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PROPER HOLOMORPHIC MAPPINGS OVER BOUNDED DOMAINS IN C^n

BY

TAI-CHUNG LEE

Abstract. Let $a_j > 1$, $i = 1, 2, \cdots, n$, be n real numbers such that $(\ln a_j)/(\ln a_i)$ are rational for $i, j = 1, \cdots, n$, and let

$$U_j^n = \{z \in C^n \mid |z_j| < a_j, \ |z_i| < 1 \text{ for } i \neq j\}.$$

Let $\Omega(a_1, \cdots, a_n)$ be the holomorphic envelope of $\bigcup_{j=1}^n U_j^n$. In this paper, we will determine all proper holomorphic mappings $\Phi : \Omega(a_1, \cdots, a_n) \to \Omega(a_1^*, \cdots, a_n^*)$.

Introduction. Consider the following problem: Given two bounded domains Ω and Ω' in C^n, determine all proper holomorphic mappings $\phi : \Omega \to \Omega'$. In this context, "proper" simply means that ϕ maps sequences in Ω converging to the boundary of Ω to sequences in Ω' converging to the boundary of Ω'.

In the case $\Omega = \Omega' = B_n$ (the unit ball in C^n), Alexander [1] showed that ϕ must be a biholomorphism. Combining the machinery of Chern and Moser [3] with the technique of Alexander [1], Burns and Shnider [2] proved that a proper self-mapping of a strictly pseudo-convex domain is a biholomorphism. On the other hand, Rudin [5] showed that all proper self-mappings of the polydisc are given by $(z_1, \cdots, z_n) \to (\phi_1(z_{i_1}), \cdots, \phi_n(z_{i_n}))$, where each ϕ_k, $1 \leq k \leq n$, is a finite Blaschke product in one complex variable and $(i_1, \cdots, i_n)$ is a permutation of $(1, 2, \cdots, n)$.

In this paper we will determine all proper holomorphic mappings $\phi : \Omega \to \Omega'$ when Ω and Ω' are the special analytic polyhedrons to be defined in the following.

Let $a_j > 1$, $j = 1, \cdots, n$, be n real numbers such that α_j/α_i are rational for $i, j = 1, \cdots, n$, where $\alpha_j = \ln a_j$,

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$$U_j^n = \{z \in C^n \mid |z_j| < a_j,\ |z_i| < 1 \text{ for } i \neq j\}.$$

It is well known that the holomorphic envelope $\Omega(a_1,\cdots, a_n)$ of $\bigcup_{j=1}^n U_j^n$ is given by

$$\begin{aligned} \Omega = \{z \in C^n \mid\ & |z_j| < a_j, && i = 1,\cdots, n, \\ & |z_j|^{\alpha_i} |z_i|^{\alpha_j} < e^{\alpha_i \alpha_j}, && i \neq j,\ i, j = 1,\cdots, n, \\ (1) \qquad & |z_i|^{\alpha_j \alpha_k} |z_j|^{\alpha_i \alpha_k} |z_k|^{\alpha_i \alpha_j} < e^{\alpha_i \alpha_j \alpha_k}, && i \neq j \neq k,\ i, j, k = 1,\cdots, n, \\ & \vdots \\ & |z_1|^{\alpha_2 \cdots \alpha_n} |z_2|^{\alpha_1 \alpha_3 \cdots \alpha_n} \cdots |z_n|^{\alpha_1 \cdots \alpha_{n-1}} < e^{\alpha_1 \cdots \alpha_n}\}. \end{aligned}$$

Since α_i/α_j are rational, we may consider that the $2^n - 1$ parts of the topological boundary of Ω are defined by $2^n - 1$ monomials ψ_j, $j = 1,\cdots, 2^n - 1$ of least degree and assume $\psi_j(z) = z_j$, $j = 1,\cdots, n$. Our main result is the following theorem.

Main theorem.

1. THEOREM. *Let $\Omega(a_1,\cdots, a_n)$, $\Omega'(a_1',\cdots, a_n')$ be two domains defined by* (1) *and let $\Phi = (\phi_1,\cdots, \phi_n)$ be a proper holomorphic map from Ω onto Ω'. Then after a permutation among coordinates* (*in the target space*), *there are n real numbers $\theta_1,\cdots, \theta_n$ and n positive integers $m_1,\cdots, m_n$ such that Φ is defined by*

$$(2) \qquad \phi_j(z_j) = e^{i\theta_j} z_j^{m_j}, \quad j = 1,\cdots, n.$$

Before we carry out the proof of Theorem 1, let us write down a key lemma.

KEY LEMMA. *Let $\{f_j\}_{j=1,2,3,\ldots}$ be a countable family of holomorphic functions from U^n* (*polydisc in C^n*) *to U* (*unit disc in C*). *Let us denote $u \in C^n$ by $u = (z, w)$, $z \in C$, $w \in C^{n-1}$. Suppose $L_j(\zeta) = \lim_{r \to 1} f_j(r\zeta, 0)$ exists for almost all $\zeta \in T$* (*unit circle in C*) *and all j. Let $E_j = \{\zeta \in T \mid |L_j(\zeta)| = 1\}$, and suppose $\bigcup_{j=1}^\infty E_j$ covers T a.e. Then there is at least one j such that f_j depends on z alone.*

Proof of the key lemma. Since the countable union $\bigcup E_j$ covers T a.e., one of the E_j must have positive measure say E_1. For fixed $\zeta \in E_1$, $0 < r < 1$, consider $f_1(r\zeta, w)$ as a family of holomorphic functions from U^{n-1} to U. Since $\lim_{r \to 1} f(r\zeta, 0) = L_1(\zeta) \in T$, by

7.3.2 of [5], $\lim_{r\to 1} f_1(r\zeta, w) = L_1(\zeta)$ uniformly on compact subsets of U^{n-1}. In particular, $\lim_{r\to 1} f_1(r\zeta, w) = L_1(\zeta)$ for all $w \in U^{n-1}$. Let $g_w(z) = f_1(z, w)$, then $g_w \in H^\infty(U)$ for all $w \in U^{n-1}$. For $w, w' \in U^{n-1}$, we have $\lim_{r\to 1} g_w(r\zeta) = \lim_{r\to 1} g_{w'}(r\zeta)$ for all $\zeta \in E_1$. Since E_1 has positive measure, $g_w(z) = g_{w'}(z)$, i.e., $f_1(z, w) = f_1(z, w')$. Which proves f_1 depends on z alone. Q.E.D.

Proof of the theorem. By the above lemma, there are n integers $1 \leq \sigma(1), \cdots, \sigma(n) \leq 2^n - 1$ such that the holomorphic functions $\psi'_{\sigma(j)}(\Phi(z))$ depend only on z_j, $j = 1, \cdots, n$ (where ψ'_j are the monomials which define the boundary of Ω'). Then the proof consists of two steps.

Step 1. We show that after a permutation among coordinates (in the target space), $\psi'_{\sigma(j)} = \psi'_j$.

Step 2. From Step 1, we have

$$\Phi(z) = (\phi_1(z_1), \phi_2(z_2), \cdots, \phi_n(z_n)).$$

Now we show $\phi_j(z_j) = e^{i\theta_j} z_j^{m_j}$, which completes the proof.

Step 1. Let us assume $\psi'_{\sigma(1)} = \psi'^{r}_1 \psi'^{s}_2, \cdots$, where $r \geq 1$. Let $z^\circ = (z_1^\circ, z_2^\circ, \cdots, z_n^\circ)$ such that

$$\psi'_{\sigma(1)}(\Phi(z^\circ)) = \psi'^{r}_1(\Phi(z^\circ))\, \psi'^{s}_2(\Phi(z^\circ)) \cdots = 0.$$

Since $\psi'_{\sigma(1)}(\Phi(z))$ depends only on z_1, we have $\psi'_{\sigma(1)}(\Phi(z_1^\circ, z'))$ $= \psi'^{r}_1(\Phi(z_1^\circ, z'))\, \psi'^{s}_2(\Phi(z_1^\circ, z')) \cdots = 0$ for all $z' \in C^{n-1}$ such that $(z_1^\circ, z') \in \Omega$. Then at least one of the factors of $\psi'_{\sigma(1)}$ must vanish for all $(z^\circ, z') \in \Omega$, and we may assume $\psi'_1(\Phi(z_1^\circ, z')) = 0$ for all $(z_1^\circ, z') \in \Omega$. We claim that ψ'_1 cannot be a factor of any $\psi'_{\sigma(j)}$, $j = 2, \cdots, n$. For instance, if $\psi'_{\sigma(2)} = \psi'^{t}_1 \psi'^{u}_2 \cdots$, where $t \geq 1$, let $|z_2| < 1$, then there are $z_3, \cdots, z_n$, such that $(z_1^\circ, z_2, z_3, \cdots, z_n) \in \Omega$. Since

$$\begin{aligned}\psi'_{\sigma(2)}&(\Phi(z_1^\circ, z_2, \cdots, z_n))\\ &= \psi'^{t}_1(\Phi(z_1^\circ, z_2, \cdots, z_n))\, \psi'^{u}_2(\Phi(z_1^\circ, z_2, \cdots, z_n)) \cdots\\ &= 0\end{aligned}$$

and $\psi'_{\sigma(2)}(\Phi(z))$ depends only on z_2, we have $\psi'_{\sigma(2)} \circ \Phi(z_2) \equiv 0$, which is a contradiction. Since there are n such functions, hence after a permutation among coordinates (in the target space), ψ'_j is the only factor of $\psi'_{\sigma(j)}$, $j=1,\cdots,n$. So the only possibility is $\psi'_{\sigma(j)}=\psi'_j$, $j=1,\cdots,n$. It follows that $\phi_j(z)=\psi'_j\circ\Phi(z)$ depends only on z_j, $j=1,\cdots,n$. This completes Step 1.

Step 2. Let

$$U=\{z\in \boldsymbol{C}\mid |z|<1\},$$
$$rU=\{z\in \boldsymbol{C}\mid |z|<r\}.$$

From Step 1 we have

$$\Phi(z_1,\cdots,z_n)=(\phi_1(z_1),\cdots,\phi_n(z_n));$$

we first show that $\phi_j(\overline{U})\supseteq\overline{U}$ and $\phi_j(a_j\overline{U}-\overline{U})\subseteq a'_j\overline{U}-\overline{U}$, which will give us (2). Before we continue the proof, we have to make the following observation. If $|z_1^\circ|\leq 1$ then

$$\begin{aligned}\Omega\cap(\{z_1^\circ\}\times \boldsymbol{C}^{n-1})&\\ =\{(z_1^\circ,z_2,\cdots,z_n)\mid\ & |z_j|<a_j, \qquad j=2,\cdots,n,\\ & |z_i|^{\alpha_j}|z_j|^{\alpha_i}<e^{\alpha_i\alpha_j}, \quad i\neq j,\ i,j=2,\cdots,n,\\ &\vdots\\ & |z_2|^{\alpha_3\cdots\alpha_n}|z_3|^{\alpha_2\alpha_4\cdots\alpha_n}\cdots|z_n|^{\alpha_2\cdots\alpha_{n-1}}<e^{\alpha_2\cdots\alpha_n}\}\\ \overset{\text{def}}{=}\{z_1^\circ\}\times\Omega_1\,.&\end{aligned}$$

If $1<|z_1^\circ|<a_1$,

$$\begin{aligned}\Omega\cap(\{z_1^\circ\}\times \boldsymbol{C}^{n-1})&\\ =\{(z_1^\circ,z_2,\cdots,z_n)\mid\ & |z_j|<a_j\,|z_1^\circ|^{-\alpha_j/\alpha_1}, \qquad j=2,\cdots,n,\\ & |z_i|^{\alpha_j}|z_j|^{\alpha_i}<e^{\alpha_i\alpha_j}\,|z_1^\circ|^{-\alpha_i\alpha_j/\alpha_1}, \quad i\neq j,\ i,j=2,\cdots,n,\\ &\vdots\\ & |z_2|^{\alpha_3\cdots\alpha_n}|z_3^\circ|^{\alpha_2\alpha_4\cdots\alpha_n}\cdots|z_n^\circ|^{\alpha_2\cdots\alpha_{n-1}}\\ &\qquad <e^{\alpha_2\cdots\alpha_n}\,|z_1^\circ|^{-\alpha_2\cdots\alpha_n/\alpha_1}\}\\ \overset{\text{def}}{=}\{z_1^\circ\}\times\Omega_{|z_1^\circ|}\,.&\end{aligned}$$

It is clear that if $|z_1^\circ|>1$, then $\Omega_1\supset\Omega_{|z_1^\circ|}$. Coming back to the proof, let us suppose for some $z_1^\circ\in\overline{U}$ and $|\phi_1(z_1^\circ)|>1$; then $\Phi=(\phi_1,\cdots,\phi_n)$ will map $\{z_1^\circ\}\times\Omega$ properly onto $\{\phi_1(z_1^\circ)\}\times\Omega'_{|\phi_1(z_1^\circ)|}$. Since $(\phi_2,\cdots,\phi_n)$ is independent of z_1, then $(\phi_2,\cdots,\phi_n)$ will map Ω_1 properly onto $\Omega'_{|\phi_1(z_1^\circ)|}$, which is a contradiction to the fact that $\Phi=(\phi_1,\cdots,\phi_n)$ maps Ω properly onto Ω'. That proves

$$\phi_1(\overline{U})\subseteq\overline{U}\,.$$

Suppose for some $|z_1^\circ|>1$ and $|\phi_1(z_1^\circ)|\leq 1$; then $\Phi=(\phi_1\cdots\phi_n)$ will map $\{z_1^\circ\}\times\Omega_{|z_1^\circ|}$ properly onto $\{\phi_1(z_1^\circ)\}\times\Omega_1'$, and hence $(\phi_2,\cdots,\phi_n)$ will map $\Omega_{|z_1|}$ properly onto Ω_1'. Then for some z_2, $|z_2|=a_2|z_1|^{-\alpha_2/\alpha_1}$, any sequence $z_2^v\to z_2$, $\lim_{v\to\infty}|\phi_2(z_2^v)|=a_2'$. On the other hand, since $|z_2|<a_2$, we must have $|\phi_2(z_2)|<a_2'$, which is a contradiction. Hence we have

$$\phi_1(a_1\overline{U}-\overline{U})\subseteq a_1'\overline{U}-\overline{U}.$$

Consider ϕ_1 as a proper map from a_1U to $a_1'U$ and $\phi_1|U$ as a proper map from U to U, and let $\phi_1^{-1}(0)=\{w_1,\cdots,w_m\}$. Then ϕ_1 can be written as

$$\phi_1(z_1)=a_1'e^{i\theta'}\prod_{j=1}^{m}\frac{z_1/a_1-w_j/a_1}{1-(\overline{w_j}/a_1)(z_1/a_1)}$$

and

$$\phi_1(z_1)=e^{i\theta}\prod_{j=1}^{m}\frac{z_1-w_j}{1-\overline{w_j}z_1}.$$

In order to have the two expressions represent the same function, we should have $w_1=w_2=\cdots=w_m=0$ and

$$\frac{a_1'e^{i\theta'}}{a_1^m e^{i\theta}}=1,$$

i.e.,

$$\phi_1(z_1)=e^{i\theta}z_1^m.$$

Similarly we can prove

$$\phi_j(z_j)=e^{i\theta'}z_j^{m'}. \qquad \text{Q.E.D.}$$

2. Corollary. *Φ exists iff there is a permutation among $(1,\cdots,n)$ and positive integers $m_1,\cdots m_n$ such that*

$$a_{\sigma(j)}'=a_j^{m_j}.$$

3. Corollary. *Let $P(\Omega)$ be the subgroup of* Aut(Ω) *consisting of the elements of permutation among coordinates. Then*

$$\mathrm{Aut}\,(\Omega)/P(\Omega)\simeq T^n.$$

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INSTITUTE OF MATHEMATICS, NATIONAL TSING HUA UNIVERSITY, HSINCHU, TAIWAN 300, R. O. C.