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A SIMPLE PROOF OF CHEVET'S THEOREM

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Abstract. A sufficient condition for the continuity of sample paths of a random field was proposed by S. Chevet. Chevet's proof is based on the notion of ε-entropy and involved with heavy computations. In this paper we shall present a simpler proof of his result using the method of interpolation. The idea is to interpolate the original paths so that the interpolated paths will be piecewise linear on any direction parallel to an axis.

1. **Introduction.** By a random field, we mean a collection of random variables, denoted by $\{X(t), t \in T\}$, defined on a probability space $(\Omega, \mathcal{F}, P)$, where T is a subset of R^k. Two fields $\{X(t), t \in T\}$ and $\{Y(t), t \in T\}$ are called *versions* or *modifications* of each other if $P\{X(t) = Y(t)\} = 1$ for every $t \in T$. For each $\omega \in \Omega$, $X(\cdot, \omega)$, considered as a function defined on T, is called a *sample path* or simply *path*. If T is a topological space, a process over T is said to be *sample-continuous* if there exists a version of it with all paths continuous. Sample-continuity has been an important question to the researchers. For $k = 1$, it is well-known that a sufficient condition was given by Kolmogorov. For $k \geq 2$, a similar condition was given by S. Chevet; see Chevet [3]. Chevet uses the notion of ε-entropy and his proof are very complicated. We shall present here a simpler proof. Our proof is based on the method of interpolation which is generally used to prove the case $k = 1$. In other words, we point out here that the method of interpolation works for general k provided that we interpolate the paths in the right way.

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2. **Chevet's Theorem.** In this section, we shall state and prove Chevet's theorem. Since continuity is a local property, we may assume, for the purpose of convenience, the indexed set T is the k-dimensional unit cube $I^k=[0,1]^k$. Before we present the main theorem, observe the following simple but useful fact.

LEMMA 2.1. *Let $e^{(i)}=(e_1^{(i)}, e_2^{(i)},\cdots, e_k^{(i)})$ be the extremal points of I^k; therefore, each $e_j^{(i)}$ is 0 or 1. Then any $t=(t_1,\cdots,t_k)\in I^k$ can be expressed as*

$$t=\sum_{i=1}^{2^k}\alpha_i(t)\,e^{(i)}\,, \tag{2.1}$$

where

$$\alpha_i(t)=\prod_{j=1}^{k}t_j^{e_j^{(i)}}(1-t_j)^{1-e_j^{(i)}}\,. \tag{2.2}$$

In particular

$$\alpha_i(t)\ge 0\,,\qquad \sum_{i=1}^{2^k}\alpha_i(t)=1\,, \tag{2.3}$$

$\alpha_i(t)$ is continuous in t and is linear when t varies on any line paralled to an axis.

Proof. We note that by (2.2) $\alpha_i(t)$ is the product probability at $e^{(i)}$ if we assign probabilities t_j at 1 and $1-t_j$ at 0 on the jth marginal. Easy to check (2.1) and (2.3) hold. |||

REMARK 2.2. If I^k is replaced by any other cube (or rectangle), Lemma 2.1 still holds if we define $\alpha_i(t)$ in a similar fashion.

To state the following theorem, let $\|\ \|$ denote the maximum norm over R^k; i.e. $\|t\|=\max(|t_1|,|t_2|,\cdots,|t_k|)$, $t=(t_1,\cdots,t_k)\in R^k$.

THEOREM 2.3. (Chevet). *Let $\{X(t),\, t\in I^k\}$ be a random field. Suppose for each $t, s\in I^k$,*

$$P(|X(t)-X(s)|\ge g(\|t-s\|))\le h(\|t-s\|)\,, \tag{2.4}$$

where g, h are nonnegative increasing functions such that for some $\varepsilon>0$,

$$\int_0^{\varepsilon}\frac{g(x)}{x}\,dx<\infty\,,\qquad \int_0^{\varepsilon}\frac{h(x)}{x^{1+k}}\,dx<\infty\,. \tag{2.5}$$

Then $\{X(t)\}$ *is sample-continuous. In fact, there is a version* $\{X(t), t \in I^k\}$ *whose paths enjoy the following Hölder condition: for almost all* $\omega \in \Omega$, *there exists a positive number* K_ω *depending on* ω *such that*

$$(2.6) \qquad |Y(t, \omega) - Y(s, \omega)| \leq K_\omega \int_0^{4\|t-s\|} \frac{g(x)}{x} dx,$$

for $t, s \in I^k$.

Proof. By (2.5), $g(t) \downarrow 0$ and $h(t) \downarrow 0$ as $t \downarrow 0$. Therefore, by (2.4), $X(t)$ is stochastically continuous; i. e.

$$X(t) \xrightarrow{P} X(s) \quad \text{as } t \to s.$$

For each $n \geq 1$, let

$$T_n = \{t = (t_1, \cdots, t_k) : \forall i,\ t_i = j2^{-n}, \text{ some integer } j,\ 1 \leq j \leq 2^n\}.$$

That is, T_n forms a 2^{-n}-grid that partitions I^k into 2^{nk} small cubes and we call any one of them a 2^{-n}-cube (see figure 1 for $k = 2$).

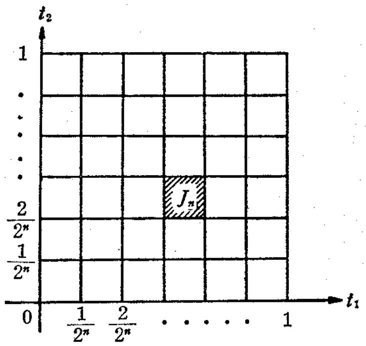

FIGURE 1.

Note that each t in I^k must belong to a 2^{-n}-cube J_n. By Lemma 2.1 and Remark 2.2, $t = \sum \alpha_i(t) e^{(i)}$, where $e^{(i)}$, $i = 1, 2, \cdots, 2^k$, are extremal points of J_n and each $\alpha_i(t)$ is defined properly as in (2.2). Now define a random field X_n by letting

$$X_n(t) = \sum_{i=1}^{2^k} \alpha_i(t) X(e^{(i)})$$

whenever $t = \sum_{i=1}^{2^k} \alpha_i(t) e^{(i)}$. This definition is free of ambiguity and the paths of $X_n(t)$ are continuous. Since by Lemma 2.1 $X_n(t)$ is linear in t on any line paralled to an axis, we have

$$\operatorname*{Max}_{t\in J_n}|X_{n+1}(t)-X_n(t)| = \operatorname*{Max}_{t\in J_n\cap T_{n+1}}|X_{n+1}(t)-X_n(t)|\,.$$

Therefore

$$\begin{aligned} &P\Big(\operatorname*{Max}_{t\in J_n}|X_{n+1}(t)-X_n(t)| \geq g(2^{-n-1})\Big)\\ &\quad = P\Big(\operatorname*{Max}_{t\in J_n\cap T_{n+1}}|X_{n+1}(t)-X_n(t)| \geq g(2^{-n-1})\Big)\\ &\quad \leq \sum_{t,s\in J_n\cap T_{n+1}} P(|X(t)-X(s)| \geq g(2^{-n-1}))\\ &\quad \leq 3^{2k}\, h(2^{-n-1})\,. \end{aligned}$$

There are 2^{nk} such J_n's. Hence,

$$\begin{aligned} &P\Big(\operatorname*{Max}_{t\in I^k}|X_{n+1}(t)-X_n(t)| \geq g(2^{-n-1})\Big)\\ &\quad \leq 3^{2k}\, 2^{nk}\, h(2^{-n-1})\,. \end{aligned}$$

But (2.5) is equivalent to the fact that $\sum_n g(2^{-n})<\infty$ and $\sum_n 2^{nk} h(2^{-n})<\infty$. By the Borel-Cantelli lemma, one concludes that $X_n(t)$ converges uniformly on I^k with probability one. Now, let

$$Y(t,\omega)=\begin{cases}\lim X_n(t,\omega) & \text{if } X_n(t,\omega) \text{ converges uniformly in } t\,,\\ 0 & \text{otherwice}\,.\end{cases}$$

Then the paths of $\{Y(t)\}$ are continuous and

$$Y(t)=X(t)\quad \text{a.s.}\qquad \text{for}\quad t\in\bigcup_{n=1}^{\infty} T_n\,.$$

For $t\notin \bigcup_{n=1}^{\infty} T_n$, since $\{X(t)\}$ is stochastically continuous, there exists a sequence $\{t_n\}$ in $\bigcup_{n=1}^{\infty} T_n$ such that

$$X(t_n)\longrightarrow X(t)\quad \text{a.s.}$$

Obviously

$$Y(t_n)\longrightarrow Y(t)\,.$$

Hence

$$Y(t)\;=\;X(t)\quad \text{a.s.}$$

Thus $\{X(t)\}$ is sample-continuous. Next, we show the field $\{Y(t)\}$ satisfies (2.6). For

$$P\Big(\operatorname*{Max}_{\|t-s\|=2^{-n}}|Y(t)-Y(s)| \geq g(\|t-s\|)\Big) \leq 3^k\, 2^{nk}\, h(2^{-n})\,,$$

by Borel-Cantelli again, one concludes that for almost all ω, there exists a positive integer $n_0(\omega)$ such that if $n \geqq n_0(\omega)$,

$$|Y(t, \omega) - Y(s, \omega)| \leq g(2^{-n}), \quad t, s \in T_n, \quad \|t - s\| = 2^{-n}.$$

For general s and t satisfying $2^{-n-1} \leq \|t - s\| < 2^{-n}$ for some $n \geq n_0(\omega)$, we choose sequences $\{t_j, j \geq n\}$ and $\{s_j, j \geq n\}$ such that t_j, $s_j \in T_j$, $\|s_{j+1} - s_j\| = 2^{-j-1}$, $\|t_{j+1} - t_j\| = 2^{-j-1}$, and $s_j \to t$, $t_j \to t$. By the path continuity of $Y(t)$,

$$Y(t, \omega) = Y(t_n, \omega) + \sum_{j=n+1}^{\infty} (Y(t_j, \omega) - Y(t_{j-1}, \omega)),$$

$$Y(s, \omega) = Y(s_n, \omega) + \sum_{j=n+1}^{\infty} (Y(s_j, \omega) - Y(s_{j-1}, \omega)).$$

Hence

$$\begin{aligned} |Y(t, \omega) &- Y(s, \omega)| \\ &\leq |Y(t_n, \omega) - Y(s_n, \omega)| \\ &\qquad + \sum_{j=n+1}^{\infty} |Y(t_j, \omega) - Y(t_{j-1} - \omega)| \\ &\qquad + \sum_{j=n+1}^{\infty} |Y(s_j, \omega) - Y(s_{j-1}, \omega)| \\ &\leq g(2^{-n}) + 2 \sum_{j=n+1}^{\infty} g(2^{-j}) \\ &\leq 2 \int_0^{4\|t-s\|} \frac{g(x)}{x} dx \end{aligned}$$

for all t, s such that $\|t - s\| < 2^{-n_0(\omega)}$. For general t, $s \in I^k$, choose $K_\omega > 0$ so that

$$|Y(t, \omega) - Y(s, \omega)| \leq K_\omega \int_0^{4\|t-s\|} \frac{g(x)}{x} dx.$$

REMARK 2.4. Since all norms over R^k are equivalent, $\| \ \|$ in Theorem 2.3 can be any other norm.

3. **Corollaries.** As consequences of Theorem 2.3, we shall show

COROLLARY 3.1 (Bernard [1]). *If a random field* $\{X(t), t \in I^k\}$ *satisfies*

$$E |X(t) - X(s)|^\alpha \leq \frac{K \|t - s\|^k}{\log \|t - s\|^{1+\beta}} \tag{3.1}$$

for some $\beta > \alpha > 0$, $K > 0$, *then it is sample-continuous.*

Proof. Let

$$g(x) = |\log x|^{-\gamma}, \qquad 1 < \gamma < \beta\alpha^{-1}$$

and

$$h(x) = Kx^k |\log x|^{-(1+\beta-\alpha\gamma)}, \qquad x > 0.$$

It is easy to see that g and h satisfy (2.5). That (2.4) is satisfied is a consequence of Chebyshev's inequality. Hence, the result follows from Theorem 2.3. |||

REMARK 3.2. Theorem 2.3 also implies that $\{X(t)\}$ in Corollary 3.1 can be modified so that the paths satisfy the following Hölder condition:

$$|X(t, \omega) - X(s, \omega)| \leq K\, |\log \|t - s\||^{-\gamma+1}$$

for any fixed γ, $1 < \gamma < \beta\alpha^{-1}$.

COROLLARY 3.3 (Totoki [5]). *If a random field* $\{X(t), t \in I^k\}$ *satisfies*

$$E\,|X(t) - X(s)|^\alpha \leq K\, \|t - s\|^{k+\beta}, \tag{3.2}$$

for some $\alpha, \beta, K > 0$, *then it is sample-continuous.*

Proof. (3.2) implies (3.1). |||

REMARK 3.4. $\{X(t)\}$ in Corollary 3.3 can be modified so that the paths enjoy the following Hölder condition:

$$|X(t, \omega) - X(s, \omega)| \leq K_\omega\, \|t - s\|^\gamma$$

for any fixed γ, $\gamma < \beta\,\alpha^{-1}$.

A random field is called *Gaussian* if its finite-dimensional distributions are multivariate normal. The path continuity of Gaussian fields or Gaussian processes in general is discussed extensively in Dudley [4]. We shall prove here

COROLLARY 3.5. *Suppose* $\{X(t)\}$ *is a Gaussian field such that* $E(X(t)) = 0$, $\forall t$ *and*

$$E\,|X(t) - X(s)|^2 \leq K\, |\log \|t - s\||^{-(3+\beta)}, \tag{3.3}$$

some K, $\beta > 0$. *Then* $\{X(t)\}$ *is sample-continuous.*

Proof. First, by direct computations we note that

$$\int_x^\infty \frac{1}{\sqrt{2\pi}} e^{-y^2/2}\,dy \le e^{-x^2/2}, \qquad x>0,$$

let

$$g(x) = |\log x|^{-(1+\beta/4)}, \qquad x>0.$$

Then

$$\begin{aligned} P(|X(t)-X(s)| &\ge g(\|t-s\|) \\ &\le \exp\{-g^2(\|t-s\|)/2E|X(t)-X(s)|^2\} \\ &\le \exp\left\{-\frac{1}{2K}|\log\|t-s\||^{1+\beta/2}\right\}, \end{aligned}$$

let $h(x)=\exp\{-1/2K\,|\log x|^{1+\beta/2}\}$, $x>0$. Apparently, g and h saisfy (2.5) and the result follows. |||

REMARK 3.6. According to Dudley [4], Corollary 3.5 is still valid if $3+\beta$ in (3.3) is replaced by $1+\beta$.

A Gaussian field $\{X(t), t\in I^k\}$ is called a *Wiener field* if $E(X(t))=0$, $\forall t$ and $E(X(t)X(s))=\Pi_{i=1}^k(t_i\wedge s_i)$, where $t=(t_1,\cdots,t_k)$, $s=(s_1,\cdots,s_k)$, $t_i\wedge s_i=\mathrm{Min}\{t_i,s_i\}$. Wiener field is sample-continuous. To show this, we need

LEMMA 3.7. *If $\{X(t), t\in I^k\}$ is a Wiener field, then*

$$E|X(t)-X(s)|^2 \le \sum_{i=1}^k |t_i-s_i|,$$

where $t=(t_1,t_2,\cdots,t_k)$, $s=(s_1,s_2,\cdots,s_k)\in I^k$.

Proof.

$$\begin{aligned} &E|X(t)-X(s)|^2 \\ &\quad= \prod_{i=1}^k t_i + \prod_{i=1}^k s_i - 2\prod_{i=1}^k (t_i\wedge s_i) \\ &\quad= \prod_{i=1}^k t_i - \prod_{i=1}^k (t_i\wedge s_i) + \left[\prod_{i=1}^k s_i - \prod_{i=1}^k (t_i\wedge s_i)\right] \\ &\quad\le \sum_{i=1}^k [|t_i - t_i\wedge s_i| + |s_i - t_i\wedge s_i|] \\ &\quad= \sum_{i=1}^k |t_i-s_i|, \end{aligned}$$

since

$$\begin{aligned}
\prod_{i=1}^{k} t_i - \prod_{i=1}^{k} (t_i \wedge s_i) \\
&= t_k\left(\prod_{i=1}^{k-1} t_i - \prod_{i=1}^{k-1} t_i \wedge s_i\right) + \prod_{i=1}^{k-1} (t_i \wedge s_i)(t_k - t_k \wedge s_k) \\
&\leq \left(\prod_{i=1}^{k-1} t_i - \prod_{i=1}^{k-1} t_i \wedge s_i\right) + (t_k - t_k \wedge s_k) \\
&\leq \sum_{i=1}^{k} (t_i - t_i \wedge s_i)
\end{aligned}$$

by induction. | | |

COROLLARY 3.8 (Chentsov [2]). ***The Wiener field is sample-continuous.***

Proof. This follows from Corollary 3.5 and Lemma 3.7 if we let $\| \ \|$ be the l_1-norm in (3.3). | | |

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