\lambda$ , put

$$m_\lambda = [\lambda^{-1/\alpha} + (\eta\sigma - c) \alpha^{-1} \lambda^{-1/2\alpha} (2 \log \log \lambda^{-1})^{1/2}] - 1.$$

Then  $\lambda m_\lambda^\alpha = 1 + o(1)$ ,  $\lambda^{1/\alpha} m_\lambda = 1 + (\eta\sigma - c) \alpha^{-1} \lambda^{1/2\alpha} (2 \log \log \lambda^{-1})^{1/2} \cdot (1 + o(1))$ , and

$$\begin{aligned} (\lambda a_{m_\lambda})^{-1} &= 1 - (\eta\sigma - c) \lambda^{1/2\alpha} (2 \log \log \lambda^{-1})^{1/2} (1 + o(1)) \\ &= 1 - (\eta\sigma - c) m_\lambda^{-1/2} (2 \log \log m_\lambda)^{1/2} (1 + o(1)), \end{aligned}$$

$$\begin{aligned} [\alpha \lambda^{1/2\alpha} (2 \log \log \lambda^{-1})^{-1/2} (N - \lambda^{-1/\alpha}) + c > \eta\sigma] \\ \subset [N > m_\lambda] = \left[ \max_{j \leq m_\lambda} a_j U_j / a_{m_\lambda} < (\lambda a_{m_\lambda})^{-1} \right] \\ = \left[ \max_{j \leq m_\lambda} a_j U_j / a_{m_\lambda} - 1 \right. \\ \quad \left. < -(\eta\sigma - c) m_\lambda^{-1/2} (2 \log \log m_\lambda)^{1/2} (1 + o(1)) \right] \\ = \left[ 1 - \max_{j \leq m_\lambda} a_j U_j / a_{m_\lambda} \right. \\ \quad \left. > (\eta\sigma - c) m_\lambda^{-1/2} (2 \log \log m_\lambda)^{1/2} (1 + o(1)) \right]. \end{aligned}$$

By (2.55) in Theorem 2,

$$\begin{aligned} P \left[ 1 - \max_{j \leq n} a_j U_j / a_n \right. \\ \left. \geq (\eta\sigma - c) n^{-1/2} (2 \log \log n)^{1/2} (1 + o(1)) \text{ i.o.} \right] = 0. \end{aligned}$$

Hence

$$(3.32) \quad \limsup_{\lambda \rightarrow 0} \alpha \lambda^{1/2\alpha} (2 \log \log \lambda^{-1})^{-1/2} (N - \lambda^{-1/\alpha}) + c \leq \sigma \text{ a.s.},$$

yielding the  $\limsup$  part of (3.24) by (3.31). The  $\liminf$  part is proved in the same way. The proofs for (3.25),  $N'$  and  $T'$  are similar. Since  $N - 1 \leq N'' \leq N'$ , (3.24) and (3.25) hold for  $N''$  and similarly for  $T''$ .

(ii) For  $0 < \eta < 1$ ,  $\eta \neq (c + a) \hat{\sigma}^{-2}$ ,  $\eta \neq c \hat{\sigma}^{-2}$ , by (2.63) of Theorem 2 (ii) (b),

$$\begin{aligned} 1 &= P \left[ \max_{j \leq n} a_j U_j < a_n - 2a_n(\eta \hat{\sigma}^2 - c) n^{-1} \log \log n \text{ i.o.} \right] \\ &= P \left[ \max_{j \leq n} a_j U_j < n^\alpha - 2(\eta \hat{\sigma}^2 - c - a) n^{\alpha-1} \log \log n \text{ i.o.} \right]. \end{aligned}$$

For all large  $n$ , put

$$\lambda_n^{-1} = n^\alpha - 2(\eta \hat{\sigma}^2 - c - a) n^{\alpha-1} \log \log n.$$

Then  $n^\alpha \lambda_n = 1 + o(1)$ ,

$$\begin{aligned} n &= \lambda_n^{-1/\alpha} (1 + 2(\eta \hat{\sigma}^2 - c - a) \alpha^{-1} n^{-1} \log \log n (1 + o(1))) \\ &= \lambda_n^{-1/\alpha} (1 + 2(\eta \hat{\sigma}^2 - c - a) \alpha^{-1} \lambda_n^{1/\alpha} \log \log \lambda_n^{-1} (1 + o(1))) \end{aligned}$$

and

$$\begin{aligned} 1 &= P\left[\max_{j \leq n} a_j U_j < \lambda_n^{-1} \text{ i. o.}\right] \\ &= P[N_{\lambda_n} > n \text{ i. o.}] \\ &= P[N_{\lambda_n} > \lambda_n^{-1/\alpha} (1 + 2(\eta \hat{\sigma}^2 - c - a) \alpha^{-1} \lambda_n^{1/\alpha} \log \log \lambda_n^{-1} (1 + o(1))) \text{ i. o.}] \\ &= P[\alpha(2 \log \log \lambda_n^{-1})^{-1} (N_{\lambda_n} - \lambda_n^{-1/\alpha}) \\ &\quad > (\eta \hat{\sigma}^2 - c - a)(1 + o(1)) \text{ i. o.}]. \end{aligned}$$

Hence

$$(3.33) \quad \limsup_{\lambda \rightarrow 0} \alpha(2 \log \log \lambda^{-1})^{-1} (N - \lambda^{-1/\alpha}) + c + a \geq \hat{\sigma}^2 \text{ a.s..}$$

For  $1 < \eta$ ,  $\eta \neq (c + a) \hat{\sigma}^{-2}$ ,  $\eta \neq c \hat{\sigma}^{-2}$ , and small  $\lambda$ , put

$$m_\lambda = [\lambda^{-1/\alpha} + 2(\eta \hat{\sigma}^{-2} - a - c) \alpha^{-1} \log \log \lambda^{-1}] - 1.$$

Then  $\lambda m_\lambda^\alpha = 1 + o(1)$ ,  $\lambda m_\lambda^\alpha = 1 + (\eta \hat{\sigma}^2 - a - c) 2m_\lambda^{-1} \log \log m_\lambda \cdot (1 + o(1))$ , and

$$(\lambda a_{m_\lambda})^{-1} = 1 - (\eta \hat{\sigma}^2 - c) 2m_\lambda^{-1} \log \log m_\lambda (1 + o(1)),$$

$$\begin{aligned} &[\alpha(2 \log \log \lambda^{-1})^{-1} (N - \lambda^{-1/\alpha}) + c + a > \eta \hat{\sigma}^2] \\ &\subset [N > m_\lambda] = \left[ \max_{j \leq m_\lambda} a_j U_j / a_{m_\lambda} < (\lambda a_{m_\lambda})^{-1} \right] \\ &= \left[ \max_{j \leq m_\lambda} a_j U_j / a_{m_\lambda} - 1 \right. \\ &\quad \left. < -(\eta \hat{\sigma}^2 - c) 2m_\lambda^{-1} \log \log m_\lambda (1 + o(1)) \right] \\ &= \left[ 1 - \max_{j \leq m_\lambda} a_j U_j / a_{m_\lambda} \right. \\ &\quad \left. > (\eta \hat{\sigma}^2 - c) 2m_\lambda^{-1} \log \log m_\lambda (1 + o(1)) \right]. \end{aligned}$$

By (2.63) of Theorem 2,

$$\begin{aligned} &P\left[1 - \max_{j \leq n} a_j U_j / a_n \right. \\ &\quad \left. \geq (\eta \hat{\sigma}^2 - c) 2n^{-1} \log \log n (1 + o(1)) \text{ i. o.}\right] = 0. \end{aligned}$$

Hence

$$(3.34) \quad \limsup_{\lambda \rightarrow 0} \alpha(2 \log \log \lambda^{-1})^{-1} (N - \lambda^{-1/\alpha}) + c + a \leq \hat{\sigma}^2 \text{ a.s.}$$

yielding (3.26) by (3.33). The same method and slight modification

will give (3.28) and (3.27) and (3.29). Similarly for  $N'$  and  $T'$ . Since  $N-1 \leq N'' \leq N'$ , (3.26) and (3.27) hold for  $N''$  and similarly (3.28) and (3.29) hold for  $T''$ .

**REMARK 3.** When  $R_n = 0$ ,  $n_\lambda = 1$  and  $a_n = n^\alpha$  with  $0 < \alpha \leq 1$ , the asymptotic normality (3.7) for  $N$ ,  $N'$  and  $N''$  has been established in Siegmund [13] and Chow and Hsiung [5] where the law of the iterated logarithm (3.24) for these random variables has also been proved. Vervaat [17] has obtained the law of the iterated logarithm for  $N''$  for the case  $\alpha = 1$ . In some sequential procedures for special cases of  $U_n$ , Bhattacharya and Mallik [2] has obtained the asymptotic normality (3.8) for the case  $\alpha = 2$  for  $T$ . They have also had some results similar to (3.13) for some special cases of  $T$ . In another context of sequential estimation, Robbins and Siegmund [12] have obtained (3.7) and (3.11) for  $\alpha = 1$  and special forms of  $U_n$  for  $N$ .

**4. The strong law of large numbers of the Marcinkiewicz-Zygmund type.** Let  $X, X_1, X_2, \dots$  be independent and identically distributed random variables with  $EX = \mu > 0$  and  $E|X|^p < \infty$  for some  $1 \leq p < 2$ . Let  $R_1, R_2, \dots$  be random variables with  $R_n = o(1)$  a.s. as  $n \rightarrow \infty$ . Put

$$(4.1) \quad S_n = \sum_1^n X_i, \quad U_n = n^{-1} S_n + R_n.$$

For  $\lambda > 0$ , let  $0 < n_\lambda = o(\lambda^{-1/\alpha})$  and  $a_1, a_2, \dots$  be positive constants, and

$$(4.2) \quad \begin{aligned} N &= N_\lambda = \inf \{n \geq n_\lambda : U_n \geq (\lambda a_n)^{-1}\}, \\ T &= T_\lambda = \inf \{n \geq n_\lambda : 0 < U_n \leq \lambda a_n\}, \end{aligned}$$

$$(4.3) \quad \begin{aligned} N' &= N'_\lambda = \sup \{n \geq n_\lambda : U_n < (\lambda a_n)^{-1}\}, \\ T' &= T'_\lambda = \sup \{n \geq n_\lambda : U_n > \lambda a_n\}, \end{aligned}$$

$$(4.4) \quad \begin{aligned} N'' &= N''_\lambda = \sum_{n_\lambda}^{\infty} I_{[U_n < (\lambda a_n)^{-1}]}, \\ T'' &= T''_\lambda = \sum_{n_\lambda}^{\infty} I_{[U_n > \lambda a_n]}. \end{aligned}$$

**THEOREM 5.** Let (4.1) hold, and  $n_0 = n_0(w)$  be a random variable such that

$$(4.5) \quad U_n > 0 \quad \text{for } n \geq n_0.$$

If  $n^{1-1/p} R_n = c\mu + o(1)$ ,  $0 < a_n = n^\alpha(1 + a n^{1/p-1} + o(n^{1/p-1}))$  for some  $\alpha > 0$ , then as  $n \rightarrow \infty$ , a.s.

$$(4.6) \quad n^{1-1/p} \left(1 - \max_{j \leq n} a_j U_j / \mu a_n\right) + c \rightarrow 0,$$

$$(4.7) \quad n^{1-1/p} \left(1 - \inf_{j \geq n} a_j U_j / \mu a_j\right) + c \rightarrow 0,$$

$$(4.8) \quad n^{1-1/p} \left(1 - \min_{n_0 \leq j \leq n} a_n U_j / \mu a_j\right) + c \rightarrow 0,$$

$$(4.9) \quad n^{1-1/p} \left(1 - \sup_{j \geq n} a_n U_j / \mu a_j\right) + c \rightarrow 0.$$

**Proof.** We can assume that  $\mu = 1$ . Put  $f_n = n^{1-\alpha-1/p}$ . Then  $n^\alpha f_n R_n = c\mu + o(1)$  a.s. and  $(a_n - n^\alpha) f_n = a + o(1)$ . By (2.23) and (2.26) in Theorem 1 (i) and (iii), a.s.

$$(4.10) \quad n^{1-1/p} \max_{j \leq n} (a_j U_j / a_n - U_n) = o(1),$$

$$(4.11) \quad n^{1-1/p} \sup_{j \geq n} (a_n U_j / a_j - U_n) = o(1).$$

By (4.10) and the Marcinkiewicz-Zygmund strong law of large numbers, as  $n \rightarrow \infty$ , a.s.

$$\begin{aligned} & n^{1-1/p} \left(1 - \max_{j \leq n} a_j U_j / a_n\right) \\ &= n^{1-1/p} (1 - U_n) + n^{1-1/p} \left(U_n - \max_{j \leq n} a_j U_j / a_n\right) \\ &= n^{-1/p} (n - S_n) - n^{1-1/p} R_n + o(1) \\ &= o(1) - c + o(1) = -c + o(1), \end{aligned}$$

yielding (4.6). By (4.11) and the same SLLN, as  $n \rightarrow \infty$ , a.s.

$$n^{1-1/p} \left(1 - \sup_{j \geq n} a_n U_j / a_j\right) = -c + o(1),$$

yielding (4.9). The proofs for (4.7) and (4.8) are similar.

**THEOREM 6.** Let (4.1)-(4.5) hold. If  $n^{1-1/p} R_n = c\mu + o(1)$ , and  $a_n = n^\alpha(1 + a n^{1/p-1} + o(n^{1/p-1}))$ , then as  $\lambda \rightarrow 0$ , a.s.

$$(4.12) \quad \alpha(\mu\lambda)^{1/p\alpha} (N - (\lambda\mu)^{-1/\alpha}) + a + c \rightarrow 0,$$

$$(4.13) \quad \alpha \left(\frac{\lambda}{\mu}\right)^{1/p\alpha} \left(T - \left(\frac{\lambda}{\mu}\right)^{-1/\alpha}\right) + a - c \rightarrow 0.$$

If  $N$  and  $T$  are replaced by  $N'$  or  $N''$  and  $T'$  or  $T''$  respectively, (4.12) and (4.13) still hold.

**Proof.** We can assume that  $\mu = 1$ . Since  $\lambda^{1/\alpha} N \rightarrow 1$  a.s., as  $\lambda \rightarrow 0$ ,  $\lambda^{1/\alpha} z = o(1)$  a.s. where  $z = N - \lambda^{-1/\alpha}$ . Therefore a.s.

$$\begin{aligned}\lambda a_N &= (1 + \lambda^{1/\alpha} z)^\alpha (1 + (a + o(1)) N^{1/p-1}) \\ &= (1 + (\alpha + o(1)) \lambda^{1/\alpha} z)(1 + (a + o(1)) N^{1/p-1}) \\ &= 1 + (\alpha + o(1)) \lambda^{1/\alpha} z + (a + o(1)) N^{1/p-1}.\end{aligned}$$

Hence a.s. by (3.6)

$$\begin{aligned}(4.14) \quad N^{1-1/p}(\lambda a_N - 1) &= (\alpha + o(1)) \lambda^{1/\alpha} N^{1-1/p} z + a + o(1) \\ &= (\alpha + o(1)) \lambda^{1/p\alpha} z + a + o(1),\end{aligned}$$

$$(4.15) \quad N^{1-1/p}(\lambda a_{N-1} - 1) = (\alpha + o(1)) \lambda^{1/p\alpha} (z - 1) + a + o(1).$$

By the definition of  $N$  and (3.6), a.s.

$$U_N - 1 \geq (\lambda a_N)^{-1} - 1 = (1 - \lambda a_N)(1 + o(1)),$$

$$U_{N-1} - 1 \leq (\lambda a_{N-1})^{-1} - 1 = (1 - \lambda a_{N-1})(1 + o(1)).$$

By (4.14) and (4.15), a.s.

$$(4.16) \quad N^{1-1/p}(U_N - 1) \geq -(\alpha + o(1)) \lambda^{1/p\alpha} z - a + o(1),$$

$$(4.17) \quad N^{1-1/p}(U_{N-1} - 1) \leq -(\alpha + o(1)) \lambda^{1/p\alpha} (z - 1) - a + o(1).$$

And

$$\begin{aligned}(4.18) \quad N^{1-1/p}(U_N - 1) &= N^{1-1/p} \left( N^{-1} \sum_1^N (X_i - 1) + R_N \right) \\ &= N^{-1/p} (S_N - N) + N^{1-1/p} R_N.\end{aligned}$$

Therefore by the Marcinkiewicz-Zygmund SLLN, and (4.16), (4.17) and (4.18), as  $\lambda \rightarrow 0$  a.s.

$$\alpha \lambda^{1/p\alpha} z + a + c \rightarrow 0,$$

yielding (4.12). The proof for (4.13) is similar, and so are the proofs for (4.12) and (4.13) when  $N$  and  $T$  are replaced by  $N'$  and  $T'$  respectively. Since  $N - 1 \leq N'' \leq N'$ , (4.12) still holds for  $N''$ , and similarly for  $T''$  in (4.13).

**5. Limit theorems for the independent case.** Let  $X_1, X_2, \dots$  be independent random variables with  $EX_n = \mu > 0$  for each  $n \geq 1$ .

Let  $R_1, R_2, \dots$  be random variables such that  $R_n = o(1)$  a.s. as  $n \rightarrow \infty$ . For each  $n \geq 1$ , put

$$(5.1) \quad S_n = \sum_1^n X_i, \quad U_n = n^{-1} S_n + R_n.$$

Assume that for some  $1 < p \leq 2$ ,

$$(5.2) \quad \sup_{n \geq 1} E |X_n|^p < \infty,$$

and that for some distribution function  $F$ , as  $n \rightarrow \infty$

$$(5.3) \quad n^{-1/p}(S_n - n\mu) \xrightarrow{\mathcal{D}} F.$$

Let  $a_1, a_2, \dots$  be increasing positive constants with  $n^{-\alpha} a_n \rightarrow 1$  as  $n \rightarrow \infty$  for some  $\alpha > 0$ . For  $\lambda > 0$ , let  $0 < n_\lambda = o(\lambda^{-1/\alpha})$  and

$$(5.4) \quad N = N_\lambda = \inf \{n \geq n_\lambda : U_n \geq (\lambda a_n)^{-1}\},$$

$$T = T_\lambda = \inf \{n \geq n_\lambda : 0 < U_n \leq \lambda a_n\},$$

$$(5.5) \quad N' = N'_\lambda = \sup \{n \geq n_\lambda : U_n < (\lambda a_n)^{-1}\},$$

$$T' = T'_\lambda = \sup \{n \geq n_\lambda : U_n > \lambda a_n\},$$

$$(5.6) \quad N'' = N''_\lambda = \sum_n^\infty I_{[U_n < (\lambda a_n)^{-1}]},$$

$$T'' = T''_\lambda = \sum_n^\infty I_{[U_n > \lambda a_n]}.$$

**THEOREM 7.** Let (5.1)–(5.3) hold and  $n_0 = n_0(w)$  be a random variable such that

$$(5.7) \quad U_n > 0 \quad \text{for } n \geq n_0.$$

Let  $f_1, f_2, \dots$  be positive constants such that  $a_n f_n = O(n^{1-1/p})$ , and  $n^{-\alpha} a_n \rightarrow 1$  as  $n \rightarrow \infty$ . If for all  $\varepsilon > 0$ ,

$$(5.8) \quad \lim_{\beta \rightarrow 1} \limsup_{n \rightarrow \infty} P \left[ \max_{n\beta \leq j \leq n/\beta} |a_j R_j - a_n R_n| f_n > \varepsilon \right] = 0,$$

$$(5.9) \quad \lim_{\beta \rightarrow 1} \limsup_{n \rightarrow \infty} P \left[ \max_{n\beta \leq j \leq n/\beta} |a_n R_j - a_j R_n| f_n > \varepsilon \right] = 0,$$

then as  $n \rightarrow \infty$ , in probability

$$(5.10) \quad \max_{j \leq n} (a_j U_j - a_n U_n) f_n = o(1),$$

$$(5.11) \quad \sup_{j \geq n} (a_n U_n - a_j U_j) f_n = o(1),$$

$$(5.12) \quad \max_{n_0 \leq j \leq n} (a_j U_n - a_n U_j) f_n = o(1),$$

$$(5.13) \quad \sup_{j \geq n} (a_n U_j - a_j U_n) f_n = o(1),$$

and if  $f_n$  in (5.10)–(5.13) is replaced by  $n^\alpha f_n/a_j$  or  $n^\alpha f_n/a_n$ , (5.10)–(5.13) still hold.

**Proof.** We shall only prove (5.11) and (5.12); the proofs for the others are similar. We can assume that  $\mu=1$ . Since  $R_n=o(1)$ ,  $n^{-\alpha} a_n \rightarrow 1$  and  $\sup_{n \geq 1} E |X_n|^\beta < \infty$ , by a strong law of large numbers, Loéve [10], let  $0 < \beta < \delta^{2/\alpha} < 1$ , there exists  $K = K(w) \geq n_0$  such that if  $j, n \geq K$

$$(5.14) \quad \delta < n^{-\alpha} a_n U_j < \delta^{-1}.$$

Therefore for  $K < j < n\beta$ ,

$$a_j U_n - a_n U_j < \delta^{-1} j^\alpha - \delta n^\alpha \leq \delta n^\alpha (\beta^\alpha \delta^{-2} - 1) < 0,$$

and since  $U_j > 0$  for  $j \geq n_0$ , for all large  $n$ ,

$$\max_{n_0 \leq j \leq K} (a_j U_n - a_n U_j) < 0.$$

Therefore there exists  $m_0 \geq K$  such that for  $n \geq m_0$ ,

$$(5.15) \quad \max_{n_0 \leq j \leq n} (a_j U_n - a_n U_j) = \max_{n\beta \leq j \leq n} (a_j U_n - a_n U_j) \text{ a.s.}$$

And for  $K \leq n < n/\beta \leq j$ , by (5.14)

$$a_n U_n - a_j U_j \leq \delta^{-1} n^\alpha - \delta j^{-\alpha} \leq \delta j^\alpha (\beta^\alpha \delta^{-2} - 1) < 0;$$

hence if  $n \geq K$

$$(5.16) \quad \sup_{j \geq n} (a_n U_n - a_j U_j) = \max_{n \leq j \leq n/\beta} (a_n U_n - a_j U_j) \text{ a.s.}$$

Let  $Y_n = X_n - 1$  and  $W_n = S_n - n$ ; since  $a_n$  is increasing,

$$(5.17) \quad \begin{aligned} & \max_{n\beta \leq j \leq n} (a_j U_n - a_n U_j) \\ & \leq \max_{n\beta \leq j \leq n} (a_j n^{-1} W_n - a_n j^{-1} W_j + a_j R_n - a_n R_j) \\ & \leq \max_{n\beta \leq j \leq n} W_n (a_j n^{-1} - a_n j^{-1}) + \max_{n\beta \leq j \leq n} a_n j^{-1} (W_n - W_j) \\ & \quad + \max_{n\beta \leq j \leq n} (a_j R_n - a_n R_j), \end{aligned}$$

$$\begin{aligned}
 & \max_{n \leq j \leq n/\beta} (a_n U_n - a_j U_j) \\
 (5.18) \quad & \leq \max_{n \leq j \leq n/\beta} (a_n n^{-1} W_n - a_j j^{-1} W_j + a_n R_n - a_j R_j) \\
 & \leq \max_{n \leq j \leq n/\beta} W_n (a_n n^{-1} - a_j j^{-1}) + \max_{n \leq j \leq n/\beta} a_j j^{-1} (W_n - W_j) \\
 & \quad + \max_{n \leq j \leq n/\beta} (a_n R_n - a_j R_j).
 \end{aligned}$$

For  $\epsilon > 0$ , by Doob's inequality and then by the inequality of Marcinkiewicz-Zygmund (Chow and Teicher [7]),

$$\begin{aligned}
 P & \left[ \max_{n\beta \leq j \leq n} a_j j^{-1} (W_n - W_j) f_n > \epsilon \right] \\
 & \leq P \left[ \max_{n\beta \leq j \leq n} |W_n - W_j| > \epsilon (a_n f_n)^{-1} \right] \\
 (5.19) \quad & \leq P \left[ \max_{n\beta \leq j \leq n} |W_n - W_j| > \epsilon \beta n (a_n f_n)^{-1} \right] \\
 & \leq \left( \frac{a_n f_n}{\epsilon \beta n} \right)^p E \left| \sum_{n\beta}^n Y_i \right|^p \leq A n^{-p} (a_n f_n)^p E \left( \sum_{n\beta}^n Y_i^2 \right)^{p/2},
 \end{aligned}$$

for some constant  $A$ . Similarly let  $m = [n/\beta]$

$$\begin{aligned}
 P & \left[ \max_{n \leq j \leq n/\beta} a_j j^{-1} (W_n - W_j) f_n > \epsilon \right] \\
 (5.20) \quad & \leq A n^{-p} (a_m f_n)^p E \left( \sum_n^m Y_i^2 \right)^{p/2}.
 \end{aligned}$$

And for any integers  $n, m$  with  $n < m$ , and since  $p \leq 2$ ,

$$E \left( \sum_n^m Y_i^2 \right)^{p/2} \leq E \left( \sum_n^m |Y_i|^p \right) \leq (m-n) \sup_{n \leq i \leq m} E |Y_i|.$$

Therefore by (5.20),  $a_n f_n = O(n^{1-1/p})$ ,  $a_n \sim n^\alpha$  and for some constant  $B$ ,

$$\begin{aligned}
 P & \left[ \max_{n \leq j \leq n/\beta} a_j j^{-1} (W_n - W_j) f_n > \epsilon \right] \\
 & \leq A \sup_i E |Y_i| n^{-p} \left( \frac{a_m}{a_n} \right)^p (a_n f_n)^p (n/\beta - n); \\
 (5.21) \quad & 0 \leq \lim_{\beta \rightarrow 1} \limsup_{n \rightarrow \infty} P \left[ \max_{n \leq j \leq n/\beta} a_j j^{-1} (W_n - W_j) f_n > \epsilon \right] \\
 & \leq \lim_{\beta \rightarrow 1} B (1/\beta - 1) = 0.
 \end{aligned}$$

Similarly by (5.19)

$$(5.22) \quad \lim_{\beta \rightarrow 1} \limsup_{n \rightarrow \infty} P \left[ \max_{n\beta \leq j \leq n} a_j j^{-1} (W_n - W_j) f_n > \epsilon \right] = 0.$$

And

$$\begin{aligned}
& P \left[ \max_{n \leq j \leq n/\beta} W_n(a_n n^{-1} - a_j j^{-1}) f_n > \varepsilon \right] \\
& \leq P \left[ W_n \geq 0, \max_{n \leq j \leq n/\beta} (n^{-1} - a_j (a_n j)^{-1}) > \varepsilon (a_n f_n)^{-1} \right] \\
& \quad + P \left[ W_n \leq 0, \max_{n \leq j \leq n/\beta} (a_j (a_n j)^{-1} - n^{-1}) > \varepsilon (a_n f_n)^{-1} \right] \\
& \leq P \left[ W_n(n^{-1} - \beta n^{-1}) > \varepsilon (a_n f_n)^{-1} \right] \\
& \quad + P \left[ W_n(\beta n^{-1} - n^{-1}) > \varepsilon (a_n f_n)^{-1} \right] \\
& = P \left[ n^{-1/p} W_n > \varepsilon(1-\beta)^{-1} n^{1-1/p} (a_n f_n)^{-1} \right] \\
& \quad + P \left[ n^{-1/p} W_n < -\varepsilon(1-\beta)^{-1} n^{1-1/p} (a_n f_n)^{-1} \right].
\end{aligned}$$

For large  $n$ , since  $a_n f_n = O(n^{1-1/p})$ , there are positive  $r_1$  and  $r_2$  such that

$$\begin{aligned}
& P \left[ \max_{n \leq j \leq n/\beta} W_n(a_n n^{-1} - a_j j^{-1}) f_n > \varepsilon \right] \\
& \leq P \left[ n^{-1/p} W_n > \varepsilon(1-\beta)^{-1} n^{1-1/p} (a_n f_n)^{-1} \right] \\
& \quad + P \left[ n^{-1/p} W_n < -\varepsilon(1-\beta)^{-1} n^{1-1/p} (a_n f_n)^{-1} \right] \\
& \leq P \left[ n^{-1/p} W_n > \varepsilon r_1 (1-\beta)^{-1} \right] \\
& \quad + P \left[ n^{-1/p} W_n < -\varepsilon r_2 (1-\beta)^{-1} \right].
\end{aligned}$$

Therefore if  $\varepsilon r_1 (1-\beta)^{-1}$  and  $-\varepsilon r_2 (1-\beta)^{-1}$  are continuity points of  $F$ , by (5.3)

$$\begin{aligned}
& \lim_{\beta \rightarrow 1} \limsup_{n \rightarrow \infty} P \left[ \max_{n \leq j \leq n/\beta} W_n(a_n n^{-1} - a_j j^{-1}) f_n > \varepsilon \right] \\
(5.23) \quad & \leq \lim_{\beta \rightarrow 1} (1 - F(\varepsilon r_1 (1-\beta)^{-1})) + \lim_{\beta \rightarrow 1} F(-\varepsilon r_2 (1-\beta)^{-1}) \\
& = 0.
\end{aligned}$$

Similarly

$$(5.24) \quad \lim_{\beta \rightarrow 1} \limsup_{n \rightarrow \infty} P \left[ \max_{n \beta \leq j \leq n} W_n(a_j n^{-1} - a_n j^{-1}) f_n > \varepsilon \right] = 0.$$

Hence, (5.15), (5.17), (5.24), (5.22) and (5.9) give (5.12). (5.16), (5.18), (5.21), (5.23) and (5.8) give (5.11). The proof of (5.16) gives that for  $0 < \beta < 1$ , there exists a random variable  $K$  such that if  $n \geq K$

$$\begin{aligned}
& \sup_{j \geq n} (a_n U_n / a_j - U_j) n^\alpha f_n \\
& = \max_{n \leq j \leq n/\beta} (a_n U_n / a_j - U_j) n^\alpha f_n \\
& = \max_{n \leq j \leq n/\beta} (a_n U_n - a_j U_j) n^\alpha f_n / a_j.
\end{aligned}$$

By (5.11), for  $\varepsilon > 0$

$$\lim_{\beta \rightarrow 1} \limsup_{n \rightarrow \infty} P \left[ \max_{n \leq j \leq n/\beta} (a_n U_n - a_j U_j) \left( \frac{n}{j} \right)^\alpha \frac{j^\alpha}{a_j} f_n > \epsilon \right] = 0$$

yielding (5.11) when  $f_n$  is replaced by  $n^\alpha f_n/a_j$ . The proofs for other cases are similar.

**THEOREM 8.** *Let (5.1)–(5.3) and (5.7)–(5.9) hold. Assume  $f_n = n^{1-1/p-\alpha}$ , and  $n^{-\alpha} a_n \rightarrow 1$ . If  $n^{1-1/p} R_n \rightarrow c\mu$  in probability, then as  $n \rightarrow \infty$ ,*

$$(5.25) \quad n^{1-1/p} \left( \max_{j \leq n} a_j U_j / a_n - \mu \right) - c\mu \xrightarrow{\mathcal{D}} F,$$

$$(5.26) \quad n^{1-1/p} \left( \inf_{j \geq n} a_j U_j / a_n - \mu \right) - c\mu \xrightarrow{\mathcal{D}} F,$$

$$(5.27) \quad n^{1-1/p} \left( \min_{n_0 \leq j \leq n} a_n U_j / a_j - \mu \right) - c\mu \xrightarrow{\mathcal{D}} F,$$

$$(5.28) \quad n^{1-1/p} \left( \sup_{j \geq n} a_n U_j / a_j - \mu \right) - c\mu \xrightarrow{\mathcal{D}} F.$$

### Proof.

$$\begin{aligned} & n^{1-1/p} \left( \max_{j \leq n} a_j U_j / a_n - \mu \right) - c\mu \\ &= n^{1-1/p} \left( \max_{j \leq n} a_j U_j / a_n - U_n \right) + n^{1-1/p} (U_n - \mu) - c\mu \\ &= \max_{j \leq n} (a_j U_j - a_n U_n) n^\alpha f_n / a_n \\ & \quad + n^{-1/p} (S_n - n\mu) + n^{1-1/p} R_n - c\mu. \end{aligned}$$

Therefore by (5.10) in Theorem 7 and (5.3), we have (5.25). The proofs for (5.26)–(5.28) are similar.

**THEOREM 9.** *Let (5.1)–(5.9) hold with  $f_n = n^{1-1/p-\alpha}$ . Assume that  $0 < a_n = n^\alpha (1 + o(n^{1/p-1}))$  and  $n^{1-1/p} R_n \rightarrow c\mu$  in probability, then as  $\lambda \rightarrow 0$*

$$(5.29) \quad \begin{aligned} \infty &> N \rightarrow \infty, \quad \infty > T \rightarrow \infty, \\ \lambda N^\alpha &\rightarrow \mu^{-1}, \quad \lambda T^\alpha \rightarrow \mu \quad \text{a.s.}, \end{aligned}$$

$$(5.30) \quad \begin{aligned} \alpha \mu (\lambda \mu)^{1/p\alpha} (N - (\lambda \mu)^{-1/\alpha}) + a\mu + c\mu &\xrightarrow{\mathcal{D}} G, \\ G(x) &= 1 - F(-x), \end{aligned}$$

$$(5.31) \quad \alpha \mu \left( \frac{\lambda}{\mu} \right)^{1/p\alpha} \left( T - \left( \frac{\lambda}{\mu} \right)^{-1/\alpha} \right) + a\mu - c\mu \xrightarrow{\mathcal{D}} F,$$

and if  $N$  and  $T$  are replaced by  $N'$  or  $N''$  and  $T'$  or  $T''$  respectively, (5.29)–(5.31) still hold.

**Proof.** Since  $\sup_{n \geq 1} E |X_n|^p < \infty$ , by a strong law of large numbers Loéve [10],  $U_n = \mu + o(1)$  a.s. as  $n \rightarrow \infty$ . It is easily seen that the proof for (5.29) will be identical to that for (3.6) in Theorem 3. If  $x$  is a continuity point of  $F$ , for  $\lambda > 0$ , let  $m = m_\lambda = [(\lambda\mu)^{-1/\alpha} + (x - c\mu - a\mu)(\alpha\mu)^{-1}(\lambda\mu)^{-1/p\alpha}]$ , then  $(\lambda\mu)^{-1/\alpha} = m(1 + o(1))$  and

$$(\lambda\mu)^{-1/\alpha} = m - (x - c\mu - a\mu)(\alpha\mu)^{-1}m^{1/p}(1 + o(1)),$$

$$(\lambda\mu)^{-1} = m^\alpha(1 - (x - c\mu - a\mu)(\alpha\mu)^{-1}m^{1/p-1}(1 + o(1)))^\alpha.$$

Therefore

$$\begin{aligned} (\lambda a_m)^{-1} &= \mu m^\alpha(1 - (x - c\mu - a\mu)(\alpha\mu)^{-1}m^{1/p-1}(1 + o(1)))^\alpha a_m^{-1} \\ &= \mu(1 - (x - c\mu - a\mu)\mu^{-1}m^{1/p-1}(1 + o(1))) \\ &\quad \cdot (1 - a m^{1/p-1}(1 + o(1))) \\ &= \mu(1 - (x - c\mu)\mu^{-1}m^{1/p-1}(1 + o(1))). \end{aligned}$$

Hence

$$\begin{aligned} P[\alpha\mu(\lambda\mu)^{1/p\alpha}(N - (\lambda\mu)^{-1/\alpha}) + a\mu + c\mu \leq x] \\ &= P[N \leq m] = P\left[\max_{j \leq n} a_j U_j \geq \lambda^{-1}\right] \\ &= P\left[\max_{j \leq n} a_j U_j/a_m \geq (\lambda a_m)^{-1}\right] \\ &= P\left[\max_{j \leq m} a_j U_j/a_m \geq \mu - (x - c\mu)m^{1/p-1}(1 + o(1))\right] \\ &= P\left[m^{1-1/p}(\max_{j \leq m} a_j U_j/a_m - \mu) - c\mu \geq -x(1 + o(1))\right] \end{aligned}$$

yielding (5.30) by (5.25) in Theorem 8. Similarly if  $x$  is a continuity point of  $x$ , let

$$m = m_\lambda = \left[ \left(\frac{\lambda}{\mu}\right)^{-1/\alpha} + \frac{(x + c\mu - a\mu)}{\alpha\mu} \left(\frac{\lambda}{\mu}\right)^{-1/p\alpha} \right]$$

then  $(\lambda/\mu)^{-1/\alpha} = m(1 + o(1))$  and

$$\left(\frac{\lambda}{\mu}\right)^{-1} = m^\alpha(1 - (x + c\mu - a\mu)(\alpha\mu)^{-1}m^{1/p-1}(1 + o(1)))^\alpha.$$

Therefore

$$\begin{aligned}
 \lambda a_m &= \mu m^{-\alpha} (1 - (x + c\mu - a\mu)(\alpha \mu^{-1}) m^{1/p-1}(1 + o(1)))^{-\alpha} a_m \\
 &= \mu (1 + (x + c\mu - a\mu) \mu^{-1} m^{1/p-1}(1 + o(1))) \\
 &\quad \cdot (1 + a m^{1/p-1}(1 + o(1))) \\
 &= \mu + (x + c\mu) m^{1/p-1}(1 + o(1)).
 \end{aligned}$$

Hence

$$\begin{aligned}
 P\left[\alpha\mu\left(\frac{\lambda}{\mu}\right)^{1/p\alpha} \left(T - \left(\frac{\lambda}{\mu}\right)^{-1/\alpha}\right) + a\mu - c\mu \leq x\right] \\
 &= P[T \leq m] = P\left[\min_{n_0 \leq j \leq m} U_j/a_j \leq \lambda\right] \\
 &= P\left[\min_{n_0 \leq j \leq m} a_m U_j/a_j \leq \lambda a_m\right] \\
 &= P\left[m^{1-1/p} \left(\min_{n_0 \leq j \leq m} a_m U_j/a_j - \mu\right) - c\mu \leq x(1 + o(1))\right]
 \end{aligned}$$

which yields (5.31) by (5.27) in Theorem 8. The proofs for  $N'$  and  $T'$  satisfying (5.29)–(5.31) are similar, and since  $N - 1 \leq N'' \leq N'$ , (5.29) and (5.30) hold for  $N''$  and similarly (5.29) and (5.31) hold for  $T''$ .

**REMARK 4.** For  $R_n = 0$ ,  $0 < \alpha \leq 1$ , (5.25) and (5.26) are given in [Chow and Hsiung, 1976]. If in particular  $F$  is a distribution function for the normal distribution when  $p = 2$ , we certainly have the asymptotic normality for the first passage times and the other related variables.

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