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FUNCTIONAL LIMIT THEOREMS FOR PROCESSES WITH POSITIVE DRIFT AND FIRST PASSAGE TIME PROCESSES

BY

I. CHANG AND C. HSIUNG

Abstract. For $c>0$, $\alpha<1$, we define $Tc=\inf\{n\geq 1: S_n>cn^\alpha\}$, where S_n is the sum of i.i.d. random variables with positive mean. In this paper we give functional forms of the central limit theorem and the law of iterated logarithm for $\max_{j\leq n} S_j j^{-\alpha}$ and Tc.

0. **Introduction.** Let $\xi_1, \xi_2, \cdots$ be a sequence of independent, identically distributed ramdom variables with $E\xi_1=\mu>0$, $\mathrm{Var}\,\xi_1=\sigma^2<\infty$. Write $S_n=\xi_1+\cdots+\xi_n$. For $c>0$, $0\leq\alpha<1$, we define the first passage time $T_c=\inf\{n\geq 1, S_n>cn^\alpha\}$. A central limit theorem for T_c as $c\to\infty$ was given by Siegmund [5]. A closely related central limit theorem for $\max_{j\leq n} S_j/j^\alpha$ was obtained later by Teicher [9], which is in fact an easy consequence of [5]. The laws of iterated logarithm for both $\max_{j\leq n} S_j/j^\alpha$ and T_c have been established by Chow & Hsiung [2]. In this paper, we will give functional forms of these theorems. Our treatment generalizes that of Vervaat [10], which discussed a special case of our results. Of course, the invariance principles of both Donsker and Strassen play an important role in this approach. It is apparent that our approach can be applied to other situations, for example, $\inf_{j\leq n} S_j/j^\beta$ for $\beta>1$ and $N_c=\inf\{n\geq 1: S_n\leq c^{-1}n^\beta\}$ for $c>0$, [3]. The above assumptions and notations will be valid throughout this paper unless mentioned otherwise.

1. **Functional limit theorems for processes with positive drift.** Let $D=D[0,\infty)$ be the set of all real valued right-continuous functions on $[0,\infty)$ which have a finite left limit at each point.

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Let C be the subset of continuous functions. We will regard D as a metric space with the metric introduced by Skorohod and Stone (c.f. [1], [7]). The concept of convergence in D can be characterized as follows: Let X_n, $X \in D$. $X_n \to X$ in D if and only if there exists a sequence of homeomorphisms λ_n of $[0, \infty)$ such that $\lambda_n(t) \to t$ and $X_n(\lambda_n(t)) \to X(t)$ uniformly on compact sets in $[0, \infty)$. Clearly, for $X \in C$, $X_n \to X$ iff $X_n(t)$ converges to $X(t)$ uniformly on compact sets in $[0, \infty)$. For $X \in D$, we define $X^\dagger(t) = \sup_{0 \le u \le t} X(u)$, $X^\dagger(t) \in D$.

Lemma 1.1. *Let $X_n \in D$, $Y \in C$, $\eta_n > 0$, $\eta_n \to 0$. Let ψ be a continuous strictly increasing function on $[0, \infty)$. Then $(X_n(t) - \psi(t))/\eta_n \to Y(t)$ uniformly on compact sets of $[0, \infty)$ implies $(X_n^\dagger(t) - \psi(t))/\eta_n \to Y(t)$ uniformly on compact sets of $[0, \infty)$.*

Proof. Let ε, t_0 be any positive numbers. Since Y is uniformly continuous, bounded on $[0, t_0]$ and ψ is continuous stictly increasing, there exists n_0 such that $Y(s) - Y(t) \le \varepsilon + (\psi(t) - \psi(s))/\eta_n$ whenever $n \ge n_0$ and $0 \le s \le t \le t_0$. By assumption there exists $n_1 \ge n_0$ such that $(X_n(s) - \psi(s))/\eta_n - Y(s) \le \varepsilon$ for all $n \ge n_1$ and all $s \in [0, t_0]$. Therefore for $n \ge n_1$, $(X_n(s) - \psi(t))/\eta_n - Y(t) \le 2\varepsilon$ for all s and t with $0 \le s \le t \le t_0$, which implies the desired result.

Theorem 1.1. *Let $X_1, X_2, \cdots$ be random elements in D, Y a random element in C and $\eta_1, \eta_2, \cdots$ positive random variables such that $\eta_n \xrightarrow{d} 0$. Let ψ be a continuous strictly increasing function on $[0, \infty)$. Then*

$$\frac{X_n - \psi}{\eta_n} \xrightarrow{d} Y \quad \textit{implies} \quad \frac{X_n^\dagger - \psi}{\eta_n} \xrightarrow{d} Y.$$

Proof. It is a consequence of Lemma 1.1 and the following theorem of Skorohod [6] and Dudley [4]:

Theorem 1.2. *Let $Y, Y_1, Y_2, \cdots$ be random elements in a separable metric space S such that Y_n converges to Y in distribution. Then there exists random element $Y', Y_1', Y_2', \cdots$ in S defined on a common probability space and having the same probability distributions as $Y, Y_1, Y_2, \cdots$ such that $Y_n' \to Y'$ a.s.*

The above approach is essentially the same as that in Vervaat [10]. Here we have adopted the convention $\xrightarrow{d}$ for convergence in distribution.

Donsker's invariance principle says that $(S(nt)-nt\mu)/\sigma\sqrt{n} \xrightarrow{d} W(t)$, where $S(nt)=\sum_{k=1}^{[nt]}\xi_k$ is a random element in D and W is the Wiener Process on $[0,\infty)$. Let $\varphi(t)$ be a continuous function on $[0,\infty)$, then

$$\frac{S(nt)\,\varphi(t)-nt\,\varphi(t)\,\mu}{\sigma\sqrt{n}} \xrightarrow{d} W(t)\,\varphi(t)\,,$$

which is a consequence of a standand theorem (see, for example, Theorem 5.1 of [1]). It follows from Theorem 1.1 that we have

THEOREM 1.3. *Let φ be a continuous function on $[0,\infty)$ such that $t\varphi(t)$ is strictly increasing. Then*

$$\frac{\sup_{0\le u\le t} S(nu)\,\varphi(u)-nt\,\varphi(t)\,\mu}{\sigma\sqrt{n}} \xrightarrow{d} W(t)\,\varphi(t)\,.$$

COROLLARY 1.1. *For $\alpha<1$,*

$$\frac{\max_{j\le n} S_j/j^{\alpha}-n^{1-\alpha}\,\mu}{\sigma\, n^{1/2-\alpha}} \xrightarrow{d} N(0,\,1)\,.$$

Proof. Let $0<\delta<1$.
Set

$$\varphi(t)=\begin{cases} t^{-\alpha} & \text{if } t\ge\delta\,,\\ \delta^{-\alpha} & \text{if } t<\delta\,.\end{cases}$$

Then it follows from Theorem 1.3 that

$$\frac{\left(\sup\limits_{0\le u\le\delta}\dfrac{S(nu)}{(n\delta)^{\alpha}}\right)\vee\left(\sup\limits_{\delta\le u\le 1}\dfrac{S(nu)}{(nu)^{\alpha}}\right)-n^{1-\alpha}\,\mu}{\sigma\, n^{1/2-\alpha}} \xrightarrow{d} N(0,\,1)\,.$$

($X\vee Y$ means $\max(X,Y)$.) By the ordinary law of iterated logarithm for almost every ω there exists $\lambda(\omega)$ such that for large n,

$$\sup_{0\le j\le n\delta} S_j/j^{\alpha}\le (n\delta)^{1-\alpha}\,\mu+\lambda\sigma(2(n\delta)^{1-2\alpha}\log\log(n\delta))^{1/2}$$

and

$$\sup_{n\delta\le j\le n} S_j/j^{\alpha}\ge n^{1-\alpha}\,\mu-\lambda\sigma(2n^{1-2\alpha}\log\log n)^{1/2}\,,$$

hence

$$\sup_{0<j\le n} S_j/j^\alpha = \left(\sup_{0\le j\le n\delta} S_j/(n\delta)^\alpha\right) \vee \left(\sup_{n\delta\le j\le n} S_j/j^\alpha\right)$$

and the desired result is obtained.

Let K be the set of real-valued absolutely continuous function g on $[0, \infty)$ such that $g(0)=0$ and $\int_0^\infty (g'(t))^2\,dt \le 1$. Strassen [8] proved that the set of random elements $\{(S(nt) - nt\mu)/\sigma\sqrt{2n\log\log n}\}_{n\ge 3}$ is a. e. relatively compact with set of limit points K. It follows from Lemma 1.1 that

THEOREM 1.4. *Let φ be a continuous function on $[0, \infty)$ such that $t\varphi(t)$ is strictly increasing. Then*

$$\left\{\frac{\sup_{0\le u\le t} S(nu)\varphi(u) - nt\varphi(t)\mu}{\sigma\sqrt{2n\log\log n}}\right\}_{n\ge 3}$$

is a. e. relatively compact with set of limit points $K\varphi = \{y\varphi \mid y \in K\}$.

For $\alpha < 1$, $0 < \delta < 1$, set

$$\varphi(t) = \begin{cases} t^{-\alpha} & t \ge \delta\,, \\ \delta^{-\alpha} & t \le \delta\,, \end{cases}$$

then by Theorem 1.4 $\{(\sup_{0\le u\le 1} S(nu)\varphi(u) - n\mu)/\sigma\sqrt{2n\log\log n}\}$ is a. e. relatively compact with set of limit points $[-1, 1]$. With a similar argument as in the proof of the previous corollary we get

COROLLARY 1.2. *For $\alpha < 1$,*

$$\left\{\frac{\max\limits_{j\le n}\dfrac{S_j}{j^\alpha} - n^{1-\alpha}\mu}{\sigma\sqrt{2n^{1-2\alpha}\log\log n}}\right\}$$

is a. e. relatively compact with set of limit point $[-1, 1]$.

REMARK 1.1. Corollaries 1.1 and 1.2 are respectively the extensions of results in [9] and [2], where only the case $0 \le \alpha < 1$ is considered.

REMARK 1.2. All the above results still hold if we replace "increasing" by "decreasing", "sup" by "inf", "$\alpha < 1$" by "$\beta > 1$", and the proofs are much the same.

2. **Functional limit theorems for first passage time processes.** Let D^+ be the subset of D which consists of nondecreasing, nonnegative unbounded functions. For $X \in D^+$, we define $X^{-1}(t) = \inf\{u \geq 0 : X(u) > t\}$ which is the generalized inverse of $X(t)$.

LEMMA 2.1. *Let $X_n \in D^+$, $Y \in C$, $\eta_n > 0$, $\eta_n \to 0$. Let ψ be a continuous strictly increasing nonnegative unbounded function on $[0, \infty)$. Then $(X_n(t) - \psi(t))/\eta_n \to Y(t)$ uniformly on compact sets of $[0, \infty)$ implies $(\psi \circ X_n^{-1}(t) - I(t))/\eta_n \to -Y \circ \psi^{-1}(t)$ uniformly on compact sets of $[0, \infty)$, where I is defied by $I(t) = t$.*

THEOREM 2.1. *Let $X_1, X_2, \cdots$ be random elements in D^+, Y a random element in C and $\eta_1, \eta_2, \cdots$ positive random variables such that $\eta_n \xrightarrow{d} 0$. Let ψ be a continuous strictly increasing nonnegative unbounded function on $[0, \infty)$. Then*

$$\frac{X_n - \psi}{\eta_n} \xrightarrow{d} Y \quad \textit{implies} \quad \frac{\psi \circ X_n^{-1} - I}{\eta_n} \xrightarrow{d} -Y \circ \psi^{-1}.$$

The proof for Lemma 2.1 is easy and hence omitted. Theorem 2.1 is a consequence of Lemma 2.1 and Theorem 1.2.

Let $S(nu)$ be defined as in §1 and let

$$X_n(t) = \sup_{0 \leq u \leq t} \frac{S(nu)\,\varphi(u)}{n}, \quad Y(t) = W(t)\,\varphi(t), \quad \eta_n = \sigma n^{-1/2},$$

where

$$\varphi(t) = \begin{cases} t^{-\alpha} & \text{if } t \geq \delta, \\ \delta^{-\alpha} & \text{if } 0 \leq t < \delta, \end{cases}$$

and $0 < \delta < 1$, $\alpha < 1$. Let $\psi(t) = t\mu\,\varphi(t)$. Theorem 1.3 says $(X_n - \psi)/\eta_n \xrightarrow{d} Y$. Therefore, Theorem 2.1 tells us that $(\psi \circ X_n^{-1} - I)/\eta_n \xrightarrow{d} -Y \circ \psi^{-1}$. If $t \geq \mu$, then for almost every ω in the underlying probability space, $X_n^{-1}(t, \omega) \geq \delta$ for large n. The reason is as follows:

$$\begin{aligned} X_n^{-1}(t) &= \inf\{u \geq 0 : S(nu)\,\varphi(u) > nt\} \\ &= \begin{cases} \inf\{0 \leq u \leq \delta : S(nu) > \delta^{\alpha} nt\} \\ \qquad\qquad \text{if there exists such } u, \\ \inf\left\{u \geq \delta : \dfrac{S(nu)}{(nu)^{\alpha}} > n^{1-\alpha} t\right\} \\ \qquad\qquad \text{if } S(nu) \leq \delta^{\alpha} nt \text{ for all } 0 \leq u < \delta. \end{cases} \end{aligned}$$

And if $t \geq \mu$, then there exists $\lambda(\omega)$ such that for all $0 \leq u < \delta$ $S(nu) \leq \mu nu + \lambda\sigma(2nu \log\log nu)^{1/2} \leq \delta^{\alpha} nt$ for large n hence $X_n^{-1}(t, \omega) \geq \delta$.

Define $T(t) = \inf\{u \geq 0 \mid S(u)/u^{\alpha} > t\}$. By the similar reason, for $t \geq \mu$ and $0 \leq u < \delta$, $S(nu)/(nu)^{\alpha} \leq n^{1-\alpha} t$ for large n. Hence for $t \geq \mu$, $X_n^{-1}(t) = T(n^{1-\alpha} t)/n \geq \delta$ for large n. From now on, let's also assume $0 < \delta < \mu \leq t$. Consider $(\psi \circ X_n^{-1} - I)/\eta_n \xrightarrow{d} -Y \circ \psi^{-1}$ on $D[\mu, \infty)$, where $D[\mu, \infty)$ is the metric space defined similarly as $D[0, \infty)$. Then we have

$$\frac{\mu\left(\dfrac{T(n^{1-\alpha} t)}{n}\right)^{1-\alpha} - t}{\sigma n^{-1/2}} \xrightarrow{d} -W\left(\left(\frac{t}{\mu}\right)^{1/(1-\alpha)}\right)\left(\frac{t}{\mu}\right)^{-\alpha/(1-\alpha)}$$

on $D[\mu, \infty)$, which implies

$$\frac{(T(nt))^{1-\alpha} - \dfrac{t}{\mu} n}{\dfrac{\sigma}{\mu} n^{(1-2\alpha)/(2(1-\alpha))}} \xrightarrow{d} -W\left(\left(\frac{t}{\mu}\right)^{1/(1-\alpha)}\right)\left(\frac{t}{\mu}\right)^{-\alpha/(1-\alpha)}$$

and by some straight-forward computations we have

$$\frac{T(nt) - \left(\dfrac{n}{\mu}\right)^{1/(1-\alpha)} t^{1/(1-\alpha)}}{\dfrac{1}{\mu}\dfrac{\sigma}{1-\alpha} n^{1/(2(1-\alpha))}\left(\dfrac{t}{\mu}\right)^{\alpha/(1-\alpha)}} \xrightarrow{d} -W\left(\left(\frac{t}{\mu}\right)^{1/(1-\alpha)}\right)\left(\frac{t}{\mu}\right)^{-\alpha/(1-\alpha)}$$

on $D[\mu, \infty)$. Therefore

THEOREM 2.2. *For* $\alpha < 1$,

$$\frac{T(nt) - \left(\dfrac{n}{\mu}\right)^{1/(1-\alpha)} t^{1/(1-\alpha)}}{\dfrac{1}{\mu}\dfrac{\sigma}{1-\alpha} n^{1/(2(1-\alpha))}} \xrightarrow{d} W\left(\left(\frac{t}{\mu}\right)^{1/(1-\alpha)}\right)$$

on $D[\mu, \infty)$.

Put $t = \mu$, we get

COROLLARY 2.1. *For* $\alpha < 1$,

$$\frac{T(n) - \left(\dfrac{n}{\mu}\right)^{1/(1-\alpha)}}{\dfrac{1}{\mu}\dfrac{\sigma}{1-\alpha}\left(\dfrac{n}{\mu}\right)^{1/(2(1-\alpha))}} \xrightarrow{d} N(0, 1).$$

Similarly, it follows from Theorem 1.4 and Lemma 2.1 that

THEOREM 2.3. *For $\alpha<1$, the set of random elements in* $D[\mu, \infty)$

$$\left\{\frac{T(nt)-\left(\frac{n}{\mu}\right)^{1/(1-\alpha)}t^{1/(1-\alpha)}}{\frac{1}{\mu}\frac{\sigma}{1-\alpha}\sqrt{2n^{1/(1-\alpha)}\log\log n^{1/(1-\alpha)}}}\right\}$$

is a.e. relatively compact with the set of limit points $\{y:[\mu, \infty)\to R \mid y(t)=g((t/\mu)^{1/(1-\alpha)}),\ g\in K\}$.

Put $t=\mu$, we get

COROLLARY 2.2. *For* $\alpha<1$,

$$\left\{\frac{T(n)-\left(\frac{n}{\mu}\right)^{1/(1-\alpha)}}{\frac{1}{\mu}\frac{\sigma}{1-\alpha}\sqrt{2\left(\frac{n}{\mu}\right)^{1/(1-\alpha)}\log\log\left(\frac{n}{\mu}\right)^{1/(1-\alpha)}}}\right\}$$

is a.e. relatively compact with the set of limit points $[-1, 1]$.

REMARK 2.1. Corollaries 2.1 and 2.2 are respectively the extensions of results in [5] and [2], where only the case $0\leq\alpha<1$ is considered.

REMARK 2.2. Similar results hold for processes related to $N_c=\inf\{n\geq 1: S_n\leq c^{-1}n^{\beta}\}$, $c>0$, $\beta>1$.

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NATIONAL CENTRAL UNIVERSITY, CHUNG-LI, TAIWAN.