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ON A DEGENERATE PRINCIPAL SERIES OF REPRESENTATIONS OF $U(2, 2)$, II

BY

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Abstract. Let $T(\rho, m; \cdot)$ ($\rho, m \in \mathbf{R} \times \mathbf{Z}$) be the degenerate principal series of representations induced from the parabolic subgroup. This paper is to analyze the intertwining relations $BT(\rho_1, m_1; g) = T(\rho_2, m_2; g)B$ for all $g \in U(2, 2)$. Our result is that B is the trivial operator if $(\rho_1, m_1) \neq (\rho_2, m_2)$. This together with the result of the previous paper yields that the series of unitary representations $T(\rho, m; \cdot)$ ($\rho, m \in \mathbf{R} \times \mathbf{Z}$, $(\rho, m) \neq (0, 2k)$) are irreducible and mutually inequivalent. The main techniques are using both Fourier transforms on the additive group and the multiplicative group. The additive Fourier transform reduces the intertwining relations down to an operator equation involving representation operator of the Weyl element which is an inversion. Then, we resolve the operator equation to an equation of special functions by applying multiplicative Fourier transforms.

In [1], a series of unitary representations $T(\rho, m; \cdot)$, $\rho \in \mathbf{R}$, $m \in \mathbf{Z}$, of $U(2, 2)$ has been carefully studied. The main result of [1] is that $T(\rho, m; \cdot)$ is irreducible if and only if (ρ, m) is not of the form $(0, 2k)$, $k \in \mathbf{Z}$. This result is obtained by deeply studying the commuting relations

$$BT(\rho, m; g) = T(\rho, m; g)B, \quad g \in U(2, 2).$$

In this paper, we analyze more general operator relations, the intertwining relations.

$$\mathcal{B}T(\rho_1, m_1; g) = T(\rho_2, m_2; g)B, \quad g \in U(2, 2).$$

Our result is that \mathcal{B} is the trivial operator if $(\rho_1, m_1) \neq (\rho_2, m_2)$. Thus the unitary representations $T(\rho, m; \cdot)$ ($\rho, m \in \mathbf{R} \times \mathbf{Z} - \{0\} \times 2\mathbf{Z}$) are irreducible and mutually inequivalent. Of the main questions concerning these representations, there remains the work of deter-

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mining the irreducible components of $T(0, 2k; \cdot)$. This problem will be treated in another paper.

The problem of analyzing the intertwining relations

$$BT(\rho_1, m_1; g) = T(\rho_2, m_2; g)B$$

is very similar in nature to the problem solved in [1]. Hence, except in the proof of our main theorem, the route and techniques used in this paper are modeled as those used in proving the irreducibility of $T(\rho, m; \cdot)$, $(\rho, m) \neq (0, 2k)$ in [1]. In proving the main theorem, we have introduced some new methods which greatly simplified the argument.

We begin with recalling some notations and results of [1]. Let \mathbf{Z} , \mathbf{R} , \mathbf{C} be the sets of all integers, real numbers and complex numbers respectively. Let $C^{2 \times 2}$ be the set of all 2×2 complex matrices. $U(2, 2)$ is the group of all 4×4 complex matrices

$$g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad a, b, c, d \in C^{2 \times 2}$$

such that $gp g^* = p$ where

$$p = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix} \text{ and } g^* = {}^t \bar{g}.$$

Let $X = \{x \in C^{2 \times 2} \mid x = x^*\}$, $\mathcal{U} = \{u(x) = \begin{bmatrix} I & x \\ 0 & I \end{bmatrix} \mid x \in X\}$,

$$\mathcal{C} = \left\{ c(a) = \begin{bmatrix} a^{*-1} & 0 \\ 0 & a \end{bmatrix} \mid a \in GL(2, \mathbf{C}) \right\},$$

$$\mathcal{V} = \left\{ v(x) = \begin{bmatrix} I & 0 \\ x & I \end{bmatrix} \mid x \in X \right\}, \quad \mathcal{B} = \mathcal{U}\mathcal{C}, \text{ and } \mathcal{B}' = \mathcal{C}\mathcal{V}.$$

Then \mathcal{U} , \mathcal{C} , \mathcal{V} , \mathcal{B} and \mathcal{B}' are subgroups of $U(2, 2)$. For the sake of convenience, we also write a for $c(a)$. If $x \in X$, we write

$$(1) \quad x = \begin{bmatrix} x_1 + x_4 & -x_2 - ix_3 \\ -x_2 + ix_3 & x_1 - x_4 \end{bmatrix}, \quad x_i \in \mathbf{R}.$$

Formula (1) gives an identification of X with \mathbf{R}^4 . Let $dx = dx_1 dx_2 dx_3 dx_4$ be the measure on X . Let F be the Fourier transform on $L^2(X)$ given by

$$(2) \quad (Ff)(x) = \hat{f}(x) = (2\pi)^{-2} \int_x e^{i(y|x)} f(y) dy$$

for $f \in L'(X) \cap L^2(X)$, where $(y|x) = \frac{1}{2} \operatorname{tr}(xy) = \sum_{i=1}^4 x_i y_i$. For a bounded linear operator B on $L^2(X)$, let $\hat{B} = FB F^{-1}$.

For every $g = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ in $U(2, 2)$ with $\det(d) \neq 0$, there is a unique decomposition

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} I & bd^{-1} \\ 0 & I \end{bmatrix} \begin{bmatrix} d^{*-1} & 0 \\ 0 & d \end{bmatrix} \begin{bmatrix} I & 0 \\ d^{-1}c & I \end{bmatrix}.$$

Therefore, $U(2, 2)$ equals \mathcal{UCO} modulo a measure zero set. In particular, for any $v(x) \in \mathcal{O}$ with $\det(xb + d) \neq 0$, we have

$$\begin{aligned} v(x) \cdot g &= \begin{bmatrix} I & b(xb + d)^{-1} \\ 0 & I \end{bmatrix} \begin{bmatrix} (xb + d)^{* -1} & 0 \\ 0 & xb + d \end{bmatrix} \\ &\quad \cdot \begin{bmatrix} I & 0 \\ (xb + d)^{-1}(xa + c) & I \end{bmatrix}. \end{aligned}$$

Thus each $g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in U(2, 2)$ acts on $\mathcal{O} \cong X$ by

$$x\tilde{g} = (xb + d)^{-1}(xa + c) \quad \text{a.e. on } X.$$

The following proposition can be proved by straight forward computation.

PROPOSITION 1. For $g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in U(2, 2)$ and $f \in L^1(X)$, we have

$$(3) \quad \int_x f((xb + d)^{-1}(xa + c)) |\det(xb + d)|^{-1} dx = \int_x f(x) dx.$$

For every $(\rho, m) \in \mathbb{R} \times \mathbb{Z}$, there corresponds a character $\chi = \chi_{\rho, m}$ of \mathcal{B} defined by $\chi(u(x)c(a)) = |\det a|^{i\rho} [\det a]^m$ where $[\det a] = (\det a)/|\det a|$. Let $\mu(a) = |\det a|^{-4}$, $\forall a = c(a) \in \mathcal{C} \cong GL(2, \mathbb{C})$.

Each $\chi_{\rho, m}$ induces a representation $T(\chi; \cdot) = T(\rho, m; \cdot)$ of $U(2, 2)$ on $L^2(X)$, such that for any $g = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ in $U(2, 2)$ and $f \in L^2(X)$

$$\begin{aligned} (4) \quad &(T(\chi; g)f)(x) \\ &= (T(\rho, m; g)f)(x) \\ &= (\chi \mu^{1/2})(xb + d) f((xb + d)^{-1}(xa + c)) \\ &= |\det(xb + d)|^{i\rho - 2} [\det(xb + d)]^m \\ &\quad \cdot f((xb + d)^{-1}(xa + c)). \end{aligned}$$

That $T(\chi; \cdot)$ is a unitary representation is an immediate consequence of Proposition (1). In this paper, we will show that if $(\rho_1, m_1) \neq (\rho_2, m_2)$, then $T(\rho_1, m_1; \cdot)$ is not equivalent to $T(\rho_2, m_2; \cdot)$. This is done by showing that if B is a bounded linear operator on $L^2(X)$ such that $BT(\rho_1, m_1; g) = T(\rho_2, m_2; g)B$, $\forall g \in U(2, 2)$ then $B = 0$.

Since $T(\rho_1, m_1; e^{i\theta} I)f = e^{2im_1\theta}f$ and $T(\rho_2, m_2; e^{i\theta} I)f = e^{2im_2\theta}f$, $\forall f \in L^2(X)$, we have $B = 0$ if $m_1 \neq m_2$ and $BT(\rho_1, m_1; e^{i\theta} I) = T(\rho_2, m_2; e^{i\theta} I)B$, $\forall e^{i\theta} I$. Thus

PROPOSITION 2. *If $m_1 \neq m_2$, then $T(\rho_1, m_1; \cdot)$ and $T(\rho_2, m_2; \cdot)$ are not equivalent.*

The remaining cases, i.e. the non-equivalence between $T(\rho_1, m_1; \cdot)$ and $T(\rho_2, m_2; \cdot)$ for $\rho_1 \neq \rho_2$ need much more elaborated work. First, we introduce more notations.

Consider the group action $(x, a) \rightarrow a^*xa$ of $GL(2C)$ on X . There are 3 orbits of positive measure. These orbits are denoted by X_1 , X_2 and X_3 which are represented by $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ and $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ respectively. Let δ_1 , δ_2 and δ_3 be their respective characteristic functions.

PROPOSITION 3. *Let B be a bounded linear operator on $L^2(X)$. Then $BT(\rho_1, m; b) = T(\rho_2, m; b)B$, $\forall b \in \mathcal{B}'$ if and only if there are constants d_1 , d_2 and d_3 such that*

$$(5) \quad \begin{aligned} (\hat{B}f)(x) &= (FBF^{-1}f)(x) \\ &= (d_1\delta_1(x) + d_2\delta_2(x) + d_3\delta_3(x)|\det x|^{i(\rho_1-\rho_2)/2}f(x)), \\ &\quad \forall f \in L^2(X). \end{aligned}$$

Proof. It can be shown that

$$(6) \quad \begin{aligned} \hat{T}(\rho, m; c(a)v(y))f(x) \\ &= |\det a|^{i\rho+2}[\det a]^m e^{-i(y|a^*xa)}f(a^*xa). \end{aligned}$$

If (5) is satisfied, then we check easily that

$$\hat{B}\hat{T}(\rho_1, m; b)f = \hat{T}(\rho_2, m; b)\hat{B}f, \quad \forall b \in \mathcal{B}'.$$

Thus

$$BT(\rho_1, m; b) = T(\rho_2, m; b)B, \quad \forall b \in \mathcal{B}'.$$

The reverse is proved in [1]. Q.E.D.

Since $\mathcal{U} = p\mathcal{O}p^{-1}$ and $U(2, 2) = \mathcal{U}\mathcal{C}\mathcal{O}$ modulo a measure zero set, we have

COROLLARY 4. *Let B be a bounded linear operator on $L^2(X)$. Then $BT(\rho_1, m; g) = T(\rho_2, m; g)B$, $\forall g \in U(2, 2)$ if and only if $\hat{B}\hat{T}(\rho_1, m; p) = \hat{T}(\rho_2, m; p)\hat{B}$ and (5) holds.*

From now on, we assume that $\rho_1 \neq \rho_2$ are two fixed real numbers. Let $\chi_1 = \chi_{\rho_1, m}$, $\chi_2 = \chi_{\rho_2, m}$ and $\sigma = (\rho_1 - \rho_2)/2$. Assume that B is a bounded linear operator on $L^2(X)$ such that $BT(\chi_1; g) = T(\chi_2; g)B$, $\forall g \in U(2, 2)$. By Corollary 4, there are constants d_1 , d_2 , and d_3 such that for $f \in L^2(X)$,

$$\hat{B}f(x) = (d_1 \delta_1(x) + d_2 \delta_2(x) + d_3 \delta_3(x)) |\det x|^{i\sigma} f(x).$$

Let $\beta(x) = (d_1 \delta_1(x) + d_2 \delta_2(x) + d_3 \delta_3(x)) |\det x|^{i\sigma}$. We will derive the conclusion $d_1 = d_2 = d_3 = 0$ by probing into the identity

$$(7) \quad \hat{B}\hat{T}(\chi_1; p) = \hat{T}(\chi_2; p)\hat{B}.$$

Since workable expression of $\hat{T}(\chi; p)$ is not available, we shall rewrite (7) as $\hat{B}FT(\chi_1; p)F^{-1} = FT(\chi_2; p)F^{-1}\hat{B}$ and resolve this equation by Mellin transform. Let

$$\mathbf{T} = \left\{ U(\theta) = \begin{bmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{bmatrix} \mid 0 \leq \theta \leq 2\pi \right\},$$

$$U_p = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},$$

$$U(\theta_1, \theta_2) = \begin{bmatrix} \cos \theta_1 & e^{i\theta_2} \sin \theta_1 \\ -e^{-i\theta_2} \sin \theta_1 & \cos \theta_1 \end{bmatrix},$$

$$\mathcal{Q} = \left\{ \omega = (\omega_1, \omega_2) = \begin{bmatrix} \omega_1 & 0 \\ 0 & \omega_2 \end{bmatrix} \mid \omega_1, \omega_2 \in \mathbb{R} \right\},$$

$$\mathcal{Q}_0 = \{\omega = (\omega_1, \omega_2) \in \mathcal{Q} \mid \omega_1 \geq \omega_2\}.$$

Then, modulo a measure zero set, the map $(\omega; \dot{u}) \mapsto u\omega u^{-1}$ is a two-to-one map of $\mathcal{Q} \times SU(2)/T$ onto X . Thus we can identify $\mathcal{Q}_0 \times SU(2)/T$ with X . If $SU(2)/T$ is represented by the cosets

$\dot{u}_p, \quad \dot{u}(\theta_1, \theta_2) \quad 0 \leq \theta_1 < \pi/2, \quad 0 \leq \theta_2 \leq 2\pi.$ Let $d\dot{u} = d\dot{u}(\theta_1, \theta_2)$
 $= \frac{1}{4} \sin 2\theta_1 d\theta_1 d\theta_2$ on $SU(2)/T$ and $d\omega = d\omega_1 d\omega_2$ on Ω_0 or Ω . We
have

$$(9) \quad \int_x f(x) dx = \int_{\Omega_0} \int_{SU(2)/T} f(u\omega u^{-1}) (\omega_1 - \omega_2)^2 du d\omega \quad \forall f \in C_c(X).$$

Let

$$L_0^2(X) = \{f \in L^2(X) \mid f(x) = f(uxu^{-1}) \quad \forall u \in SU(2)\}$$

$$L_1^2(\Omega) = \{\psi \in L^2(\Omega) \mid \psi(\omega') = -\psi(\omega) \quad \forall \omega \in \Omega\}$$

where $\omega' = (\omega_1, \omega_2)' = (\omega_2, \omega_1)$. The space $L_0^2(X)$ is invariant under F , \hat{B} , and $T(x; p)$. Let $\phi : L_0^2(X) \rightarrow L_1^2(\Omega)$ be defined by

$$(10) \quad (\phi f)(\omega) = \frac{\sqrt{\pi}}{2} (\omega_1 - \omega_2) f(\omega).$$

It is easily checked that

$$(11) \quad (\phi^{-1} \psi)(uxu^{-1}) = \frac{2}{\sqrt{\pi}} (\omega_1 - \omega_2)^{-1} \psi(\omega).$$

Using (9), we verify easily that ϕ and ϕ^{-1} are isometries. By straight-forward computation, we have

PROPOSITION 5. For $\psi \in L_1^2(\Omega)$

$$(12) \quad \begin{aligned} (\phi F \phi^{-1} \psi)(\omega) &= -\frac{i}{2\pi} \int_{\Omega} e^{i(\eta|\omega)} \psi(\eta) d\eta, \\ (\phi T(x; p) \phi^{-1} \psi)(\omega) &= \chi(\omega) (\det \omega)^{-1} \psi(-\omega^{-1}), \\ (\phi \hat{B} \phi^{-1} \psi)(\omega) &= \beta(\omega) \psi(\omega). \end{aligned}$$

Let Ω^* denote the collection of $\omega \in \Omega$ for which $\omega_1 \omega_2 \neq 0$. Let $\hat{\Omega}^*$ denote the group of unitary characters of the multiplicative group Ω^* . A typical element λ of $\hat{\Omega}^*$ is given by

$$(13) \quad \lambda(\omega) = |\omega_1|^{i\xi_1} [\omega_1]^{\varepsilon_1} |\omega_2|^{i\xi_2} [\omega_2]^{\varepsilon_2},$$

where $\xi_1, \xi_2 \in \mathbf{R}$, $\varepsilon_1, \varepsilon_2 = 0$ or 1 and $[\omega_i] = \omega_i / |\omega_i|$. We also denote λ by $(\xi_1, \varepsilon_1; \xi_2, \varepsilon_2)$ and identify $\hat{\Omega}^*$ with $(\mathbf{R} \times \mathbf{Z}_2) \times (\mathbf{R} \times \mathbf{Z}_2)$.

The map $f \mapsto |\det \omega|^{1/2} f$ is an isometry from $L^2(\Omega)$ onto $L^2(\Omega^*, d\omega / |\det \omega|)$. Combined with the Fourier transform of

the multiplicative group \mathcal{Q}^* , we have the Mellin transform $M : L^2(\mathcal{Q}) \rightarrow L^2(\hat{\mathcal{Q}}^*)$ such that

$$(14) \quad \psi^2(\lambda) = (M\psi)(\lambda) = \frac{1}{8\pi} \int_{\mathcal{Q}} |\det \omega|^{1/2} \lambda(\omega) \psi(\omega) \frac{d\omega}{|\det \omega|}.$$

$\forall \psi \in L^1(\mathcal{Q}) \cap L^2(\mathcal{Q})$, $\lambda \in \hat{\mathcal{Q}}^*$. Let \mathcal{H} be the image of $L^2_1(\mathcal{Q})$ under M . We shall compute $\tilde{F} = MFM^{-1}$, $\tilde{F}^{-1} = MF^{-1}M^{-1}$, $\tilde{T}(\chi; p) = MT(\chi; p)M^{-1}$ and $\tilde{A} = M\hat{A}M^{-1}$ on \mathcal{H} . First, we note that

1. Let $\zeta_1 = (0, 0; 0, 0)$, $\zeta_2 = (0, 1; 0, 0)$, $\zeta_3 = (0, 0; 0, 1)$ and $\zeta_4 = (0, 1; 0, 1)$. Then $\beta(\omega) = (\sum_{i=1}^4 c_i \zeta_i(\omega)) |\det \omega|^{\epsilon}$, where $c_1 = -(d_1 + 2d_2 + d_3)$, $c_2 = c_3 = \frac{1}{4}(d_1 - d_3)$, $c_4 = \frac{1}{4}(d_1 - 2d_2 + d_3)$.
2. $\chi_{\rho, m}(\omega) = |\omega_1|^{i\rho} [\omega_1]^m |\omega_2|^{i\rho} [\omega_2]^m$.
3. $[\det \omega] = \zeta_4(\omega)$.

We are going to show $c_1 = c_2 = c_3 = c_4 = 0$. For $(\xi, \epsilon) \in \mathbf{R} \times \mathbf{Z}_2$, let

$$k_1(\xi, \epsilon) = \begin{cases} \frac{2}{\sqrt{2\pi}} \Gamma\left(\frac{1}{2} + i\xi\right) \cos\left(\frac{1}{4} + \frac{i\xi}{2}\right) \pi = 2^{i\xi} \frac{\Gamma(\frac{1}{4} + i\xi/2)}{\Gamma(\frac{1}{4} - i\xi/2)}, & \epsilon = 0, \\ \frac{2i}{\sqrt{2\pi}} \Gamma\left(\frac{1}{2} + i\xi\right) \sin\left(\frac{1}{4} + \frac{i\xi}{2}\right) \pi = i 2^{i\xi} \frac{\Gamma(\frac{3}{4} + i\xi/2)}{\Gamma(\frac{3}{4} - i\xi/2)}, & \epsilon = 1. \end{cases}$$

For $\lambda = (\xi_1, \epsilon_1; \xi_2, \epsilon_2) \in \hat{\mathcal{Q}}^*$, define

$$k_2(\lambda) = k_2(\xi_1, \epsilon_1; \xi_2, \epsilon_2) = -k_1(\xi_1, \epsilon_1) k_1(\xi_2, \epsilon_2).$$

PROPOSITION 6. For any $\phi \in L^2(\mathcal{Q})$, let $\tilde{\phi} = M\phi \in \mathcal{H}$, the image of ϕ under the Mellin transform M . Then for any $\lambda \in \hat{\mathcal{Q}}^*$

$$(15) \quad \begin{aligned} \tilde{F}\tilde{\phi}(\lambda) &= k_2(\lambda) \tilde{\phi}(\lambda^{-1}), \\ \tilde{F}^{-1}\tilde{\phi}(\lambda) &= -\lambda(-I) k_2(\lambda) \tilde{\phi}(\lambda^{-1}), \\ \tilde{B}\tilde{\phi}(\lambda) &= \sum_1^4 c_j \tilde{\phi}(\lambda \zeta_j \sigma), \\ \tilde{T}(\chi; p)\tilde{\phi}(\lambda) &= \lambda(-I) \tilde{\phi}(\chi^{-1} \lambda^{-1} \zeta_4). \end{aligned}$$

Proof.

$$\begin{aligned}\widetilde{F\phi}(\lambda) &= MF\phi(\lambda) = (8\pi)^{-1} \int_{\Omega} |\det \omega|^{1/2} \lambda(\omega) F\phi(\omega) |\det \omega|^{-1} d\omega \\ &= (16\pi^2 i)^{-1} \int_{\Omega} |\det \omega|^{1/2} \lambda(\omega) \int_{\Omega} e^{i(\eta|\omega)} \phi(\eta) d\eta |\det \omega|^{-1} d\omega.\end{aligned}$$

Let $\phi \in C_c^\infty(\Omega)$, for any $\varepsilon > 0$, $e^{-\varepsilon(|\omega_1|+|\omega_2|)} |\det \omega|^{-1/2} \lambda(\omega) e^{i(\eta|\omega)} \phi(\eta)$ is integrable on $\Omega \times \Omega$ with respect to $d\omega d\eta$. By Lebesgue dominated convergence theorem and Fubini theorem,

$$\begin{aligned}\widetilde{F\phi}(\lambda) &= \lim_{\varepsilon \rightarrow 0} (16\pi^2 i)^{-1} \int_{\Omega} e^{-\varepsilon(|\omega_1|+|\omega_2|)} |\det \omega|^{-1/2} \lambda(\omega) \\ &\quad \cdot \int_{\Omega} e^{i(\eta|\omega)} \phi(\omega) d\omega d\eta \\ &= \lim_{\varepsilon \rightarrow 0} (16\pi^2 i)^{-1} \int_{\Omega} \left(\phi(\eta) \int_{\Omega} e^{-\varepsilon(|\omega_1|+|\omega_2|)} \right. \\ &\quad \left. \cdot |\det \omega|^{-1/2} \lambda(\omega) e^{i(\eta|\omega)} d\omega \right) d\eta \\ &= (16\pi^2 i)^{-1} \int_{\Omega} \left(\phi(\eta) \lim_{\varepsilon \rightarrow 0} \int_{\Omega} e^{-\varepsilon(|\omega_1|+|\omega_2|)} \right. \\ &\quad \left. \cdot |\det \omega|^{-1/2} \lambda(\omega) e^{i(\eta|\omega)} d\omega \right) d\eta, \\ \lambda(\omega) &= |\omega_1|^{i\xi_1} [\omega_1]^{\varepsilon_1} |\omega_2|^{i\xi_2} [\omega_2],\end{aligned}$$

$$\begin{aligned}\int_{\Omega} e^{-\varepsilon(|\omega_1|+|\omega_2|)} |\det \omega|^{-1/2} \lambda(\omega) e^{i(\eta|\omega)} d\omega \\ &= \int_{\Omega} e^{-\varepsilon(|\omega_1|+|\omega_2|)} e^{i(\eta_1\omega_1 + \eta_2\omega_2)} \\ &\quad \cdot |\omega_1|^{i\xi_1-1/2} [\omega_1]^{\varepsilon_1} |\omega_2|^{i\xi_2-1/2} [\omega_2]^{\varepsilon_2} d\omega_1 d\omega_2 \\ &= \int_{-\infty}^{\infty} e^{-\varepsilon|\omega_1|+i\eta_1\omega_1} |\omega_1|^{i\xi_1-1/2} [\omega_1]^{\varepsilon_1} d\omega_1 \\ &\quad \cdot \int_{-\infty}^{\infty} e^{-\varepsilon|\omega_2|+i\eta_2\omega_2} |\omega_2|^{i\xi_2-1/2} [\omega_2]^{\varepsilon_2} d\omega_2, \\ \int_{-\infty}^{\infty} e^{-\varepsilon|\omega_1|+i\eta_1\omega_1} |\omega_1|^{i\xi_1-1/2} [\omega_1]^{\varepsilon_1} d\omega_1 \\ &= \int_{-\infty}^{\infty} e^{\varepsilon\omega_1+i\eta_1\omega_1} (-\omega_1)^{i\xi_1-1/2} (-1)^{\varepsilon_1} d\omega_1 \\ &\quad + \int_{-\infty}^{\infty} e^{-\varepsilon\omega_1+i\eta_1\omega_1} \omega_1^{i\xi_1-1/2} d\omega_1 \\ &= \int_{-\infty}^{\infty} e^{-\varepsilon\omega_1-i\eta_1\omega_1} \omega_1^{i\xi_1-1/2} (-1)^{\varepsilon_1} d\omega_1 \\ &\quad + \int_{-\infty}^{\infty} e^{-\varepsilon\omega_1+i\eta_1\omega_1} \omega_1^{i\xi_1-1/2} d\omega_1,\end{aligned}$$

$$\begin{aligned} \int_0^\infty e^{-\epsilon\omega_1 - i\eta_1\omega_1} \omega_1^{i\xi_1 - 1/2} d\omega_1 &= \int_0^\infty e^{-(\epsilon + i\eta_1)\omega_1} \omega_1^{(1/2 + i\xi_1) - 1} d\omega_1 \\ &= (\epsilon - i\eta_1)^{-(1/2 + i\xi_1)} \Gamma\left(\frac{1}{2} + i\xi_1\right) \end{aligned}$$

Principal value. Similarly

$$\begin{aligned} \int_0^\infty e^{-(\epsilon + i\eta_1)\omega_1} \omega_1^{(1/2 + i\xi_1) - 1} d\omega_1 &= (\epsilon + i\eta_1)^{-(1/2 + i\xi_1)} \Gamma\left(\frac{1}{2} + i\xi_1\right), \\ (\epsilon + i\eta_1)^{-(1/2 + i\xi_1)} &= e^{-(1/2 + i\xi_1)(\log|\epsilon + i\eta_1| + i\arg(\epsilon + i\eta_1))}, \end{aligned}$$

$$\lim_{\epsilon \rightarrow 0} (\log|\epsilon + i\eta_1| + i\arg(\epsilon + i\eta_1)) = \log|\eta_1| + i\operatorname{sgn}(\eta_1)\frac{\pi}{2},$$

$$\lim_{\epsilon \rightarrow 0} (\epsilon + i\eta_1)^{-(1/2 + i\xi_1)} = |\eta_1|^{-(1/2 + i\xi_1)} e^{-i(1/2 + i\xi_1)\operatorname{sgn}(\eta_1)\pi/2}.$$

Similarly

$$\lim_{\epsilon \rightarrow 0} (\epsilon - i\eta_1)^{-(1/2 + i\xi_1)} = |\eta_1|^{-(1/2 + i\xi_1)} e^{i(1/2 + i\xi_1)\operatorname{sgn}(\eta_1)\pi/2}.$$

$$\begin{aligned} \lim_{\epsilon \rightarrow \infty} \int_{-\infty}^\infty e^{-\epsilon|\omega_1| + i\eta_1\omega_1} |\omega_1|^{i\xi_1 - 1/2} [\omega_1]^{\epsilon_1} d\omega_1 &= |\eta_1|^{-(1/2 + i\xi_1)} \cdot \epsilon_1 = 0, \\ &= |\eta_1|^{-(1/2 + i\xi_1)} (e^{-i(1/2 + i\xi_1)[\eta_1]\pi/2} \\ &\quad + e^{i(1/2 + i\xi_1)[\eta_1]\pi/2}) \Gamma\left(\frac{1}{2} + i\xi_1\right) \\ &= |\eta_1|^{-(1/2 + i\xi_1)} 2 \cos\left(\frac{1}{2} + i\xi_1\right) \frac{\pi}{2} \Gamma\left(\frac{1}{2} + i\xi_1\right) \\ &= |\eta_1|^{-(1/2 + i\xi_1)} [\eta_1]^{\epsilon_1} 2 \cos\left(\frac{1}{4} + \frac{i\xi_1}{2}\right) \pi \cdot \Gamma\left(\frac{1}{2} + i\xi_1\right). \end{aligned}$$

$$\begin{aligned} \lim_{\epsilon \rightarrow 0} \int_{-\infty}^\infty e^{-\epsilon|\omega_1| + i\eta_1\omega_1} |\omega_1|^{i\xi_1 - 1/2} [\omega_1]^{\epsilon_1} d\omega_1 &= |\eta_1|^{-(1/2 + i\xi_1)} \cdot \epsilon_1 = 1, \\ &= |\eta_1|^{-(1/2 + i\xi_1)} (-e^{-i(1/2 + i\xi_1)[\eta_1]\pi/2} \\ &\quad + e^{i(1/2 + i\xi_1)[\eta_1]\pi/2}) \Gamma\left(\frac{1}{2} + i\xi_1\right) \\ &= |\eta_1|^{-(1/2 + i\xi_1)} 2i \sin\left([\eta_1]\left(\frac{1}{4} + i\frac{1}{2}\xi_1\right)\pi\right) \cdot \Gamma\left(\frac{1}{2} + i\xi_1\right) \\ &= |\eta_1|^{-(1/2 + i\xi_1)} [\eta_1]^{\epsilon_1} 2i \sin\left[\frac{1}{4} + i\frac{1}{2}\xi_1\right] \pi \cdot \Gamma\left[\frac{1}{2} + i\xi_1\right], \end{aligned}$$

where Γ is the usual gamma function.

$$\begin{aligned} \therefore \lim_{\epsilon \rightarrow 0} \int_{-\infty}^\infty e^{-\epsilon|\omega_1| + i\eta_1\omega_1} |\omega_1|^{i\xi_1 - 1/2} [\omega_1]^{\epsilon_1} d\omega_1 &= |\eta_1|^{-(1/2 + i\xi_1)} [\eta_1]^{\epsilon_1} \sqrt{2\pi} k_1(\xi_1, \epsilon_1), \end{aligned}$$

$$\lim_{\varepsilon \rightarrow 0} \int_{-\infty}^{\infty} e^{-\varepsilon |\omega_2| + i \eta_2 \omega_2} |\omega_2|^{i \xi_2 - 1/2} [\omega_2]^{\varepsilon_2} d\omega_2 \\ = |\eta_2|^{-(1/2 + i \xi_2)} [\eta_2]^{\varepsilon_2} \sqrt{2\pi} k_1(\xi_2, \varepsilon_2).$$

Notice

$$\lambda^{-1}(\eta) = |\eta_1|^{-i\xi_1} [\eta_1]^{\varepsilon_1} |\eta_2|^{-i\xi_2} [\eta_2]^{\varepsilon_2},$$

$$\therefore \lim_{\varepsilon \rightarrow 0} (16\pi^2 i)^{-1} \int_{\Omega} \phi(\eta) \int_{\Omega} e^{-\varepsilon(|\omega_1| + |\omega_2|)} \\ \cdot |\det \omega|^{-1/2} \lambda(\omega) e^{i(\eta|\omega)} d\omega d\eta \\ = (16\pi^2 i)^{-1} \int_{\Omega} \phi(\eta) |\eta_1|^{-(1/2 + i\xi_1)} \\ \cdot [\eta_1]^{\varepsilon_1} |\eta_2|^{-(1/2 + i\xi_2)} [\eta_2]^{\varepsilon_2} 2\pi k_2(\xi_1, \varepsilon_1; \xi_2, \varepsilon_2) \\ = (8\pi)^{-1} \int_{\Omega} \phi(\eta) \lambda^{-1}(\eta) k_2(\lambda) d\eta = k_2(\lambda) M\phi(\lambda^{-1}).$$

The calculation of \tilde{F}^{-1} is similar to F . We proceed to calculate \tilde{B} .

$$\tilde{B}\tilde{\phi}(\lambda) = (8\pi)^{-1} \int_{\Omega} |\det \omega|^{-1/2} \lambda(\omega) \beta(\omega) \phi(\omega) d\omega \\ = (8\pi)^{-1} \int_{\Omega} |\det \omega|^{-1/2} \lambda(\omega) \left(\sigma(\omega) \sum_{j=1}^4 c_j \zeta_j(\omega) \right) \phi(\omega) d\omega \\ = (8\pi)^{-1} \sum_{j=1}^4 c_j \int_{\Omega} |\det \omega|^{-1/2} (\lambda \sigma \zeta_j)(\omega) \phi(\omega) d\omega \\ = \sum_{j=1}^4 c_j \tilde{\phi}(\lambda \sigma \zeta_j).$$

The calculation of $\tilde{T}(x; p)$ is straightforward.

From (7) $\hat{B}FT(x_1; p)F^{-1} = FT(x_2; p)F^{-1}\hat{B}$ on $L^2(\Omega)$ if and only if $\tilde{B}\tilde{F}\tilde{T}(x_1; p)\tilde{F}^{-1} = \tilde{F}\tilde{T}(x_2; p)\tilde{F}^{-1}\tilde{B}$ on \mathcal{H} .

PROPOSITION 7. $\tilde{B}\tilde{F}\tilde{T}(x_1; p)\tilde{F}^{-1} = \tilde{F}\tilde{T}(x_2; p)\tilde{F}^{-1}\tilde{B}$ on \mathcal{H} if and only if for $j = 1, 2, 3, 4$

$$(16-j) \quad c_j k_2(\lambda) k_2(\chi_2^{-1} \zeta_4 \lambda) = c_j k_2(\zeta_j \lambda \sigma) k_2(\chi_1^{-1} \zeta_4 \zeta_j \lambda \sigma) \\ a.e. \text{ for } \lambda \in \hat{\Omega}^*.$$

Proof. Since $\zeta_j^{-1} = \zeta_j$ and $\zeta_4(-1) = 1$,

$$\begin{aligned} & \tilde{B}\tilde{F}\tilde{T}(x_1; p)\tilde{F}^{-1}\tilde{\phi}(\lambda) \\ &= \sum_{j=1}^4 c_j \tilde{F}\tilde{T}(x_1; p)\tilde{F}^{-1}\tilde{\phi}(\lambda \sigma \zeta_j) \\ &= - \sum_{j=1}^4 c_j k_2(\lambda \sigma \zeta_j) k_2(\lambda \sigma \zeta_j \chi_1^{-1} \zeta_4) \tilde{\phi}(\lambda^{-1} \sigma^{-1} \chi_1 \zeta_j \zeta_4), \end{aligned}$$

$$\begin{aligned} \tilde{F} \tilde{T}(\chi_2; p) \tilde{F}^{-1} \tilde{B} \tilde{\phi}(\lambda) \\ = -k_2(\lambda) k_2(\lambda \chi_2^{-1} \zeta_4) \sum_{j=1}^4 c_j \tilde{\phi}(\lambda^{-1} \sigma \chi_2 \zeta_4 \zeta_j). \end{aligned}$$

Notice that $(\sigma^{-1} \chi_1)(\omega) = (\sigma \chi_2)(\omega) = |\det \omega|^{i(\rho_1 + \rho_2)/2}$. By comparing the coefficients of both sides, the proposition is proved. Consider (16-1), then

$$\begin{aligned} & c_1 k_2(\lambda) k_2(\chi_2^{-1} \zeta_4 \lambda) \\ &= c_1 k_2(\xi_1, \varepsilon_1; \xi_2, \varepsilon_2) \\ & \quad \cdot k_2(-\rho_2 + \xi_1, m_2 + \varepsilon_1 + 1; -\rho_2 + \xi_2, m_2 + \varepsilon_2 + 1) \\ &= -c_1 k_1(\xi_1, \varepsilon_1) k_1(\xi_2, \varepsilon_2) \\ & \quad \cdot k_1(-\rho_2 + \xi_1, m_2 + \varepsilon_1 + 1) k_1(-\rho_2 + \xi_2, m_2 + \varepsilon_2 + 1), \end{aligned}$$

$$\begin{aligned} & c_1 k_2(\zeta_1 \lambda \sigma) k_2(\chi_1^{-1} \zeta_4 \zeta_1 \lambda \sigma) \\ &= c_1 k_2\left(\frac{\rho_1 - \rho_2}{2} + \xi_1, \varepsilon_1; \frac{\rho_1 - \rho_2}{2} + \xi_2, \varepsilon_2\right) \\ & \quad \cdot k_2\left(-\frac{\rho_1 + \rho_2}{2} + \xi_1, m_1 + \varepsilon_1 + 1; \right. \\ & \quad \left. -\frac{\rho_1 + \rho_2}{2} + \xi_2, m_1 + \varepsilon_2 + 1\right) \\ &= -c_1 k_1\left(\frac{\rho_1 - \rho_2}{2} + \xi_1, \varepsilon_1\right) k_1\left(\frac{\rho_1 - \rho_2}{2} + \xi_2, \varepsilon_2\right) \\ & \quad \cdot k_1\left(-\frac{\rho_1 + \rho_2}{2} + \xi_1, m_1 + \varepsilon_1 + 1\right) \\ & \quad \cdot k_1\left(-\frac{\rho_1 + \rho_2}{2} + \xi_2, m_1 + \varepsilon_2 + 1\right). \end{aligned}$$

Consider ξ_1 and ξ_2 to be complex, then k_1 is a meromorphic function with isolated poles determined by its first argument. Its poles are different by an even integral multiple of i . Since $\rho_1 \neq \rho_2$, $(\rho_1 - \rho_2)/2 \neq 0$, $(\rho_1 + \rho_2)/2 \neq \rho_2$ and ρ_1, ρ_2 are real numbers, the poles of $k_2(\lambda) k_2(\chi_2^{-1} \zeta_4 \lambda)$ do not coincide with the poles of $k_2(\zeta_1 \lambda \sigma) k_2(\chi_1^{-1} \zeta_4 \zeta_1 \lambda \sigma)$. Thus the function equation holds only if $c_1 = 0$.

For other function equations we notice that the only difference is the ζ_j in the arguments. Since ζ_j is of the form $(0, \varepsilon_1; 0, \varepsilon_2)$, as meroporphic functions all left sides of (16-2), (16-3) and (16-4) have same poles as the left side of (16-1), and all right sides of

(16-2), (16-3) and (16-4) have same poles as the right side of (16-1). Hence the equations (16-j) hold if and only if $c_j = 0$ $j = 1, 2, 3, 4$.

The above result together with Corollary 4, we have

THEOREM. $(\rho_1, m_1) \neq (\rho_2, m_2)$, then the representations $T(\rho_1, m_1; \cdot)$ and $T(\rho_2, m_2; \cdot)$ are not equivalent.

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