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ASYMPTOTIC PROPERTIES OF PERTURBED RANDOM VOLTERRA INTEGRAL EQUATIONS

BY

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Abstract. Presented herein is the asymptotic behavior of the random solutions $y(t;\ \omega)$ and $x(t;\ \omega)$ of the random Volterra integral equations $y(t;\ \omega)=f(t;\ \omega)+\int_0^t k(t,\ s;\ \omega)\,ds$ and $x(t;\ \omega)=f(t;\ \omega)+\int_0^t k(t,\ s;\ \omega)[x(s;\ \omega)+g(s,\ x(s;\ \omega))]\,y(s;\ \omega)\,ds$, respectively. The existence of the resolvent kernel in the random resolvent equation $r(t,\ s;\ \omega)=k(t,\ s;\ \omega)+\int_s^t k(t,\ u;\ \omega)\,r(u,\ s;\ \omega)\,du$ is studied under suitable restrictions on the random kernel $k(t,\ s;\ \omega)$, and then the limiting behavior of $\|x(t;\ \omega)-y(t;\ \omega)\|_{L_2(\Omega,A,P)}$ as $t\to\infty$ is considered. Some applications to systems theory are indicated.

1. **Introduction.** The mathematical descriptions of phenomena in many scientific research areas frequently result in random Volterra integral equations, for example, see Bharucha-Reid [1], Padgett [4, 5] and Tsokos and Padgett [7]. This is particularly true of systems theory when a system involves a random parameter (Tsokos [6], Wei [8]). In this paper we consider the random linear Volterra integral equation

$$y(t;\ \omega)=f(t;\ \omega)+\int_0^t k(t,\ s;\ \omega)\,y(s;\ \omega)\,ds \tag{1.1}$$

and the perturbed form

$$x(t;\ \omega)=f(t;\ \omega)+\int_0^t k(t,s;\ \omega)[x(s;\ \omega)+g(s,x(s;\ \omega))]\,ds, \tag{1.2}$$

where $t\in R_+=[0,\ \infty)$, $\omega\in\Omega$, the sample space of a probability space $(\Omega,\ A,\ P)$, $y(t;\ \omega)$ and $x(t;\ \omega)$ are unknown stochastic processes, $k(t,\ s;\ \omega)$ is the stochastic kernel defined for $0\le s\le t<\infty$, $f(t;\ \omega)$ is a known stochastic process, $g(t,\ x)$ is the random nonlinear perturbing higher order term defined for $t\in R_+$ and

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$x \in R^n$. Under some restrictions on the random perturbing term $g(t, x(t;\ \omega))$, we wish to compare the random solutions of (1.1) and (1.2).

In a recent paper, Jordan and Wheeler [2] studies equations of the form (1.1) and (1.2) in the deterministic case. However, due to the random nature of certain physical situations in many branches of the applied mathematical sciences, stochastic models are necessarily more descriptive and realistic. We extend the study of Jordan and Wheeler to the stochastic setting and also extend the result to allow a larger class of the unknown stochastic process $x(t;\ \omega)$.

In §2 some function spaces which are useful in considering random equations are defined, a theorem of Padgett [5] on the existence and uniqueness of the random solutions is stated. Also we investigate the almost sure existence of the resolvent kernel associated with $k(t, s;\ \omega)$, which is defined by the random resolvent equation

$$(1.3) \qquad r(t, s;\ \omega) = k(t, s;\ \omega) + \int_s^t k(t, u;\ \omega)\, r(u, s;\ \omega)\, du$$

where $0 \leq s \leq t < \infty$. §3 will contain the results concerning the limiting properties of the random solutions. In §4 applications of the results to a general system with random parameters will be presented.

2. **Random solutions and the resolvent kernel.** A stochastic process $x_j(t;\ \omega)$, $t \in R_+$, $j = 1, 2, \cdots, n$ is said to belong to the space $L_2(\Omega, A, P)$ or to be a second order process if for each $t \in R_+$, we have

$$\|x_j(t;\ \omega)\|_{L_2}^2 = E[\,|x_j(t;\ \omega)|^2] = \int_\Omega |x_j(t;\ \omega)|^2\, dp < \infty .$$

The collection of all equivalent n-component random vectors $x(t;\ \omega) = (x_1(t;\ \omega), \cdots, x_n(t;\ \omega))$ constitutes a separable Banach space with norm

$$\|x(t;\ \omega)\| = \|x(t;\ \omega)\|_{L_2^n} = \max_{1 \leq j \leq n} \|x_j(t;\ \omega)\|_{L_2},$$

the space will be denoted by $L_2^n(\Omega, A, P)$.

The space $L_\infty(\Omega, A, P)$ is defined to be the space of all P-essentially bounded stochastic processes $y(t;\ \omega)$ with norm

$$|||y(t;\ \omega)||| = P\text{-ess}\sup_{\omega\in\Omega}|y(t;\ \omega)| = \inf_{\Omega_0}\{\sup_{\Omega-\Omega_0}|y(t;\ \omega)|\}$$

where Ω_0 is such that $P(\Omega_0)=0$.

Let $C=C(R_+, L_2^n(\Omega, A, P))$ be the space of all continuous functions from R_+ into $L_2^n(\Omega, A, P)$ with the topology of uniform convergence on compact subsets of R_+. The space C is a Fréchet space (Yosida 1965) with metric defined by Fréchet combination of the sequence of seminorms

$$\|x(t;\ \omega)\|_n = \sup_{0\le t\le n}\|x(t;\ \omega)\|.$$

For Banach spaces B, D contained in C, the pair (B, D) is said to be ***admissible*** with respect to a linear operator U if $U(B)\subset D$. The space B is said to be ***stronger than*** C if every convergent sequence in B also converges in C. We define $BC=BC(R_+, L_2^n(\Omega, A, P))$ to be the Banach space of all bounded continuous functions from R_+ into $L_2^n(\Omega, A, P)$ with norm

$$\|x(t;\ \omega)\|_{BC} = \sup_{t\in R_+}\|x(t;\ \omega)\|.$$

We denote by $GC(R_+, L_2^n(\Omega, A, P))$ the Banach space of all continuous functions from R_+ into $L_2^n(\Omega, A, P)$ such that there exist a positive constant Γ and a positive continuous function $G(t)$ on R_+ satisfying $\|x(t;\ \omega)\|\le\Gamma G(t)$ for all $t\in R_+$. The norm in $GC(R_+, L_2^n(\Omega, A, P))$ is given by

$$\|x(t;\ \omega)\|_{GC} = \sup_{t\in R_+}\{\|x(t;\ \omega)\|/G(t)\}.$$

Note that $BC\subset GC\subset C$ and $BC=GC$ *if* $G(t)\equiv 1$. By a ***random solution*** $x(t;\ \omega)$ of the equation (1.1) or (1.2) we will mean an element of C which satisfies the equation P-almost everywhere (or almost surely).

The following result of Padgett [5] is stated for reference.

THEOREM A. *Under the following conditions, there is a unique random solution of* (1.2):

(i) *For each pair* $(t, s)\in\Delta=\{(t, s)\ :\ 0\le s\le t<\infty\}$,

$k(t, s; \omega) \in L_\infty(\Omega, A, P)$ *and the function* $k : \Delta \to L_\infty(\Omega, A, P)$ *is continuous.*

(ii) *B and D are Banach spaces stronger than C and the pair (B, D) and (D, D) are admissible with respect to the integral operator U defined by*

$$(Ux)(t; \omega) = \int_0^t k(t, s; \omega)\, x(s; \omega)\, ds .$$

(iii) *$x(t; \omega) \to g(t, x(t; \omega))$ is an operator from*

$$S(\rho) = \{x(t; \omega) : x(t; \omega) \in D,\ \|x(t; \omega)\|_D \leq \rho\}$$

into B satisfying the Lipschitz condition

$$\|g(t, x(t; \omega)) - g(t, y(t; \omega))\|_B \leq \lambda\, \|x(t; \omega) - y(t; \omega)\|_D ,$$

where ρ and λ are constants.

(iv) *$f(t; \omega) \in D$, $(1 + \lambda)\|U\| < 1$, and*

$$\|f(t; \omega)\|_D + \|U\| \cdot \|g(t, 0)\|_B \leq \rho[1 - \|U\|(1 + \lambda)] ,$$

where $\|U\|$ is the norm of the operator U.

The existence of the resolvent kernel is essential to this study. The following theorem gives conditions:

THEOREM 2.1. *Under the following conditions, there exists a kernel $r(t, s; \omega) \in L_\infty(\Omega, A, P)$ which solves the resolvent equation* (1.3) *almost surely for almost all s, t with $0 \leq s \leq t < \infty$.*

(i) *Same as condition* (i) *in Theorem* A.

(ii) $\int_0^T |||k(t, s; \omega)|||^2\, dt < \infty$ *for almost all $s \in [0, T]$.*

Proof. Let $T > 0$ be fixed. For $0 \leq t \leq s \leq T$, let

$$A(t) = \int_0^T |||k(t, s; \omega)|||^2\, ds ,$$

$$B(t) = \int_t^T |||k(s, t; \omega)|||^2\, ds ,$$

$$C(t, s) = \int_s^t A(u)\, du$$

and define

$$r_1(t, s; \omega) = k(t, s; \omega) ,$$

$$r_{n+1}(t, s; \omega) = \int_s^t k(t, u; \omega)\, r_n(u, s; \omega)\, du .$$

Define $r_n(t, s; \omega) = 0$ if $T \geq s \geq t \geq 0$. First, we show by induction that for $n \geq 0$ and $0 \leq s \leq t \leq T$,

$$r_{n+1}(t, s; \omega) \in L_\infty(\Omega, A, P)$$

and

$$|||r_{n+2}(t, s; \omega)||| \leq [A(t)B(s)C(t, s)^n/n!]^{1/2} \tag{2.1}$$

For $n = 0$, the results are immediated by Hölder inequality. By the Hölder inequality again and induction, we have

$$\begin{aligned} |r_{n+2}(t, s; \omega)| &\leq \int_s^t |||k(t, u; \omega)||| \cdot |||r_{n+1}(u, s; \omega)|||\, du \\ &\leq \left[\int_s^t |||k(t, u; \omega)|||^2\, du\right]^{1/2}\left[\int_s^t |||r_{n+1}(u, s; \omega)|||^2\right]^{1/2} \\ &\leq [A(t)B(s)]^{1/2}\left[\int_0^t A(u)C(u, s)^{n+1}/(n-1)!\, du\right]^{1/2} \\ &= [A(t)B(s)C(t, s)^n/n!]^{1/2}, \end{aligned}$$

where the inequalities are taken almost surely.

Now, define $r(t, s; \omega) = \sum_{n=1}^{\infty} r_n(t, s; \omega)$ if $0 \leq s \leq t \leq T$ and $r(t, s; \omega) = 0$ if $0 \leq t < s \leq T$. By (2.1) it follows

$$\begin{aligned} |||r(t, s; \omega)||| &\leq \sum_{n=1}^{\infty} |||r_n(t, s; \omega)||| \\ &\leq |||k(t, s; \omega)||| + \sum_{n=2}^{\infty} [A(t)B(s)D^n/n!]^{1/2}, \end{aligned}$$

where $D = \int_0^T A(u)\, du$. The series converges by ratio test since $[(D^{n+1}/(n+1)!)/(D^n/n!)]^{1/2} = [D/(n+1)]^{1/2} < 1$ for any $n > D$. Moreover, it is clear that $r(t, s; \omega) \in L_\infty(\Omega, A, P)$. Using Lebesgue Dominate Convergence Theorem, we have for almost all ω,

$$\begin{aligned} \int_s^t k(t, u; \omega)\, r(u, s; \omega)\, du &= \int_s^t k(t, u; \omega)\left[\sum_{n=1}^{\infty} r_n(u, s; \omega)\right] du \\ &= \sum_{n=1}^{\infty} \int_s^t k(t, u; \omega)\, r_n(u, s; \omega)\, du \\ &= \sum_{n=2}^{\infty} r_n(t, s; \omega) \\ &= r(t, s; \omega) - k(t, s; \omega). \end{aligned}$$

Thus $r(t, s; \omega)$ solves the resolvent equation almost surely for $0 \leq s \leq t \leq T$. Since $T \geq 0$ is arbitrary, the proof is completed. ///

We now will rewrite (1.2) to a equivalent form containing the existing random resolvent kernel. Left multiplying both side of (1.1) by $r(t, s; \omega)$ and integrating, we obtain

$$\begin{aligned}\int_0^t r(t, u; \omega)\, y(u; \omega)\, du - \int_0^t r(t, u; \omega)\, f(u; \omega)\, du \\ &= \int_0^t r(t, u; \omega)\Big[\int_0^u k(u, s; \omega)\, y(s; \omega)\, ds\Big] du \\ &= \int_0^t \Big[\int_s^t r(t, u; \omega)\, k(u, s; \omega)\, du\Big] y(s; \omega)\, ds \\ &= \int_0^t [r(t, s; \omega) - k(t, s; \omega)]\, y(s; \omega)\, ds .\end{aligned}$$

Hence

$$\int_0^t k(t, s; \omega)\, y(s; \omega)\, ds = \int_0^t r(t, u; \omega)\, f(u; \omega)\, du ,$$

and (1.1) is equivalent to

$$y(t; \omega) = f(t; \omega) + \int_0^t r(t, u; \omega)\, f(u; \omega) du, \qquad t \in R_+ . \tag{2.2}$$

Now (1.2) can be written in the form

$$x(t; \omega) = F(t; \omega) + \int_0^t k(t, s; \omega)\, x(s; \omega)\, ds ,$$

where

$$F(t; \omega) = f(t; \omega) + \int_0^t k(t, s; \omega)\, g(s, x(s; \omega))\, ds .$$

Applying (2.2) to this equation, we obtain

$$\begin{aligned}x(t; \omega) &= F(t; \omega) + \int_0^t r(t, s; \omega) F(s; \omega)\, ds \\ &= \Big[f(t; \omega) + \int_0^t k(t, s; \omega)\, g(s, x(s; \omega))\, ds\Big] \\ &\quad + \int_0^t r(t, s; \omega)\, f(s; \omega)\, ds \\ &\quad + \int_0^t r(t, s; \omega) \int_0^s k(s, u; \omega)\, g(u, x(u; \omega))\, du\, ds \\ &= \Big[f(t; \omega) - \int_0^t r(t, s; \omega)\, f(s; \omega)\, ds\Big] + \int_0^t k(t, u; \omega) \\ &\quad - \int_u^t r(t, s; \omega)\, k(s, u; \omega)\, ds\Big] g(u, x(u; \omega))\, du \\ &= y(t; \omega) + \int_0^t r(t, s; \omega)\, g(s, x(s; \omega))\, ds ,\end{aligned} \tag{2.3}$$

where $x(t;\ \omega)$ is the random solution of the nonlinear perturbed equation (1.2) and $y(t;\ \omega)$ is the random solution of the corresponding linear equation (1.1).

3. **Asymptotic behavior of the random solutions.** Now the following theorem on the asymptotic properties of the random solutions is readily proved.

THEOREM 3.1. *Let $y(t;\ \omega)$ and $x(t;\ \omega)$ denote the random solutions of* (1.1) *and* (1.2), *respectively. Suppose*

(i) *g satisfies*

$$|g(t,\ x(t;\ \omega))| \leq d(t;\ \omega)\|x(t;\ \omega)\|$$

for all $t \geq 0$ and $\omega \in \Omega$, where $d(t;\ \omega) \in GC$ and satisfying

$$\int_T^{T+1} \|d(s;\ \omega)\|\, ds \longrightarrow 0 \qquad as \quad T \to \infty\,; \tag{3.1}$$

(ii) *$r(t,\ s;\ \omega)$ satisfies*

(a) $\int_0^t |||r(t,\ s;\ \omega)|||\, G(s)^2\, ds \leq B < \infty$ *for all* t;

(b) $\sup_{t\geq T}\int_0^{t-T} |||r(t,\ s;\ \omega)|||\, G(s)^2\, ds \to 0$ *as* $T \to \infty$; *and*

(c) *for all $\varepsilon > 0$ there exists $\delta > 0$ such that if $A \subset R_+$ with $m(A) < \delta$, $\int_A |||r(t,\ s;\ \omega)|||\, G(s)^2\, ds < \varepsilon$, where m denotes Lebesgue measure. Then $\|x(t;\ \omega) - y(t;\ \omega)\| \to 0$ as $t \to \infty$ if $y \in GC$.*

Proof. First, we show that $y(t;\ \omega) \in GC$ imples $x(t;\ \omega) \in GC$. Let $L = \|d(t;\ \omega)\|_{BC}$ and $M = \|y(t;\ \omega)\|_{BC}$. By hypothesis (ii), choose T such that $\sup_{t\geq T}\int_0^{t-T} |||r(t,\ s;\ \omega)|||\, G(s)^2\, ds < \frac{1}{4} L$ and choose $\delta > 0$ such that $\int_A |||r(t,\ s;\ \omega)|||\, G(s)^2\, ds < \frac{1}{4} N$ whenever $A \subset [0,\ t]$ and $m(A) < \delta$. For $t \geq T$, let $A(t) = \{s : t - T \leq s \leq t,\ \|d(s;\ \omega)\| \geq \frac{1}{4} B\}$. Then part (b) of hypothesis (ii) implies that there exists $T_1 > T$ so that $m(A(t)) < \delta$ whenever $t > T_1$. Since $x(t;\ \omega) \in C$, the compactness of $[0,\ T_1]$ implies that there exists $J > 1$ such that $\|x(t;\ \omega)\| \leq J$ on $[0,\ T_1]$. Choosing $P > 4M + 3J$, we have $\|x(t;\ \omega)\| < P$ on $[0,\ \infty)$. For if not, there exists $t > T_1$ such that $\|x(s;\ \omega)\| < P$ for $0 \leq s < t$ and $\|x(t;\ \omega)\| = P$. But

$$\|x(t;\ \omega)\| \leq M + \left\{\int_\Omega \left[\int_0^t |||r(t,\ s;\ \omega)|||\, |g(s,\ x(s;\ \omega))|\, ds\right]^2 dP\right\}^{1/2},$$

and

$$\int_0^t |||r(t,\ s;\ \omega)||| \cdot |g(s,\ x(s;\ \omega)|\ ds$$
$$= \int_{[0,\ t-T]\cup([t-T,\ t]-A(t))\cup A(t)} |||r(t,\ s;\ \omega)||| \cdot |g(s,\ x(s;\ \omega)|\ ds$$
$$\leq 3(1+P)/4.$$

So $\|x(t;\ \omega)\| \leq M + 3(1+P)/4 < P$, a contradiction.

It remains to prove that $\|x(t;\ \omega) - y(t;\ \omega)\| \to 0$ as $t \to \infty$. Equation (2.3) implies $\|x(t;\ \omega) - y(t;\ \omega)\| \leq \left\{\int_\Omega \left[\int_0^t |||r(t,\ s;\ \omega)||| \cdot |g(s,\ x(s;\ \omega))|\ ds\right]^2 dP\right\}^{1/2}$. Let $N = \|x(t;\ \omega)\|_{BC}$. Using hypothesis (ii) again, choose T such that $\sup_{t\geq T} \int_0^{t-T} |||r(t,\ s;\ \omega)|||ds < \varepsilon/3L(1+N)$ and choose $\delta > 0$ such that $\int_A |||r(t,\ s;\ \omega)|||\ ds < \varepsilon/3L(1+N)$ whenever $A \subset [0,\ t]$ and $m(A) < \delta$, where ε is an arbitrary positive number. Define for $t \geq T$,

$$A(t) = \{s : t - T \leq s \leq t,\ \|d(s;\ \omega)\| \geq \varepsilon/3B(1+N)\}.$$

Since $d(t;\ \omega)$ is diminishing, there exists $T_1 > T$ such that $m(A(t)) < \delta$ whenever $t \geq T_1$. Then for $t \geq T_1$,

$$\int_0^t |||r(t,\ s;\ \omega)||| \cdot |g(s;\ \omega)|\ ds$$
$$\leq (1+N)\left\{L\left[\int_0^{t-T} |||r(t,\ s;\ \omega)|||\ ds + \int_A |||r(t,\ s;\ \omega)|||\ ds\right] + \int_{[t-T,\ t]-A} |||r(t,\ s;\ \omega)|||\ d(s;\ \omega)\ ds\right\}$$
$$< (1+N)\{L[2\varepsilon/3L(1+N)] + [\varepsilon/3B(1+N)] \cdot B\} = \varepsilon.$$

This completes the proof. ///

4. **Application to systems theory.** Random integral equations occur frequently and naturally in control systems theory (Tsokos and Padgett [7]). In general, consider H as a black box (for example, a stochastic electrical or mechanical system) with input and output terminals. With the general hypothesis that H is linear, time invariant and nonanticipatory, H can be reprossented by

$$Hf(t;\ \omega) = \int_0^t h(t-s;\ \omega) f(s;\ \omega)\ ds,$$

where f is a random input signal, h is the stochastic impulse function associated with H. If G is a random linear operator defined by $Gf(t;\ \omega) = Af(t;\ \omega)$, where A is a constant, then we form a feedback stochastic control system by connecting H with the box G, as illustrated in Figure 1.

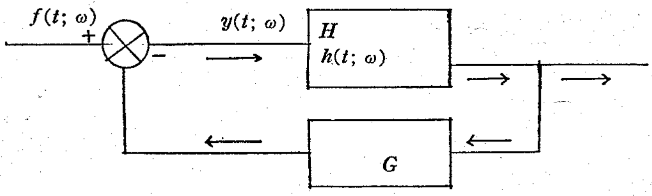

FIGURE 1.

The system thus can be expressed by the following random linear Volterra integral equation:

$$y(t;\ \omega) = f(t;\ \omega) - \int_0^t k(t-s;\ \omega)\, y(s;\ \omega)\, ds, \tag{4.1}$$

where $k(t;\ \omega) = A \cdot h(t;\ \omega)$.

On the other hand, a class of nonlinear stochastic servomechanisms can be described by the differential equation

$$\dot{x}(t;\ \omega) = [A(\omega) + b(\omega)]\, x(t;\ \omega) + b(\omega)\, g(t,\ x(t;\ \omega)), \qquad t \geq 0,$$

where x is an unknown random state variables, $A(\omega)$ is a random variable subject to error (which is frequently the case in applied problems, where $A(\omega)$ can often be written in the form $A(\omega) = \alpha + \varepsilon(\omega)$, with α assumed to be known and the perturbing element $\varepsilon(\omega)$ a random variable), $b(\omega)$ is a random variable and g is the characteristic function of the servomotor. Integrating this equation, we have

$$x(t;\omega) = f(t;\omega) + \int_0^t k(t-s;\omega)[x(s;\omega) + g(s;x(s;\omega))]\, ds, \tag{4.2}$$

where $f(t;\ \omega) = e^{A(\omega)(t)}\, x(0;\ \omega)$, $k(t;\ \omega) = b(\omega)\, e^{A(\omega)(t)}$ if $t \geq 0$ and $k(t;\ \omega) = 0$ almost surely if $t < 0$.

To compare the systems (4.1) and (4.2), we cite Theorem 3.1. Under the appropriate conditions, the two systems are consistent in the long run (as $t \to \infty$). The rate of convergence of $\|x(t;\ \omega) - y(t;\ \omega)\|$ is closed related to the hypothesis of Theorem 3.1 and can be determined up to a given tolerance limit. Moreover, the hypothesis of Theorem 3.1 would have to be interpreted in the framework of the particular system in question.

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