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THE SPECTRAL THEORY OF A REDUCED SYMMETRIC HYPERBOLIC SYSTEM WITH DISCRETE WAVE FRONTS(1)

BY

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Abstract. The distribution of the resolvent set and the continuous spectrum of a selfadjoint operator reduced from an exterior problem for some symmetric hyperbolic system with dispersion term is characterized by the set of real roots of the polynomial symbol for the unperturbed system considered in the same exterior domain and with star-like analytic wave fronts; there is no need of assuming convexity and nonvanishing curvature. Moreover, the continuous spectrum is absolutely continuous. However, an ambiguity in classification occurs for a finite number of points on the real line, whose locations are determined by the dispersion term of the unperturbed system. On the other hand, if any eigenvalues exist for either the perturbed or the unperturbed system, they belong to that finite number of points. Boundary conditions are the same for both systems. The perturbation arises not at the boundary but anywhere in the exterior domain and diminishes quickly enough at infinity.

1. Introduction and main result. Let Ω be a domain in R^N with a bounded simply connected complement and piecewise C^1-boundary $\partial\Omega$. With the imposed conditions listed below, consider the reduced differential operator of the exterior problem

$$L_i\, u(x) = A_{i0}^{-1}(x) \sum_{j=1}^{N} A_{ij}(x)\, D_j\, u(x) + A_{i0}^{-1}(x)\, A_{i,N+1}(x)\, u(x) \quad \text{on } \Omega \qquad (i = 0, 1); \tag{1}$$

$$B(x)\, u(x) = b(x)\, u(x) \quad \text{on } \partial\Omega \quad \text{(almost everywhere)}, \tag{2}$$

for some symmetric hyperbolic system of N' first order partial differential equations with dispersion term $A_{i0}^{-1}(x)\, A_{i,N+1}(x)$:

Assumption (a) For $i = 0, 1$, matrices $A_{ij}(x)$ are Hermitian

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symmetric and of order $N' > 1$, $j = 0, \cdots, N$; $A_{0j}(x) = A_{0j}$ are constant matrices, $j = 0, \cdots, N+1$; $A_{i0}(x)$ are positive definite and uniformly bounded:

$$C_0 |y|^2 \leq A_{i0}(x)\, y \cdot y^* \leq C_1 |y|^2 \quad (y \in R^{N'} \backslash \{0\};\ C_0,\ C_1\text{: constants});$$

$A_{1,N+1}(x)$ is continuous almost everywhere; and L_i are formally selfadjoint:

$$\sum_{j=1}^{N} D_j A_{ij}(x) = A_{i,N+1}(x) - A_{i,N+1}^*(x) \quad \text{on} \quad \Omega\,,$$

where $D_j = -\sqrt{-1}\ \partial/\partial x_j$ and A_{ij}^* is the adjoint of A_{ij}.

Assumption (b) For $1 \leq j \leq N$, $A_{ij}(x) - A_{0j}$ are continuous almost everywhere on the closure $\bar{\Omega}$ of Ω (then $A_{1j}(x) = A_{0j}$ on $\partial\Omega$ a.e.); for each unit inward normal $n(x)$ of $\partial\Omega$ at x where $\partial\Omega$ is C^1, let $b(x)$ be a scalar real-valued function and let $B(x)$ be an $N' \times N'$ matrix such that $B(x) - b(x)\, I$ is integrable on $\partial\Omega$ and

$$B(x) - B^*(x) = -\sqrt{-1} \sum_{j=1}^{N} A_{0j}\, n_j(x)\,. \tag{3}$$

Denote by $T(\lambda)$ the matrix $\lambda A_{00} - A_{0,N+1}$, by S the set of real λ at which $T(\lambda)$ is singular, by $b(y, \lambda)$ the product of all irreducible factors on the real number field in (y, t) of the determinant of $\sum_{j=1}^{N} A_{0j}\, T^{-1}(\lambda)\, y_j - I$, and by $N(b(\cdot, \lambda))$ the real null set of $b(\cdot, \lambda) : N(b(\cdot, \lambda)) = \{y \in R^N : b(y, \lambda) = 0\}$.

Assumption (c) When $\lambda \notin S$ and is real, $N(b(\cdot, \lambda))$ is bounded and $y \cdot \mathrm{grad}_y b(y, \lambda) \neq 0$ on $N(b(\cdot, \lambda)) \backslash \{0\}$.

Assumption (d) For $a > N$, let C be a constant such that

$$(4)_j \qquad |A_{1j}(x) - A_{0j}| \leq C(1 + |x|)^{-a} \quad \text{on} \quad \Omega \qquad (0 \leq j \leq N+1);$$

$$(4)_0^{-1} \qquad |D_k A_{10}^{-1}(x)| \leq C(1 + |x|)^{-a} \quad \text{on} \quad \Omega \qquad (1 \leq k \leq N)\,.$$

(Here restriction $(4)_0^{-1}$ is removed if $A_{10}(x) = A_{00}$ on Ω.)

The operators L_i are studied in the Hilbert spaces $H_{i,c}$ with norms $\|\cdot\|_{i,c}$ induced by the inner products

$$(u, v)_{i,c} = \int u(x) \cdot [W_{i,c}(x)\, v(x)]^*\, dx \qquad (x \in \Omega)\,, \tag{5}$$

with weight $W_{i,c}(x) = (1 + |x|)^c A_{i,0}(x)$. The index c will be deleted if $c = 0$.

MAIN THEOREM. I. *Under assumptions* (a) *and* (b), L_i *is a self-adjoint operator in the Hilbert space* H_i *with domain*

$$D(L_i) = \{u : D^\alpha u \in H_i, \ |\alpha| \leq 1, \ u \in H_i(\partial\Omega) \ \& \ [B(x) - b(x) I] u(x) = 0 \text{ on } \partial\Omega\}. \tag{6}$$

II. *Suppose assumptions* (a), (b), (c), *and* (d) *with constant* C *in* (d) *not greater than a constant* κ *which depends only on* a, N, C_0, C_i *and the matrices* A_{0j}, $1 \leq j \leq N+1$ (cf. Theorem 2.5). *Suppose that the kernel* $\ker(b(x) I - B(x))$ *of* $b(x) I - B(x)$ *is included in that of* $b(x) I - B^*(x)$ *for almost all* $x \in \partial\Omega$. *Then the set* C_ρ *of complex numbers* $\lambda \notin S$ *such that* $N(b(\cdot, \lambda))$ *is contained in* $\{0\}$, *including nonreal complex numbers, belongs to the resolvent set of* L_i; *the set* S *contains all the eigenvalues of* L_i *whenever they exist; the set* C_σ *of real numbers* $\lambda \notin S$ *with* $N(b(\cdot, \lambda))$ *consisting of more than one element is a part of continuous spectrum and is absolutely continuous.*

It is important to make the following remarks. The character, actually the classification, of the set S is not clear through the employed argument. However, it can be clarified by an entirely different approach. The set $R \backslash S$ may not just consist of the continuous spectrum of L_i and may include parts of the resolvent set of L_i for a particular problem, say the Dirac system. Here the possibility of $A_{0,N+1} = 0$ or $\partial\Omega = \emptyset$ is not excluded; even then S is still not empty and equal to $\{0\}$. The boundary operator $B(x)$ is not uniquely determined. The boundary condition (2) is so general that it includes conditions of Neumann type, Robin type, or those combined with Dirichlet type for a group of differential equations in physics. The details are stated in the Appendix §4 at the end of the paper.

The spectral theory for the reduced operator of a symmetric hyperbolic system of the first order has been studied by Mochizuki [6], Schulenberger and Wilcox [9], Wilcox [10], and many others referred to in their bibliographies. But for the perturbed system, they considered either the whole space problem, or a perturbation on the leading coefficient $A_{i0}(x)$ only, or a perturbation on a finite region. For the unperturbed system, they considered a whole space problem instead of an exterior problem as here and either proved or assumed the convexity and nonvanishing curvature of $N(b(\cdot;\lambda))$ and the coerciveness inequality which are not used here. The assumption $a>N$ and smallness of C can be omitted for $a>1$ and any $C>0$ through another approach in a forthcoming article; other problems with $a>1$, in particular second order equations, have been studied by Agmon [1], Ikebe-Uchiyama [4], Mochizuki [7], all of whom give more detailed references. The condition that $N'>2$ is essential for uniqueness of the solution to the free-space problem satisfying a certain radiation condition. In other words, the problem considered here is at least either that of a system of two first order equations or a second order scalar equation.

The rest of the paper is designed as follows: In the next section the spectral theory for the operator L_i is clarified by the properties of a certain elementary solution of the free space unperturbed system. Actually, the system with nonhomogeneous boundary condition (i.e. more general than in the main theorem) is studied in detail in that section. The elementary solution is discussed in §3. Finally, in the Appendix the Dirac system and a system form of the wave equation are discussed as examples.

2. Spectral theory and inhomogeneous boundary value problem.

The problem with inhomogeneous boundary condition,

$$(2.1)_i \qquad L_i v = \lambda v + f \quad \text{on} \quad \Omega ,$$

$$(2.2) \qquad B(x)\,v(x) = b(x)\,v(x) + g(x) \quad \text{on} \quad \partial\Omega \qquad (g \in L^1(\partial\Omega)) ,$$

is studied through an integral equation with the benefit of explicit properties for some elementary solution related to the unperturbed operator L_0 considered on the whole space R^N.

Let χ be the characteristic function of the set $\bar{\Omega}$. Denote by $\delta(\partial\Omega)$ the Dirac-delta function with support $\partial\Omega$.

LEMMA 2.1. *Under assumptions* (a) *and* (b), *if* $v \in D(L_i)$ *is a solution to system* $(2.1)_i$ *and* (2.2), *then* $w=\chi v$ *is a solution of the following distribution equation*:

$$(2.3)_i \qquad (L_i - \lambda I)\,w = \chi f - \delta(\partial\Omega)\,\{[b(x)I - B^*(x)]v + g(x)\} \quad \text{on} \quad R^n .$$

Conversely, if $w \in D(L_i)$ *is a solution of system* $(2.3)_i$, *then* w *is a solution of* $(2.1)_i$ *and* (2.2).

Proof. Assume $v \in D(L_i)$ is a solution of $(2.1)_i$ and (2.2). Then, for any C_0^∞-function ϕ,

$$((L_i - \lambda I)\, w,\ \phi) = (w,\ (L_i^* - \bar{\lambda} I)\, \phi)$$
$$= \int w(x) \cdot [(L_i^* - \bar{\lambda} I)\, \phi(x)]^*\, dx \qquad (x \in \Omega)\,.$$

Let $\Omega_r = \{x \in \Omega : |x| \leq r\}$. Then by the divergence theorem

$$
\begin{aligned}
&((L_i - \lambda I)\, w,\ \phi) \\
&\quad = \lim \int v(x) \cdot [(L_i^* - \bar{\lambda} I)\, \phi(x)]^*\, dx \qquad (x \in \Omega_r,\ r \to \infty) \\
&\quad = \int (L_i - \lambda I)\, v(x) \cdot \phi^*(x)\, dx \qquad (x \in \Omega) \\
&\qquad + \int \sqrt{-1} \sum_{j=1}^{N} A_{0j}\, n_j(x)\, v(x) \cdot \phi^*(x)\, dx \qquad (x \in \partial\Omega) \\
&\quad = (\chi(L_i - \lambda I)\, v,\ \phi) - (\delta(\partial\Omega)\, [B(x) - B^*(x)]\, v,\ \phi) \\
&\quad = (\chi f,\ \phi) - (\delta(\partial\Omega) \{[b(x)\, I - B^*(x)]\, v(x) + g(x)\},\ \phi)\,.
\end{aligned}
\tag{2.4}
$$

This gives the relation $(2.3)_i$.

Conversely, assume $w \in D(L_i)$ is a solution of $(2.3)_i$. Let $v = w$ on $\bar{\Omega}$. Then on Ω we have $(2.1)_i$. For any C_0^∞-function ϕ we have relation (2.4). On the other hand,

$$
\begin{aligned}
&(\chi f - \delta(\partial\Omega) \{[b(x)\, I - B^*(x)]\, v + g(x)\},\ \phi) \\
&\quad = (\chi(L_i - \lambda I)\, v,\ \phi) \\
&\qquad - (\delta(\partial\Omega) \{[b(x)\, v + g(x)] - B^*(x)\, v\},\ \phi)\,.
\end{aligned}
$$

This and relation (2.4) imply

$$(\delta(\partial\Omega)\, B(x)\, v,\ \phi) = (\delta(\partial\Omega) \{b(x)\, v + g(x)\},\ \phi)\,,$$

which yields (2.3). The proof is complete.

System (2.2) has a solution if and only if $g(x)$ is orthogonal to the manifold $M(B^* - bI)$ of all solutions of the system $B^*(x)\, u(x) = b(x)\, u(x)$. If $g(x)$ is orthogonal to $M(B^* - bI)$, system (2.2) may not have a unique solution and we denote by v_g the general solution to (2.2). Therefore, instead of (2.3), we consider from now on the distribution equation

$$(2.5)_i \qquad \begin{aligned}&(L_i - \lambda I)\,(\chi v)\\ &\qquad = \chi f - \delta(\partial\Omega)\{[b(x)\,I - B^*(x)]\,v_g(x) + g(x)\} \quad \text{on} \quad R^N.\end{aligned}$$

The expression $(2.5)_1$ is equivalent to either of the following two relations:

$$(2.6) \qquad \begin{aligned}&(L_0 - \lambda I)\,(\chi v)\\ &\quad = \chi f - \delta(\partial\Omega)\{[bI - B^*]\,v_g + g\}\\ &\qquad + A_{10}^{-1}(A_{10} - A_{00})\,A_{00}^{-1}\Big[\sum_{j=1}^{N} A_{0j}\,D_j(\chi v) + A_{0,N+1}(\chi v)\Big]\\ &\qquad + A_{10}^{-1}\Big[\sum_{j=1}^{N} (A_{0j} - A_{1j})\,D_j(\chi v) + (A_{0,N+1} - A_{1,N+1})\,(\chi v)\Big],\end{aligned}$$

$$(2.6)' \qquad \begin{aligned}&(L_0\,\lambda I)\,(\chi v)\\ &\quad = A_{00}^{-1}A_{10}\,\chi f - \delta(\partial\Omega)\,A_{00}^{-1}A_{10}\,\{[bI - B^*]\,v_g + g\}\\ &\qquad + \sum_{j=1}^{N} A_{00}^{-1}[A_{0j} - A_{1j}]\,D_j(\chi v) + A_{00}^{-1}[A_{0,N+1} - A_{1,N+1}]\,(\chi v)\\ &\qquad + \lambda A_{00}^{-1}[A_{10} - A_{00}]\,(\chi v)\,,\end{aligned}$$

which in turn have expressions as integral equations through some elementary solution of $\sum_{j=1}^{N} A_{0j}\,T^{-1}(\lambda)\,D_j - zI$. Let us mention here certain properties of the elementary solution that will be proved in the next section.

Let $\tilde{x}$ denote a unit vector. For real $\lambda \notin S$, the eigenvalues of the matrix

$$(2.7) \qquad A_0(\tilde{x})\,T^{-1}(\lambda) = \sum_{j=1}^{N} A_{0j}\,T^{-1}(\lambda)\,\tilde{x}_j\,,$$

whenever they exist, can be listed as follows:

$$(2.8) \qquad \begin{aligned}&r_1(\tilde{x};\,\lambda) > \cdots > r_m(\tilde{x};\,\lambda)\,;\\ &r_j(-\tilde{x};\,\lambda) = -r_{m-j+1}(\tilde{x};\,\lambda)\,, \qquad 1 \leq j \leq m\,;\\ &r_j(\tilde{x};\,\lambda) = 0 \qquad \text{if } m \text{ is odd and } j = (m+1)/2\,;\end{aligned}$$

moreover, $r_j(t\tilde{x};\,\lambda)$ are homogeneous in t of degree 1 and uniformly continuous over any compact interval of the real line disjoint from S. Denote by $r_j(\tilde{x};\,\lambda)$ the inverse of $r_j(\tilde{x};\,\lambda)$ for $j \neq (m+1)/2$ and the 0-function if m is odd and $j = (m+1)/2$; by $P_j(\tilde{x};\,\lambda)$ the projection of $C^{N'}$ onto the eigenspace relating to $r_j(\tilde{x};\,\lambda)$; and by $P_\pm(\tilde{x};\,\lambda) = \sum_{j=1}^{[m/2]} P_j(\pm\tilde{x};\,\lambda)$.

LEMMA 2.2. *Suppose assumptions* (a) *and* (c). *Then there exists one and only one elementary solution* $E(x; \lambda, z)$ *of* $\sum_{j=1}^{N} A_{0j} T^{-1}(\lambda) D_j - zI$, $\lambda \notin S$, *with the following properties*:

(a) $E(x; \lambda, z)$ *is an analytic function in* $(R^N \setminus \{0\}) \times (C \setminus S) \times (C \setminus R)$.

(b) *For each* $x \neq 0$ *and any real* $\lambda \notin S$, *the limits*

$$E(x; \pm\lambda) = \lim E(x; \lambda, 1 + \sqrt{-1}\,\tau) \qquad (\tau \to \pm 0) \tag{2.9}$$

exist and the convergence is uniform in every compact set of $(R^N \setminus \{0\}) \times (R \setminus S)$. *Moreover,* $E(x; \pm\lambda)$ *are continuous functions of* (x, λ) *in* $(R^N \setminus \{0\}) \times (R \setminus S)$.

(c) *For* $\lambda_0 \notin S$, $E(x; \lambda, 1 + \sqrt{-1}\,\tau)$ *behaves at infinity in* x *like*

$$E(x; \lambda, 1 + \sqrt{-1}\,\tau) \begin{cases} = O(|x|^{(1-N)/2}); \\ \neq o(|x|^{(1-N)/2}), \end{cases} \qquad \textit{if} \quad \tau = \pm 0; \tag{2.10}$$

$$[I - P_{\pm}(\tilde{x}; \lambda)] E(x; \lambda, 1 + \sqrt{-1}\,\tau) \begin{cases} = O(|x|^{-(1+N)/2}); \\ \neq o(|x|^{-(1+N)/2}), \end{cases} \quad \textit{if} \quad \tau = \pm 0, \tag{2.11$_\pm$}$$

uniformly in (λ, τ) *with* $|\tau| > \varepsilon'$ *for any* $\varepsilon' > 0$ *and* $\lambda \in [\lambda_0 - \varepsilon, \lambda_0 + \varepsilon]$ *which is small on an interval that is disjoint from* S.

(d) *For* λ *such that* $\lambda \notin S$ *and* $N(b(\cdot; \lambda)) \subset \{0\}$, $E(\cdot; \lambda)$ *is rapidly decreasing and smooth.*

(e) $D_k E(x; \lambda, 1 + \sqrt{-1}\,\tau)$ *also satisfy "the radiation conditions"* (2.10) *and* (2.11) *for each* $k = 1, \cdots, N$. (The radiation conditions (2.10) and (2.11) are first employed by Mochizuki [6].)

From the proof (in the next section) of the foregoing assertion, it is easy to obtain the following result, which will be proved also in the next section.

LEMMA 2.3. *In addition to assumptions* (a), (b), (c), *and* (d), *assume that the Fourier transform* $f^{\sim}(\xi)$ *of the force function* χf *and its derivatives* $D^{\alpha} f^{\sim}(\xi)$, $|\alpha| \leq 2$ *are locally integrable and dying out at infinity or* $\chi f \in H_{0,c}$ *with* $c > 2N - 1$. *Then for every* $v_g \in L^1(\partial\Omega)$, *system* $(2.1)_0$ *and* (2.2), *or equivalently* $(2.5)_0$, *has one and only one* *solution satisfying the incoming* $(+)$ (*or outgoing* $(-)$) *radiation conditions* (2.10) *and* $(2.11)_{\pm}$.

The uniqueness in Lemma 2.3 and assertion (c) of Lemma 2.2, in particular property (2.10), lead to the integral equation which is another expression of $(2.1)_1$ and (2.2) with solution considered in $D(L_1)$.

LEMMA 2.4. *In addition to assumptions* (a) *through* (d), *suppose that* $f \in H_{0,c}$ *with* $c > 2N-1$. *Then for every* $v_g \in L^1(\partial\Omega)$, *system* $(2.1)_1$ *and* (2.2) *or equivalently system* (2.6) (*or* (2.6)′) *has a solution such that its first derivatives and itself fulfill the radiation condition if and only if there is a solution satisfying either of the following two systems*

$$
\begin{aligned}
\chi v = {} & E(\cdot;\lambda) * (\chi f)^* - E(\cdot;\lambda) * (\delta(\partial\Omega)\{[bI-B^*]v_g + g\})^* \\
& - \sum_{j=1}^{N} [D_j E(\cdot;\lambda)] * [A_{10}^{-1}(A_{0j}-A_{1j})(\chi v) \\
& \qquad + A_{10}^{-1}(A_{10}-A_{00})A_{00}^{-1}A_{1j}(\chi v)]^* \\
& + E(\cdot;\lambda) * [A_{10}^{-1}(A_{0,N+1}-A_{1,N+1}^*)(\chi v) \\
& \qquad + A_{10}^{-1}(A_{10}-A_{00})A_{00}^{-1}A_{0,N+1}(\chi v)]^* \\
& + E(\cdot;\lambda) * \left[\sum_{j=1}^{N} (D_j A_{10}^{-1}) A_{1j}(\chi v)\right]^*,
\end{aligned}
\tag{2.12}
$$

$$
\begin{aligned}
\chi v = {} & E(\cdot;\lambda) * (\chi A_{00}^{-1}A_{10}f)^* \\
& - E(\cdot;\lambda) * (\delta(\partial\Omega)A_{00}^{-1}A_{10}\{[bI-B^*]v_\rho + g\})^* \\
& + E(\cdot;\lambda) * [A_{00}^{-1}\{(A_{0,N+1}-A_{1,N+1}^*) - \lambda(A_{00}-A_{10})\}(\chi v)]^* \\
& + \sum_{j=1}^{N} [D_j E(\cdot;\lambda)] * [A_{00}^{-1}(A_{1j}-A_{0j})(\chi v)]^* .
\end{aligned}
\tag{2.12)$'$}
$$

Proof. Assume that v is a solution of (2.6) and satisfies the radiation condition. Then, applying the elementary solution $E(\cdot;\lambda)$ through convolution from the left on both sides of the adjoint of (2.6) yields

$$
\begin{aligned}
\chi v = {} & E(\cdot;\lambda) * \{\chi f\}^* - E(\cdot;\lambda) * \{\delta(\partial\Omega)[bI-B^*)v_g + g]\}^* \\
& + \sum_{j=1}^{N} \int E(\cdot - y;\lambda)\{[A_{00}^{-1}A_{0j} - A_{10}^{-1}(y)A_{1j}(y)] \\
& \qquad \cdot D_j v(y)\}^* \, dy, \qquad (y \in \Omega) \\
& + \int E(\cdot - y;\lambda)\{[A_{10}^{-1}(y) - A_{00}]A_{00}^{-1}A_{0,N+1} \\
& \qquad + A_{10}^{-1}[A_{0,N+1} - A_{1,N+1}(y)]v(y)\}^* \, dy .
\end{aligned}
\tag{2.13}
$$

Every term on the right side is well-defined because of $(4)_j$ in assumption (d), of Lemma 2.2(e), of the radiation condition in Lemma 2.2(c) for $E(\cdot;\lambda)$, and of the radiation condition for v. Consider the first integrals as the limits when $r\to\infty$ of the integrals over the region Ω_r. Then the divergence theorem, formal self-adjointness of L_1 in assumption (a), and assumption (b) and (d) imply that

$$\begin{aligned}
&\sum_{j=1}^{N}\int E(x-y;\lambda)\{[A_{00}^{-1}A_{0j}-A_{10}^{-1}(y)A_{1j}(y)]D_j v(y)\}^* dy \qquad (y\in\Omega)\\
&\quad=-\sum_{j=1}^{N}\int D_j E(x-y;\lambda)(\{A_{10}^{-1}(y)[A_{0j}-A_{1j}(y)]\\
&\qquad\qquad +A_{10}^{-1}(y)[A_{10}(y)-A_{00}]A_{00}^{-1}A_{0j}\}v(y))^* dy\\
&\qquad+\int E(x-y;\lambda)\left\{\sum_{j=1}^{N}[D_j A_{10}^{-1}(y)]A_{1j}(y)v(y)\right\}^* dy \qquad (y\in\Omega)\\
&\qquad+\int E(x-y;\lambda)\{A_{10}^{-1}(y)[A_{1,N+1}(y)\\
&\qquad\qquad -A_{1,N+1}^*(y)]v(y)\}^* dy \qquad (y\in\Omega),
\end{aligned}$$

which along with (2.13) lead to expression (2.12).

Conversely, assume v is a solution of (2.12). Then the expression (2.12) itself and the radiation conditions for $E(x;\lambda)$ imply that v satisfies the related radiation conditions. Next, formal adjointness for L_1 in assumption (a) and the application of $L_0-\lambda I$ from the left on both sides of (2.12) give relation (2.6). A similar argument applies to equivalence between (2.6)′ and (2.12)′. This completes the proof of the lemma.

Employment of the fixed point theorem affords the existence of solutions to (2.12). The assertion is also true for (2.12)′ provided that $A_{10}(x)\equiv A_{00}$, but condition $(4)_0^{-1}$ can be dropped. This follows from the proof of the theorem below.

THEOREM 2.5. *Suppose that assumptions* (a) *through* (d) *hold with constant* $C<\kappa$ *where*

$$\kappa=[2N(C_0^2+1)]^{-1}[\{b^2+4(a-N)N(C_0^2+1)(C_E^2C_0)^{-1/2}\}^{1/2}-b];$$

$$b=C_0(N+1)+C_0^2\sum_{j=1}^{N+1}|A_{0j}|+\sum_{j=1}^{N}|A_{0j}|^2;$$

and constant C_E *is given in Theorem* 3.1, *and suppose that* $f\in H_{0,c}$ *with* $c>2N-1$. *Then for every* $L^1(\partial\Omega)$*-solution* v_g *of* (2.2), *there* *exists one and only one solution to system* (2.1)$_i$ *and* (2.2) *satisfying the radiation conditions.*

Proof. For the case $i=0$, the last two terms on the right side of (2.12) vanish. Relation (2.12) and Lemma 2.2 give the assertion (also of Lemma 2.3 with the second assumption for f). Next assume $i=1$ and show that integral equation (2.12) has exactly one solution fulfilling the radiation condition. It suffices to argue that in the Hilbert space $H_{0,-1-\varepsilon}$ the linear transformation $\boldsymbol{E}$ with kernel

$$
\begin{aligned}
&\boldsymbol{E}(x, y; \lambda)\\
&\quad = -\sum_{j=1}^{N} \chi(y)\, D_j\, E(x-y;\lambda)\\
&\qquad \cdot A_{10}^{-1}(y)\,[\{A_{0j}-A_{1j}(y)\} + \{A_{10}(y)-A_{00}\}\, A_{00}^{-1}\, A_{1j}(y)]\\
(2.14)\qquad &\qquad + \chi(y)\, E(x-y;\lambda)\\
&\qquad \cdot A_{10}^{-1}(y)\,[\{A_{0,N+1}-A_{1,N+1}(y)\} + \{A_{10}(y)-A_{00}\}\, A_{00}^{-1} A_{0,N+1}]\\
&\qquad + \sum_{j=1}^{N} E(x-y;\lambda)\,[D_j\, A_{10}^{-1}(y)]\, A_{1j}(y)
\end{aligned}
$$

is a contraction mapping, since if there is a fixed point, expression (2.12) and Lemma 2.2 imply the radiation condition for the solution.

Assume $v \in H_{0,-1-\varepsilon}$. Then, by inequality $(4)_j$ and $(4)_0^{-1}$ in assumption (d) and by estimate (2.10) and (2.11) in Lemma 2.2(c) and (e) for $E(x;\lambda)$, we have

$$
(2.15)\qquad |\boldsymbol{E}v(x)| \le \bar{C}\int (1+|y|)^{-a}(1+|x-y|)^{-(N-1)/2}\,|v(y)|\,dy \qquad (y\in R^N)\,;
$$

$$
(2.16)\qquad \bar{C} = CC_E\Big\{CN(C_0^2+1) + C_0(N+1) + C_0^2\sum_{j=1}^{N+1}|A_{0,j}| + \sum_{j=1}^{N}(A_{0j})\Big\}.
$$

The following Peetre inequality is well known (cf. Friedrichs [2] or Kumano-Go [5]):

$$
(1+|x|)^c \le (1+|x-y|)^{|c|}(1+|y|)^c \qquad (c\in R)\,,
$$

and together with the Schwarz inequality implies that

$$
\begin{aligned}
&\int (1+|y|)^{-a}(1+|x-y|)^{-(N-1)/2}\,|v(y)|\,dy\\
&\qquad \le C_0^{1/2}\,\varepsilon^{-1/2}(1+|x|)^{-(N-1)/2}\,\|v\|_{0,c}\\
&\qquad\qquad (c=2(N-a)-1+\varepsilon;\ \varepsilon>0)\,.
\end{aligned}
$$

Choose $\varepsilon = a - N$. We then obtain the estimate

$$\|Ev\|_{0,c} \leq \bar{C}C^{1/2}(a-N)^{-1}\|v\|_{0,c} \qquad (c = N - a - 1).$$

Therefore, E is a contraction mapping in the Hilbert space $H_{0,-1-\varepsilon}$ ($\varepsilon = a - N$), provided $\bar{C}C_0^{1/2} < a - N$, which follows from the constant C in the hypothesis.

Proof of main theorem. Because the set of all functions u such that $D^\alpha u \in H_i$, $|\alpha| \leq 1$ and $u(x) = 0$ on $\partial\Omega$ is dense in H_i, the domain $D(L_i)$ of L_i given in (6) is dense in H_i. By assumption (a), the differential operators L_i and L_i^* in terms of A's are the same for each $i = 0, 1$. The divergence theorem and assumption (b) imply that $D(L_i^*) = D(L_i)$ and $L_i = L_i^*$. This is the first assertion.

Because L_i is selfadjoint in H_i, nonreal complex numbers belong to the resolvent set of L_i. By Lemma 2.2(d) and proof of Theorem 2.5, the set of real $\lambda \notin S$ with $N(b(\cdot, \lambda)) \subset \{0\}$ is included in the resolvent set of L_i. It suffices to consider the set $R\backslash(S \cup C_\rho)$ that is absolutely continuous with respect to L_i in H_i. Study now, instead of problem (2.1)$_i$ and (2.2), the corresponding problem with homogeneous boundary condition:

$$(2.17)_i \qquad L_i v = \lambda v + f \quad \text{on} \quad \Omega,$$

$$(2.18) \qquad B(x)\, v(x) = b(x)\, v(x) \quad \text{on} \quad \partial\Omega,$$

with the assumption that every solution of (2.18) which is integrable almost everywhere on $\partial\Omega$ belongs to $\ker(B^*(x) - b(x)\, I)$. Therefore, because $g = 0$, the equivalent problem (2.5)$_i$ is reduced to the equation below, which is equivalent to (2.17)$_i$ and (2.18):

$$(2.19) \qquad (L_i - \lambda I)(\chi v) = \chi f \quad \text{on} \quad R^n,$$

where χ has value 1 on $\bar{\Omega}$ and 0 on the complement of $\bar{\Omega}$. By Theorem 2.5, system (2.19) has one and only one solution satisfying the radiation condition for each f in $H_{0,c}$, $c > 2N - 1$, which is dense in H_i. Hence $R\backslash(S \cup C_\rho)$ is a subset of the continuous spectrum of L_i. Moreover, since the solution of (2.19) is obtained through the contraction mapping E with kernel (2.14), which depends on λ through the elementary function $E(\cdot, \lambda)$, and since by Lemma 2.2(c), the constant C_E obtained from asymptotic property (2.10) for $E(\cdot, \lambda)$ is independent of λ (cf. Theorem 3.1), the solution is differential in real λ on each compact interval disjoint from $S \cup C_\rho$. Therefore, $R \backslash (S \cup C_\rho)$ is absolutely continuous, and the main theorem is proved.

3. **The elementary solution** $E(\cdot, \lambda)$ **of** $L_0 - \lambda I$. Under assumptions (a) and (c), property (2.8) and assertions (a) through (e) in Lemma 2.2 are proved. The fact that there is no need of assuming $N(b(\cdot; \lambda))$ to be convex and to have nonvanishing curvature is demonstrated in detail here.

Actually, after derivations of the property of eigenvalues for matrix $A_0(\tilde{x})\, T^{-1}(\lambda)$ in (2.7) and corresponding projection, the following results are shown for the system

$$L_0 u = \lambda u + f \quad \text{on} \quad R^N. \tag{3.1}$$

THEOREM 3.1. *Suppose assumptions* (a) *and* (c).

(i) *For every nonreal complex number* λ *or every real number* $\lambda \notin S$ *with* $N(b(\cdot, \lambda)) \subset \{0\}$ (*namely*, $\lambda \in C_\rho$) *and for* $f \in H_0(R^N)$, *system* (3.1) *has exactly one solution in* $H_0(R^N)$.

(ii) *For real* $\lambda \notin S \cup C_\rho$, *if* f *is an* $H_0(R^N)$*-function such that its Fourier transform* $f^\sim$ *vanishes on* $N(b(\cdot, \lambda))$, *then system* (3.1) *has one and only one solution in* $H_0(R^N)$. (*The set of such functions* f *is dense in* $H_0(R^N)$.)

(iii) *For real* $\lambda \notin S \cup C_\rho$, *if* f *is a distribution such that* $D^\alpha f^\sim$ *are integrable on* R^N *and vanish at infinity for each* α, $|\alpha| \leq 2$ *and* $f^\sim(y) \equiv 0$ *on* $N(b(\cdot; \lambda))$, *then there exists a unique incoming solution* u^+ (*outgoing solution* u^-) *to system* (3.1) *such that at infinity*,

$$\begin{aligned} & u^\pm(x; \lambda, z) \\ &\quad = (2\sqrt{-1})^{-1} \pi(-4z\pi\sqrt{-1}\,|x|)^{(1-N)/2} \sum_{j=1}^{[m-2]} [\pm r_j(\tilde{x}; \lambda)]^{(1+N)/2} \\ &\qquad \cdot \exp\{-2\sqrt{-1}\,|x|\, r_j(\tilde{x}; \lambda)\}\, T^{-1}(\lambda)\, P_j(\pm\tilde{x}; \lambda) \\ &\qquad \cdot f^\sim(-z\tilde{x} r_j(\tilde{x}; \lambda))\, H(\operatorname{sgn} \operatorname{Im} z) + O(|x|^{-(1+N)/2}) \end{aligned} \tag{3.2'}$$

$$= O(|x|^{(1-N)/2}) \quad but \quad \neq o(|x|^{(1-N)/2}) \qquad if \quad z = 1; \tag{3.2}$$

$$\{I - P_\pm(\tilde{x}; \lambda)\}\, T(\lambda)\, u^\pm(x; \lambda, z) = O(|x|^{-(1+N)/2}), \tag{3.3}$$

where $H(t)$ *is the Heaviside function and* $\operatorname{sgn} t = 1$ *if* $t \geq 0$ *and* $\operatorname{sgn} t = -1$ *if* $t < 0$. *Moreover, such functions* $u^{\pm}$ *satisfy the property for* E *in Lemma* 2.2(c).

(iv) *The constant* C_E *in Theorem* 2.5 *is given by*

$$C_E = (\pi/2)(2\pi)^{(1-N)/2} \operatorname{Max} \sum [r_j(\tilde{x};\ \lambda)]^{(1+N)/2} \cdot |T^{-1}(\lambda) P_j(\pm\tilde{x};\ \lambda) \cdot (1,\cdots,1)^t| ,$$

where the maximum is taken on real $\lambda \notin S \cup \{0\}$ *and* $\tilde{x} \in S^{N-1}$ *and the summation runs on* $j = 1,\cdots,[m/2]$.

It is clear that assertions in Lemma 2.2 are consequences of the theorem by taking $f(x) = \delta(x)$. Before the proof of the theorem, we analyze the regularity of $T(\lambda)$ in terms of λ and the eigenvalues of matrix $A_0(x)\, T^{-1}(\lambda)$ which is defined in (2.7).

LEMMA 3.2. *Suppose assumptions* (a) *and* (c).

(i) *The set* S *of values for* λ *such that* $T(\lambda)$ *is singular consists of a finite number of real points (for the case* $A_{0,N+1} = 0$, $S = \{0\}$.)

(ii) *For nonreal complex number* λ, *the matrix* $A_0(\tilde{x})\, T^{-1}(\lambda)$ *has no real eigenvalues.*

(iii) *For real* $\lambda \notin S$, *the eigenvalues* $r_j(\tilde{x};\ \lambda)$ *of* $A_0(\tilde{x})\, T^{-1}(\lambda)$ *preserve property* (2.8), *are real-valued homogeneous in* $\tilde{x}$ *of degree* 1, *and are analytic in* $(\tilde{x}, \lambda) \in S^{N-1} \times (R \backslash S)$.

Proof. By the same argument for a Hermitian matrix possessing only real eigenvalues, the positive definiteness of A_{00} and the Hermitian property of A_{00} and $A_{0,N+1}$ lead to assertion (i). Similarly, we have assertion (ii). Assume now that λ is real and not in the set S. Then, by the same argument of $T(\lambda)$, eigenvalues of matrix $A_0(y)\, T^{-1}(\lambda)$ are real-valued, whenever the eigenvalues exist. Denote by $b_j(y, s;\ \lambda)$ the irreducible factors of (y, s)-polynomial $\det\{A_0(y)\, T^{-1}(\lambda) - sI\}$ with parameter λ and by $b(y, s;\ \lambda)$ the product of these $b_j(y, s;\ \lambda)$. Then $b(y;\ \lambda) = b(y, 1;\ \lambda)$. Because $\det\{A_0(y)\, T^{-1}(\lambda) - sI\}$ is analytic in real $\lambda \notin S$, the employment of the argument by Wilcox [10, pp. 53–55] with $b_j(\cdot;\ \lambda)$ and $b(\cdot;\ \lambda)$

replacing Q_j and Q, respectively, implies the rest of the assertions. The proof of the lemma is complete.

Proof of Theorem 3.1. For complex λ such that $N(b(\cdot, \lambda))=\emptyset$, the matrix $A_0(y)-T(\lambda)$ is regular and $\{A_0(y)-T(\lambda)\}^{-1}=O(|y|^{-1})$ at infinity. Hence, we have assertion (i) except the case with $N(b(\cdot; \lambda))=\{0\}$. Assume in the rest of the argument that $\lambda \notin S$ is real and $N(b(\cdot; \lambda))\neq\emptyset$. Then matrix $A_0(\tilde{y})\,T^{-1}(\lambda)$, $\tilde{y}\in S^{N-1}$, has the following spectral representation:

$$\{A_0(\tilde{y})\,T^{-1}(\lambda)-zI\}^{-1}=\sum_{j=1}^{m} P_j(\tilde{y};\ \lambda\,[r_j(\tilde{y};\ \lambda)-z]^{-1}. \tag{3.4}$$

However, the projections $P_j(\tilde{y}; \lambda)$ in $C^{N'}$ cnto the eigenspaces relating the eigenvalues $r_j(\tilde{y}; \lambda)$ can be constructed by

$$P_j(\tilde{y}, \lambda)=-(2\pi\sqrt{-1})^{-1}\oint [A_0(\tilde{y})\,T^{-1}(\lambda)-zI]^{-1}dz \quad (z\in\Gamma_j), \tag{3.5}$$

where Γ_j are small circles with centers $r_j(\tilde{y}; \lambda)$. Hence $P_j(y; \lambda)$ are analytic in $(y; \lambda)\in(R^N\backslash\{0\})\times(C\backslash S)$, homogeneous of degree 0 in y and

$$P_j(-y;\ \lambda)=P_{m-j+1}(y;\ \lambda); \qquad P_j(-y;\ \lambda)=P_j(y;\ \lambda)$$

as $j=(m+1)/2$ if m is cdd.

If $N(b(\cdot; \lambda))=\{0\}$, then m is cdd; $r_j(y; \lambda)\neq 0$ for all $j\neq(m+1)/2$; and $r_j(y; \lambda)=0$ as $j=(m+1)/2$. Thus relation (3.4) gives

$$\begin{aligned}[A_0(y)&-zT(\lambda)]^{-1}\\ &=-T^{-1}(\lambda)\,P_{(m+1)/2}(\tilde{y};\ \lambda)\,z^{-1}\\ &\quad+\sum_{j=1}^{m}{}'\,T^{-1}(\lambda)\,P_j(\tilde{y};\ \lambda)\,[r_j(y;\ \lambda)-z]^{-1},\end{aligned} \tag{3.6}$$

where the summation runs on $j=1,\cdots, m$ with $j\neq(m+1)/2$; moreover,

$$T^{-1}(\lambda)\,P_{(m+1)/2}(\tilde{y};\ \lambda)=O(1)$$

and

$$T^{-1}(\lambda)\,P_j(\tilde{y};\ \lambda)\,[r_j(y;\ \lambda)-z]^{-1}=O(|y|^{-1})$$

at infinity for all complex numbers z. Therefore the smoothness on the right side of (3.6) gives $u\in H_0(R^N)$ whenever $f\in H_0(R^N)$ by letting $u^{\sim}(y)=[A_0(y)-T(\lambda)]^{-1}f^{\sim}(y)$. This completes the proof for assertion (i) of the theorem.

Assume that $N(b(\cdot;\lambda))$ consists of more than a point. The summation on the right side of (3.6) with $z=1$ and $j\neq(m+1)/2$ possesses singularity at $N(r_j(\cdot;\lambda)-1)$, which are the manifolds with radius $r_j(\tilde{y};\lambda)=1/r_j(\tilde{y};\lambda)$ at each direction $\tilde{y}\in S^{N-1}$. It means that (3.6) has another expression

$$\begin{aligned}&[A_0(y)-zT(\lambda)]^{-1}\\ (3.7)\qquad &\quad=-T^{-1}(\lambda)\,P_{(m+1)/2}(\tilde{y};\lambda)\,z^{-1}\\ &\qquad+\sum_{j=1}^{m}{}'\,T^{-1}(\lambda)\,P_j(\tilde{y};\lambda)\,[r-zr_j(\tilde{y};\lambda)]^{-1}\qquad(r=|y|).\end{aligned}$$

If $H_0(R^n)$-function f has Fourier transform $f^{\sim}(y)$ vanishing on $N(b(\cdot;\lambda))$, the right side and then the left side of the following expression is well-defined and belongs to $H_0(R^N)$:

$$\begin{aligned}&[A_0(r\tilde{y})-zT(\lambda)]^{-1}f^{\sim}(r\tilde{y})\\ (3.8)\qquad &\quad=-T^{-1}(\lambda)\,P_{(m+1)/2}(\tilde{y};\lambda)\,z^{-1}f^{\sim}(r\tilde{y})\\ &\qquad+\sum_{j=1}^{m}{}'\,r_j(\tilde{y};\lambda)\,T^{-1}(\lambda)\,P_j(\tilde{y};\lambda)\,[r-zr_j(\tilde{y};\lambda)]^{-1}f^{\sim}(r\tilde{y}).\end{aligned}$$

Therefore, let u be its inverse Fourier transform with $z=1$ and then u is an $H_0(R^N)$-function and a solution to system (3.1). That is the existence part of assertion (ii).

Assume, in addition to $N(b(\cdot;\lambda))$ consisting of more than a point, that $D^{\alpha}f^{\sim}$ are integrable on R^N and vanish at infinity for each α, $|\alpha|\leq 1$ and $f^{\sim}(y)\neq 0$ on $N(b(\cdot;\lambda))$. Let u be the inverse Fourier transform of $[A_0(r\tilde{y})-T(\lambda)]^{-1}f^{\sim}(r\tilde{y})$ which is given in (3.8) and has singularity, for each term under the summation, on the manifold $\{r_j(\tilde{y};\lambda)\tilde{y}\}$, $j\neq(m+1)/2$, if $z=1$. The integrations are in the sense of Cauchy principal value. Denote by u_j the inverse Fourier transform of the jth term on the right side of (3.8). If m is odd and $j=(m+1)/2$,

$$u_j(x;\ \lambda) = -2^{-1}(2\pi)^{-N}T^{-1}(\lambda)$$

$$(3.9)_{(m+1)/2} \qquad \cdot \int \exp\{-\sqrt{-1}\,x\cdot y\}\,P_j(\tilde{y};\ \lambda)\,f^{\sim}(y)\,dy \qquad (y\in R^N).$$

By the analyticity of $P_{(m+1)/2}$ and by conditions on $f^{\sim}(y)$, this representation gives $u_{(m+1)/2}\in H_0(R^N)$. Assume $j\neq(m+1)/2$ in the rest. Write u_j in the following form:

$$(3.9)_j \qquad u_j(x;\ \lambda,\ z) = v_j(x,\ \lambda,\ z) + w_j(x;\ \lambda,\ z)$$

$$\begin{aligned} w_j(y;\ \lambda,\ z) &\\ = 2^{-1}(2\pi)^{-N}\int &\exp\{-\sqrt{-1}\,x\cdot y\} \\ (3.10)_j \qquad &\cdot [r - zr_j(\tilde{y};\ \lambda)]^{-1}r_j(\tilde{y};\ \lambda)\,P_j(\tilde{y};\ \lambda) \\ &\cdot \{[f^{\sim}(y) - f^{\sim}(\tilde{y}r_j(\tilde{y};\ \lambda))(r_j(\tilde{y};\ \lambda)/|y|)^{N-1}] \\ &+ [1-h_j(|y|,\ \tilde{y})]f^{\sim}(\tilde{y}r_j(\tilde{y};\ \lambda))(r_j(\tilde{y};\ \lambda)/|y|)^{N-1}]\}\,dy \\ &\qquad (y\in R^N); \end{aligned}$$

$$(3.11)_j \quad v_j(x;\ \lambda,\ z) = (4\pi)^{-1}\int T^{-1}(\lambda)\,U_j(r,\ x,\ \lambda,\ z)\,dr \qquad (r\in R)$$

$$\begin{aligned} &U_j(r;\ x,\ \lambda,\ z) \\ (3.12)_j \qquad &= \int \exp\{-\sqrt{-1}\,rx\cdot\tilde{y}\}\,[r - zr_j(\tilde{y},\ \lambda)]^{-1}\,V_j(\tilde{y};\ r,\ \lambda)\,dy \\ &\qquad (\tilde{y}\in S^{N-1}) \end{aligned}$$

$$(3.13)_j \quad V_j(\tilde{y};\ r,\ \lambda) = (2\pi)^{1-N}[r_j(\tilde{y};\ \lambda)]^N P_j(\tilde{y};\ \lambda)\,h_j(r,\ \tilde{y})\,f^{\sim}(r\tilde{y}),$$

where h_j is a smooth function with $h_j(r,\tilde{y})=0$ for $|r-r_j(\tilde{y};\ \lambda)|>\varepsilon$ $(\varepsilon>0)$ and $h_j(r_j(\tilde{y};\ \lambda),\ \tilde{y})=1$. On w_j in $(3.10)_j$ for each j and on v_j in $(3.11)_j$ for each j that $r_j(\tilde{y},\lambda)<0$, the integrand is regular (in particular, on $r=r_j(\tilde{y};\ \lambda)$) and vanishes faster at infinity than $f^{\sim}$. Hence $w_j\in H_0(R^N)$ for all j and $v_j\in H_0(R^N)$ for j such that $r_j(\tilde{y},\ \lambda)<0$.

Without loss of generality, for j with $r_j(y,\ \lambda)>0$ assume that $\tilde{x}=(0,\cdots,0,1)$ and then $\tilde{x}\cdot\tilde{y}=\tilde{y}_N$. By Morse [8, p. 179], there exists the local coordinate $s'=(s_1,\cdots,s_{N-1})$ at $\pm\tilde{x}$ such that $\tilde{y}_N=\pm\{1-\sum_{k=1}^{N-1}s_k^2\}$. This region can be chosen so small that the

Jacobian $J(s')$ of the transformation $\tilde{y}\to s'$ has values greater than 1/2 and 1 at $\pm\tilde{x}$. Let $\phi_j(s)$ be a smooth function with values between 0 and 1, supported on this region, and 1 at $\pm\tilde{x}$; let $\psi_j(\tilde{y})=\phi_j(s'(\tilde{y}))$; and let

$$W_j(s';\ r,\ \lambda)=V_j(\tilde{y}(s');\ r,\ \lambda)\,\phi_j(s')J(s')\,. \tag{3.14$_j$}$$

Then $(3.12)_j$ and $(3.13)_j$ imply

$$\begin{aligned} &U_j(r;\ x,\ \lambda,\ z)\\ &\quad\equiv X_j(r;\ x,\ \lambda,\ z)+Y_j(r;\ x,\ \lambda,\ z)\\ &\quad\equiv \exp\{\mp\sqrt{-1}\,r\,|x|\}\int\exp\Big\{\pm\sqrt{-1}\sum_{k=1}^{N-1}|x|\,rs_k^2\Big\}\\ &\qquad\cdot W_j(s';\ r,\ \lambda)\,[r-zr_j(\tilde{y}(s');\ \lambda)]^{-1}\,ds'\\ &\qquad+(2\pi)^{1-N}\int\exp\{-\sqrt{-1}\,rx\cdot\tilde{y}\}\,r_j(\tilde{y};\ \lambda)\,P_j(\tilde{y};\ \lambda)\\ &\qquad\cdot[r-zr_j(\tilde{y};\ \lambda)]^{-1}f^{\sim}(\tilde{y}r_j(\tilde{y};\ \lambda))\,[1-\psi_j(\tilde{y})]\,d\tilde{y}\,. \end{aligned} \tag{3.15$_j$}$$

The upper sign is for $\tilde{y}(0)=\tilde{x}$ and $j<(m+1)/2$ and the lower sign is for $\tilde{y}(0)=-\tilde{x}$ and $j>(m+1)/2$. By Taylor's formula with the remainder, with $D_k=\partial/\partial s_k$,

$$\begin{aligned} &W_j(s';\ r,\ \lambda)\,[r-zr_j(\tilde{y}(s');\ \lambda)]^{-1}\\ &\quad=V_j(\tilde{x};\ r,\ \lambda)\,[r-zr_j(\tilde{x};\ \lambda)]^{-1}\\ &\qquad+\sum_{k=1}^{N-1}s_k\,Z_1(r;\ \tilde{x},\ k)+\sum s'^{\alpha}Z_2(r;\ \tilde{x},\ \alpha)\\ &\qquad\qquad\qquad\qquad(|\alpha|=2,\ 0\le\alpha_k\le 1)\\ &\qquad+\sum_{k=1}^{N-1}s_k^2 Z_3(r;\ \tilde{x},\ k)+s'^{\beta}Z_4(r;\ \tilde{x},\ \beta)\qquad(|\beta|=3)\ ; \end{aligned} \tag{3.16$_j$}$$

$$\begin{aligned} &Z_3(r;\ \tilde{x},\ k)\\ &\quad=[r-zr_j(\tilde{x};\ \lambda)]^{-1}[\operatorname{grad}_{\tilde{x}}V_j(\tilde{x};\ r,\ \lambda)\cdot D_k^2\tilde{y}(0)\\ &\qquad+\operatorname{grad}_{\tilde{x}}\{\operatorname{grad}_{\tilde{x}}V_j(\tilde{x},\ r,\ \lambda)\cdot D_k\tilde{y}(0)\}\cdot D_k\tilde{y}(0)]\\ &\qquad+z[r-zr_j(\tilde{x};\ \lambda)]^{-2}\\ &\qquad\cdot[V_j(\tilde{x};\ r,\ \lambda)\operatorname{grad}_{\tilde{x}}\{\operatorname{grad}_{\tilde{x}}r_j(\tilde{x};\ \lambda)\cdot D_k y(0)\}\cdot D_k y(0)\\ &\qquad+V_j(\tilde{x};\ r,\ \lambda)\operatorname{grad}_{\tilde{x}}r_j(\tilde{x};\ \lambda)\cdot D_k^2\tilde{y}(0)\\ &\qquad+2\{\operatorname{grad}_{\tilde{x}}r_j(\tilde{x};\ \lambda)\cdot D_k\tilde{y}(0)\}\\ &\qquad\cdot\{\operatorname{grad}_{\tilde{x}}V_j(\tilde{x};\ r,\ \lambda)\cdot D_k\tilde{y}(0)\}]\\ &\qquad+2z^2[r-zr_j(\tilde{x};\ \lambda)]^{-3}V_j(\tilde{x};\ r,\ \lambda)\\ &\qquad\cdot[\operatorname{grad}_{\tilde{x}}r_j(\tilde{x};\ \lambda)\cdot D_k\tilde{y}(0)]^2\,. \end{aligned} \tag{3.16$'_j$}$$

Relations $(3.15)_j$, $(3.16)_j$ and $(3.16)'_j$ give

$$
(3.17)_j \qquad
\begin{aligned}
&X_j(r, x, \lambda, z)\\
&\quad = [(\pm\sqrt{-1}\,|x|\,r)/\pi]^{(1-N)/2} \exp\{\mp\sqrt{-1}r\,|x|\}\\
&\qquad \cdot \Big\{V_j(\pm\tilde{x};\ r,\ \lambda)[r - zr_j(\pm\tilde{x};\ \lambda)]^{-1}\\
&\qquad + [(\pm\sqrt{-1}\,|x|\,r)/\pi]^{1/2}[(\pm\sqrt{-1}\,|x|\,r)^{-3/2}(2/\sqrt{\pi})]\\
&\qquad \cdot \sum_{k=1}^{N-1} Z_3(r;\ \pm\tilde{x},\ k)\\
&\qquad + O(|x|^{-(3+N)/2}r^{-2}[r - zr_j(\pm\tilde{x};\ \lambda)]^{-1})\Big\}
\end{aligned}
$$

at infinity in $|x|$ and r, where we use the results that integrations on the real line for $\exp\{-s^2\}$, $s\exp\{-s^2\}$, and $s^2\exp\{-s^2\}$ are $\pi^{1/2}$, 0, and $2\pi^{-1/2}$, respectively, and where the upper sign is used for $j<(m+1)/2$ and the lower sign is used for $j>(m+1)/2$. In order to determine the asymptotic behavior of

$$
(3.18)_j \quad V_{j0}(x;\ \lambda,\ z) = (4\pi)^{-1}\int T^{-1}(\lambda)\,X_j(r;\ x,\ \lambda,\ z)\,dr \qquad (r\in R),
$$

defined through $(3.11)_j$, $(3.15)_j$, and $(3.17)_j$, recall the well-known formulas (cf. Gel'fand-Shilov [3, p. 360]) for Fourier transforms as below:

$$
(3.19) \qquad
\begin{aligned}
&\int \exp\{\pm\sqrt{-1}rs\}\,[r - (a+\sqrt{-1}b)]^{-1}\,dr \qquad (r\in R)\\
&\quad = -\sqrt{-1}\,\pi(\operatorname{sgn} b)\,H(\mp s\operatorname{sgn} b)\\
&\qquad\qquad \cdot \exp\{\pm\sqrt{-1}[a+\sqrt{-1}b]s\}\,;
\end{aligned}
$$

$$
(3.20) \qquad
\begin{aligned}
&\int \exp\{\pm\sqrt{-1}rs\}\,[r - (a+\sqrt{-1}b]^{-k}\,dr \qquad (r\in R,\ k>1)\\
&\quad = \pi(\operatorname{sgn} b)\,|s|\,(\pm\sqrt{-1}s)^{k-2}[(k-1)!]^{-1}\\
&\qquad\qquad \cdot \exp\{\pm\sqrt{-1}s(a+\sqrt{-1}b)\}\,.
\end{aligned}
$$

Let $f_{0j}(\cdot;\ \tilde{x})$ be the Fourier transform in $r\in R$ of the function $r^{(1-N)/2}[r_j(\tilde{x};\ \lambda)]^N P_j(\tilde{x};\ \lambda)\,h_j(r\tilde{x})\,f^{\sim}(r\tilde{x})$; $f_{ij}(\cdot;\ \tilde{x})$ $(i=1, 2, 3)$ be those of products of $r^{-(1+N)/2}$ with coefficients of $[r - zr_j(\tilde{x};\ \lambda)]^{-1}$, $z[r - zr_j(\tilde{x};\ \lambda)]^{-2}$, and $2z^2[r - zr_j(\tilde{x};\ \lambda)]^{-3}$, respectively. Using the upper sign for $0<j<(m+1)/2$ and the lower sign for $m>j>(m+1)/2$ concludes the estimate at infinity in x for $(3.18)_j$

$$
\begin{aligned}
v_{j0}(x;\ \lambda,\ z)& \\
= \mp 4^{-1}\sqrt{-1}&(\pm 4m\sqrt{-1}\ |x|)^{(1-N)/2}\ T(\lambda) \\
&\cdot \Big\{\int_{-\infty}^{|x|} \exp\{\mp\sqrt{-1}\, z r_j(\pm\tilde{x};\ \lambda)(|x|-t)\} \\
&\cdot f_{0j}(t,\ \pm\tilde{x})\, dt \cdot [\pm H(\pm \operatorname{Im} z r_j(\pm\tilde{x};\ \lambda))] \\
&+ (-1)^{(N-1)/2}\int_{|x|}^{\infty} \exp\{\pm\sqrt{-1}\, z r_j(\pm\tilde{x};\ \lambda)(|x|-t)\} \\
&\cdot f_{0j}(t,\ \pm\tilde{x})\, dt \cdot [\mp H(\mp \operatorname{Im} z r_j(\pm\tilde{x};\ \lambda))] \\
&+ 2(\pm\sqrt{-1}\,\pi\ |x|)^{-1} \\
&\cdot \int_{-\infty}^{|x|} \exp\{\mp\sqrt{-1}\, z r_j(\pm\tilde{x};\ \lambda)(|x|-t)\} \\
&\cdot f_{1j}(t,\ \pm\tilde{x})\, dt\, [\pm H(\pm \operatorname{Im} z r_j(\pm\tilde{x};\ \lambda))] \\
&+ 2(\mp\sqrt{-1}\,\pi\ |x|)^{-1}\,(-1)^{(N-1)/2} \\
&\cdot \int_{|x|}^{\infty} \exp\{\pm\sqrt{-1}\, z r_j(\pm\tilde{x};\ \lambda)(|x|-t)\} \\
&\cdot f_{1j}(t,\ \pm\tilde{x})\, dt \cdot [\mp H(\mp \operatorname{Im} z r_j(\pm\tilde{x};\ \lambda)] \\
&+ 2(\pm\pi\ |x|)^{-1} z \int_{-\infty}^{\infty} \exp\{\mp\sqrt{-1}\, z r_j(\pm\tilde{x};\ \lambda)\, t\} \\
&\cdot |t|\ f_{2j}(|x|-t,\ \pm\tilde{x})\, dt \\
&\pm 2z^2(\pm\sqrt{-1}\,\pi\,|x|)^{-1}\int_{-\infty}^{\infty} \exp\{\mp\sqrt{-1}\, z r_j(\pm\tilde{x};\ \lambda)\, t\} \\
&\cdot |t|\ t f_{3j}(|x|-t,\ \pm\tilde{x})\, dt \\
&\pm 2z^2(\pm\sqrt{-1}\,\pi\,|x|)^{-1}\int_{-\infty}^{\infty} \exp\{\mp\sqrt{-1}\, z r_j(\pm\tilde{x};\ \lambda)\, t\} \\
&\cdot |t|\ t f_{3j}(|x|-t,\ \pm\tilde{x})\, dt\Big\} \\
&+ O(|x|^{-(3+N)/2})\,.
\end{aligned}
\tag{3.21$_j$}
$$

Denote by $v_{j1}(x;\ \lambda,\ z)$ the function defined in $(3.18)_j$ with X_j replaced by Y_j. Applying the same argument of v_{j0} to v_{j1} and taking into account $1-\psi_j=0$ on a small neighborhood of $\pm\tilde{x}$ lead to the fact that

$$
v_{j1}(x;\ \lambda,\ z) = O(|x|^{-(3+N)/2}) \tag{3.22$_j$}
$$

at infinity. Because $f_{ij}(t, \tilde{x})$ $(i=1, 2, 3)$ decay at infinity in t with high index, the asymptotic property of $w_j(\cdot;\ \lambda,\ z)$ and relations $(3.9)_j$ and $(3.21)_j$ imply

$$
\begin{aligned}
&u_j(x;\ \lambda,\ z) \\
&\quad = \mp 2^{-1}\pi\sqrt{-1}\,(-4z\pi\sqrt{-1}\,|x|)^{(1-N)/2}[r_j(\pm\tilde{x};\ \lambda)]^{(1+N)/2} \\
&\qquad \cdot \exp\{\mp\sqrt{-1}\,zr_j(\pm\tilde{x};\ \lambda)\,|x|\}\,T^{-1}(\lambda)\,P_j(\pm\tilde{x};\ \lambda) \\
&\qquad \cdot f^{\sim}(\mp z\tilde{x}r_j(\pm\tilde{x};\ \lambda))\,[\pm H(\pm\operatorname{Im} zr_j(\pm\tilde{x};\ \lambda))] \\
&\qquad \mp 4^{-1}\sqrt{-1}\,(\pm 4\pi\sqrt{-1}\,|x|)^{(1-N)/2}\,T^{-1}(\lambda) \\
&\qquad \cdot \left\{\int_{|x|}^{\infty}\exp\{\mp\sqrt{-1}\,zr_j(\pm\tilde{x};\ t)(|x|-t)\}\right. \\
&\qquad \cdot f_{0j}(t,\ \pm\tilde{x})\,dt\,[\pm H(\pm\operatorname{Im} zr_j(\pm\tilde{x};\ \lambda))] \\
(3.23)_j \qquad &\qquad + (-1)^{(N-1)/2}\int_{|x|}^{\infty}\exp\{\pm\sqrt{-1}\,zr_j(\pm\tilde{x};\ t)(|x|-t)\} \\
&\qquad \cdot f_{0j}(t,\ \pm\tilde{x})\,dt\,[\mp H(\mp\operatorname{Im} zr_j(\pm\tilde{x};\ \lambda))] \\
&\qquad \mp 4\sqrt{-1}\,|x|^{-1}\exp\{\mp\sqrt{-1}\,zr_j(\pm\tilde{x};\ \lambda)\,|x|\} \\
&\qquad \cdot f_{1j}^{\sim}(\mp zr_j(\pm\tilde{x};\ \lambda),\ \pm\tilde{x})\,[\pm H(\pm\operatorname{Im} zr_j(\pm\tilde{x};\ \lambda)] \\
&\qquad \pm 2z(\pi\,|x|)^{-1}\int_{-\infty}^{\infty}\exp\{\mp\sqrt{-1}\,zr_j(\pm\tilde{x};\ \lambda)\,t\} \\
&\qquad \cdot |t|\,f_{2j}(|x|-t,\ \pm\tilde{x})\,dt \\
&\qquad + 2z^2(\sqrt{-1}\,\pi\,|x|)^{-1}\int_{-\infty}^{\infty}\exp\{\mp\sqrt{-1}\,zr_j(\pm\tilde{x};\ \lambda)\,t\} \\
&\qquad \left.\cdot |t|\,tf_{3j}(|x|-t,\ \pm\tilde{x})\,dt\right\} \\
&\qquad + O(|x|^{-(3+N)/2})
\end{aligned}
$$

at infinity, where the upper sign (lower sign) is used for $0<j<(m+1)/2$ $((m+1)/2<j<m)$. Taking into account $r_j(\pm\tilde{x};\ t)>0$ if $j<(m+1)/2$ and $r_j(\pm\tilde{x},\ t)<0$ if $j>(m+1)/2$ yields assertion (3.2)$'$, which gives (3.2) by considering the coefficient of $|x|^{(1-N)/2}$ as an almost periodic function. Relation (3.3) follows from expression (3.2)$'$ and the properties of projections $P_j(\pm\tilde{x};\ \lambda)$.

The constant C_E is finite and independent of λ since $T(\lambda)=O(|\lambda|)$, $r_j(\tilde{x};\ \lambda)=O(|\lambda|^{-1})$, and $P_j(\tilde{x};\ \lambda)=O(1)$ for large $|\lambda|$ uniformly in $\tilde{x}$ and since these three functions are smooth.

Thus the assertions of Theorem 3.1 are proved except the parts involving uniqueness. For uniqueness, assume u is a solution of $L_0 u=\lambda u$ on R^N. Then each entry of its Fourier transform $u^{\sim}$ has support contained in $N(b(\cdot,\ \lambda))$. If $N(b(\cdot,\ \lambda))=\emptyset$, including the case λ is nonreal, then $u^{\sim}=0$ and so $u=0$. If $N(b(\cdot,\ \lambda))=\{0\}$, then $u^{\sim}$ is a linear combination of derivatives of the Dirac-delta function and u is a polynomial which does not belong to $H_0(R^N)$ except $u=0$. Therefore, the assertions about uniqueness in the whole theorem are reduced to the case that $N(b(\cdot; \lambda))$ includes more than one point. By assumption (d), the partition of unity relating to $N(b(\cdot; \lambda))$ implies that the support of $u^{\sim}$ can be assumed to be contained in an $(N-1)$-dimensional manifold $M=\{\tilde{y}r_1(\tilde{y}, \lambda)\}$, say. Since the manifold is compact, $u^{\sim}$ can be a distribution of finite order only. Because other cases can be proved similarly, it suffices to consider that $u^{\sim}$ is smooth and then L_2-integrable over M. We claim that if $u\neq 0$, then $u\notin H_{0,-1}(R^N)$ but $u\in H_{0,-1-\varepsilon}(R^N)$ with any $\varepsilon>0$. (The following proof is essentially given by S. Agmon in 1971 Summer Institute on Partial Differential Equations at Berkeley, California.) Without loss of generality, assume that M has coordinates $y'=(y_1,\cdots, y_{N-1})$ on which the Jacobian J of the transformation is smooth and regular. Denote by $v(y')$ the product $u^{\sim}J$ in the coordinates y'. Then $v\in H_0(R^{N-1})$ and

$$\int |u(x', x_N)|^2\, dx' = \int |v(y')|^2\, dy' \qquad (x', y' \in R^{N-1}),$$

which is a nonzero constant. Hence we have the claim.

On the other hand, we claim that if $u^{\pm}$ is an incoming $(+)$ (or outgoing $(-)$) solution of $(L_0-\lambda I)u^{\pm}=0$ on R^N, then $u^{\pm}$ belongs to $H_0(R^N)$. Indeed, the proof can be seen from Mochizuki [6, pp. 238–239], which is recalled as follows. Since the argument for outgoing solutions is the same, it suffices to show the assertion for incoming solutions. Put G_r as the set of $x\in R^N$ such that $|x|<r$. Then, with $v=T^{-1}(\lambda)u$ the divergence theorem gives

$$\begin{aligned} 0 &= \int \Big\{\sum A_{0j}\, T^{-1}(\lambda)\, D_j v(x)\cdot v^*(x) - v(x) \\ &\qquad \cdot \Big[\sum A_{0j}\, T^{-1}(\lambda)\, Dv(x)\Big]^*\Big\}\, dx \qquad (x\in G_r) \\ &= \sqrt{-1}\int \sum_{j=1}^{N} A_{0j}\, T^{-1}(\lambda)\, \tilde{x}_j\, v(x)\cdot v^*(x)\, dS(x) \qquad (|x|=r). \end{aligned}$$

Because $\sum_{k=1}^{N} A_{0j}\, T^{-1}(\lambda)\, \tilde{x}_k P_j(\pm\tilde{x}; \lambda) = \pm[r_j(\tilde{x}; \lambda)]^{-1} P_j(\pm\tilde{x}; \lambda)$, $0<j<(m+1)/2$, and $=0$ as $j=(m+1)/2$ if m is odd, and because $P_+(\tilde{x}; \lambda)\oplus P_-(\tilde{x}; \lambda) = I\ominus P_j(\tilde{x}; \lambda)$ $(j=(m+1)/2)$, it follows that

$$0 = \sum_{j=1}^{[m/2]} \int [r_j(\tilde{x};\ \lambda)]^{-1} P_j(\tilde{x};\ \lambda)\, v(x) \cdot v^*(x)\, dS(x)$$

$$- \sum_{j=1}^{[m/2]} \int [r_j(\tilde{x};\ \lambda)]^{-1} P_j(-\tilde{x};\ \lambda)\, v(x) \cdot v^*(x)\, dS(x) \quad (|x| = r)$$

$$\equiv J_1 - J_2\, ;$$

$$J_1 \geq \int [r_1(\tilde{x};\ \lambda)]^{-1} P_+(\tilde{x};\ \lambda)\, v(x) \cdot v^*(x)\, dS(x)$$

$$\geq \alpha \int |P_+(\tilde{x};\ \lambda)\, v(x)|^2\, dS(x) \qquad (|x| = r)\, ,$$

$$J_2 \leq \int [r_j(\tilde{x};\ \lambda)]^{-1} P_-(\tilde{x};\ \lambda)\, v(x) \cdot v^*(x)\, dS(x)$$

$$\leq \beta \int |[I - P_+(\tilde{x};\ \lambda)]\, v(x)|^2\, dS(x)$$

$$= O(|x|^{-1-\varepsilon}) \quad \text{at infinity} \qquad (\varepsilon > 0)\, ,$$

where $\alpha = [\max r_1(\tilde{x};\ \lambda)]^{-1} \neq 0$, $\beta = [\min r_j(\tilde{x};\ \lambda)]^{-1} \neq 0$, $j = [m/2]$. Hence, by $J_1 = J_2$,

$$\int |v(x)|^2\, dS(x)$$

$$= \int |P_+(\tilde{x};\ \lambda)\, v(x)|^2 + |[I - P_+(\tilde{x};\ \lambda)]\, v(x)|^2\, dS(x) \quad (|x| = r)$$

$$\leq \text{const} \cdot J_1 + O(r^{-1-\varepsilon}) = O(r^{-1-\varepsilon}) \quad \text{at infinity} \qquad (\varepsilon > 0)\, .$$

Hence the second assertion is concluded by noting that $T^{-1}(\lambda)$ is invertible for $\lambda \notin S$.

The combination of the previous two claims leads to the fact that all incoming $(+)$ (outgoing $(-)$) solutions to $(L_0 - \lambda I)\, u = 0$ on R^N, with $N(b(\cdot, \lambda))$ consisting of more than one point, vanish identically. This furnishes the uniqueness, and the proof of Theorem 3.1 is complete.

4. **Appendix.** The Dirac system and then a system form of the wave equation in R^3 are studied to illustrate the boundary operator and the radiation condition introduced in the previous sections.

The Dirac system: $D_t\, u = L_0\, u \equiv \sum_{j=1}^{3} A_{0j}\, D_j\, u + A_{04}\, u$, where $D_t = -\sqrt{-1}\, \partial/\partial t$, $D_j = -\sqrt{-1}\, \partial/\partial x_j$, and the four 4×4 matrices are

$$
(4.1)\qquad
\begin{aligned}
&A_{01} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \qquad
A_{02} = \sqrt{-1}\begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \\
&A_{03} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}, \qquad
A_{04} = -a\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}
\end{aligned}
$$

with given constant a. Before specifying interesting cases, impose a general boundary operator as below:

$$
(4.2)\qquad B = \begin{bmatrix} B_{11}(x) & B_{12}(x) & B_{13}(x) & B_{14}(x) \\ \overline{B_{12}(x)} & B_{22}(x) & B_{23}(x) & B_{24}(x) \\ \alpha & \beta & B_{33}(x) & B_{34}(x) \\ \gamma & \delta & \overline{B_{34}(x)} & B_{44}(x) \end{bmatrix}
$$

where $\alpha = \overline{B_{13}(x)} - \sqrt{-1}\,n_3(x)$, $\beta = \overline{B_{23}(x)} - n_2(x) - \sqrt{-1}\,n_1(x)$, $\gamma = \overline{B_{14}(x)} + n_2(x) - \sqrt{-1}\,n_1(x)$, $\delta = \overline{B_{24}(x)} + \sqrt{-1}\,n_3(x)$ and where $n(x) = (n_1(x), n_2(x), n_3(x))$ is the unit normal of $\partial\Omega$ at x and $B_{ij}(x)$ $(1 \leq i, j \leq 3)$ are complex-valued functions with $B_{ii}(x)$ $(i = 1, 2, 3)$ being real-valued. It is easy to give an example such that the condition in the main theorem for the boundary operator does not hold:

$$
(4.3)\qquad \ker(b(x)I - B(x)) \subset \ker(b(x)I - B^*(x)) \quad \text{for almost all } x \in \partial\Omega,
$$

where $b(x)$ is a real-valued function on $\partial\Omega$; say, $B_{11}(x) \equiv B_{22}(x) \equiv \alpha_1$, $B_{33}(x) \equiv B_{44}(x) \equiv \alpha_2$ are real constants; $B_{12}(x) \equiv \beta_1$ and $B_{34}(x) \equiv \beta_2$ are complex constants; $B_{13}(x)$, $B_{14}(x)$, $B_{23}(x)$, and $B_{24}(x)$ vanish; and $b(x)$ is $\alpha_1 \pm \beta_1$ (or $\alpha_2 \pm \beta_2$) with $\beta_2^2 \neq (\alpha_2 - \alpha_1 \mp \beta_1)^2$. Then vectors $v = (v_1, v_2, v_3, v_4)$ with $v_3 = v_4 = 0$ and $\beta_1[v_1 \mp v_2] = 0$ are elements of $\ker[b(x)I - B(x)]$; vectors v with $v_1 = v_2 = 0$ and $\beta_2[v_3 \mp v_4] = 0$ form the space $\ker[b(x)I - B^*(x)]$. Both kernels are nontrivial and intersect only at $v = 0$.

Next let $B_{13} = (2\sqrt{-1})^{-1} n_3(x)$, $B_{14}(x) = -n_2(x)/2 + (2\sqrt{-1})^{-1} n_1(x)$,

$B_{23}(x) = n_2(x)/2 + (2\sqrt{-1})^{-1} n_1(x)$, $B_{24}(x) = \sqrt{-1}\, n_3(x)/2$, and let all other $B_{ij}(x)$'s vanish. The eigenvalues are $\pm\sqrt{-1}/2$ of multiplicity 2. Then choose $b(x)$ to be any complex constant not equal to $\pm\sqrt{-1}/2$. For any boundary operator $B(x)$, adding any Hermitian matrix to it will satisfy the relation (3) between the boundary operator and the coefficient matrices. In addition to this kind of non-uniqueness for boundary operators, there are other types which differ from the previous one by a non-Hermitian matrix. For example, choose $B_{ii}(x)$ $(i = 1, 2, 3, 4)$ to be real constants C_i such that $\sum_{i=1}^{4} C_i = 0$ and sum of all triple-products of distinct elements of them vanishes; $B_{12} = B_{34} = 0$; $B_{13}(x) = (2\sqrt{-1})^{-1} n_3(x)$; $B_{14}(x) = -n_2(x)$; $B_{23}(x) = n_2(x)$; and $B_{24}(x) = \sqrt{-1}\, n_3/2$. The matrix has no real constant eigenvalues. Set $b(x)$ to be any real constant. These examples of boundary operators include all three usual types of boundary operators in physics: Dirichlet, Neumann, and Robin. They deserve the names through the usual wave equation rewritten in the system form which will be discussed later.

For the free space problem in the last section $S = \{\pm a\}$; the resolvent set of L_0 is $\{C\backslash R\} \cup \{\lambda : -a < \lambda < a\}$; and the continuous spectrum consists of real λ with $|\lambda| > a$. Actually, because the determinant of $\sum A_{0j}\, y_j + A_{04} \mp aI$ is just $|y|^4$, the values $\lambda = \pm a$ belong to continuous spectrum of L_0. For radiation conditions it is clear that $r_j(y;\lambda) = -r_2(y;\lambda) = \sqrt{\lambda^2 - a^2}/|y|$ for real $\lambda : |\lambda| > a$. Let $d = [(\lambda - a)/(\lambda + a)]^{+1/2}$. Then $P_1(\tilde{y};\lambda) = I - P_2(\tilde{y};\lambda)$ where

$$(4.4) \qquad P_1(\tilde{y};\lambda) = \frac{1}{2}\begin{bmatrix} 1 & 0 & \tilde{y}_3 d & \alpha \\ 0 & 1 & \beta & \tilde{y}_3 d \\ \tilde{y}_3/d & \gamma & 1 & 0 \\ \delta & -y_3/d & 0 & 1 \end{bmatrix},$$

where $\alpha = -(\tilde{y}_1 - \sqrt{-1}\,\tilde{y}_2)\, d$, $\beta = (\tilde{y}_1 + \sqrt{-1}\,\tilde{y}_2)\, d$, $\gamma = (\tilde{y}_1 - \sqrt{-1}\,\tilde{y}_2)/d$, and $\delta = -(\tilde{y}_1 + \sqrt{-1}\,\tilde{y}_2)/d$.

The wave equation in system: $D_t u = L_0 u = \sum_{j=1}^{3} A_{0j} D_j u + A_{04} u$, with $u^t = (D_t w, D_1 w, D_2 w, D_3 w, aw)$ and

$$(4.5)\qquad \sum_{j=1}^{3} A_{0j} D_j + A_{04} = \begin{bmatrix} 0 & D_1 & D_2 & D_3 & 0 \\ D_1 & 0 & 0 & 0 & 0 \\ D_2 & 0 & 0 & 0 & 0 \\ D_3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} + a \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix},$$

where w satisfies the Klein-Gordon equation $\partial^2 w/\partial t^2 = \Delta w - a^2 w$. This is just the usual wave equation if $a = 0$. The reduced system of (4.5) is then $L_0 v = \lambda v$ with $v = (\lambda\hat{w}, D_1\hat{w}, D_2\hat{w}, D_3\hat{w}, a\hat{w})$ and $\hat{w}$ fulfilling the equation $\Delta\hat{w} = (a^2 - \lambda^2)\hat{w}$. Relation (3) indicates that $B_{j1}(x) = \overline{B_{1j}(x)} - \sqrt{-1}\, n_{j-1}(x)$ $(j=2, 3, 4)$ and $B_{ij}(x) = \overline{B_{ji}(x)}$ otherwise. Particularly, diagonal entries $B_{ii}(x)$ are real. The following special choice satisfies condition (4.3) and is nontrivial: Let $B_{1j}(x) = (2\sqrt{-1})^{-1} n_{j-1}(x)$ $(j = 2, 3, 4)$; $B_{ii}(x) \equiv b(x)$ $(i = 1, 2, 3, 4, 5)$; $B_{ij}(x) \equiv 0$ otherwise. Then the kernel of $b(x)I - B(x)$ is equal to that of $b(x)I - B^*(x)$, and consists of $v(x)$ such that $v_1(x) \equiv 0$, $(v_2(x), v_3(x), v_4(x))$ is orthogonal to $n(x)$, and $v_5(x)$ has no further restriction. The kernel is of dimension 3.

The following is the study of the boundary conditions for the system relating to the Dirichlet, Neumann, Robin conditions, respectively, for the Klein-Gordon equation. The Dirichlet boundary condition $w = 0$ on $\partial\Omega$ implies $D_t w = 0$ and $\hat{w} = 0$ on $\partial\Omega$. To fit relation (3), the relating adjoint boundary condition is of Neumann type, which is seen from the example indicated below, $b(x) \equiv 0$,

$$B = \sqrt{-1} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ n_1(x) & 0 & 0 & 0 & 0 \\ n_2(x) & 0 & 0 & 0 & 0 \\ n_3(x) & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix};$$

(4.6)

$$B^* = \sqrt{-1} \begin{bmatrix} 0 & n_1(x) & n_2(x) & n_3(x) & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

This does not satisfy condition (4.3). Also, if the adjoint boundary operator is interchanged with the boundary operator, one case of the Neumann boundary condition is obtained which does not satisfy condition (4.3). However, in view of the fact that $\boldsymbol{n}(x)\cdot Dw(x,t)=0$ on $\partial\Omega$ for any $t>0$ implies $\boldsymbol{n}(x)\cdot D_t Dw(x,t)=0$ for any $t>0$ or $\boldsymbol{n}(x)\cdot D\tilde{w}(x,\lambda)=0$ for $\lambda\neq 0$ on $\partial\Omega$, we have the following boundary operator for the system meeting condition (4.3):

$$(4.7)\qquad B=-\sqrt{-1}\begin{bmatrix} \sqrt{-1}\,b(x) & \frac{n_1(x)}{2} & \frac{n_2(x)}{2} & \frac{n_3(x)}{2} & 0 \\ \frac{n_1(x)}{2} & \sqrt{-1}\,b(x) & 0 & 0 & \frac{\lambda n_1(x)}{2a} \\ \frac{n_2(x)}{2} & 0 & \sqrt{-1}\,b(x) & 0 & \frac{\lambda n_2(x)}{2a} \\ \frac{n_3(x)}{2} & 0 & 0 & \sqrt{-1}\,b(x) & \frac{\lambda n_3(x)}{2a} \\ 0 & \frac{-\bar{\lambda} n_1(x)}{2\bar{a}} & \frac{-\bar{\lambda} n_2(x)}{2\bar{a}} & \frac{-\bar{\lambda} n_3(x)}{2\bar{a}} & \sqrt{-1}\,b(x) \end{bmatrix}.$$

For the Robin boundary condition $\sqrt{-1}\,\boldsymbol{n}(x)\cdot Dw(x,t)=\beta w(x,t)$ on $\partial\Omega$ for any $t>0$, the similar boundary operator can be chosen as below:

$$(4.8)\qquad B=-\sqrt{-1}\begin{bmatrix} \sqrt{-1}\,b(x) & \frac{n_1(x)}{2} & \frac{n_2(x)}{2} & \frac{n_3(x)}{2} & \frac{\sqrt{-1}\,\beta}{2a} \\ \frac{n_1(x)}{2} & \sqrt{-1}\,b(x) & 0 & 0 & \frac{\lambda n_1(x)}{2a} \\ \frac{n_2(x)}{2} & 0 & \sqrt{-1}\,b(x) & 0 & \frac{\lambda n_2(x)}{2a} \\ \frac{n_3(x)}{2} & 0 & 0 & \sqrt{-1}\,b(x) & \frac{\lambda n_3(x)}{2a} \\ \frac{\sqrt{-1}\,\beta}{2\bar{a}} & \frac{-\bar{\lambda} n_1(x)}{2\bar{a}} & \frac{-\bar{\lambda} n_2(x)}{2\bar{a}} & \frac{-\bar{\lambda} n_3(x)}{2\bar{a}} & \sqrt{-1}\,b(x) \end{bmatrix}.$$

It is interesting to point out that both boundary operators in (4.7) and (4.8) depend on the parameter λ which is treated as the variable for the Laplace transform corresponding to time variable t. Only for system (4.5) of first order equations, without relating to the single Klein-Gordon equation, there is a boundary operator satisfying condition (4.3), which is a combination of Dirichlet and Neumann boundary conditions:

$$(4.9)\quad B=\begin{bmatrix} \beta+b(x) & \frac{-\sqrt{-1}\,n_1(x)}{2} & \frac{-\sqrt{-1}\,n_2(x)}{2} & \frac{-\sqrt{-1}\,n_3(x)}{2} & \alpha \\ \frac{-\sqrt{-1}\,n_1(x)}{2} & b(x) & 0 & 0 & 0 \\ \frac{-\sqrt{-1}\,n_2(x)}{2} & 0 & b(x) & 0 & 0 \\ \frac{-\sqrt{-1}\,n_3(x)}{2} & 0 & 0 & b(x) & 0 \\ \bar{\alpha} & 0 & 0 & 0 & b(x) \end{bmatrix},$$

where β is real. The kernels of both boundary operators $B-b(x)I$ and $B^*-b(x)I$ consist of vectors $v=(v_1, v_2, v_3, v_4, v_5)$ with $v_1(x)=0$ and $n(x)\cdot(v_2(x), v_3(x), v_4(x))=2\sqrt{-1}\,\alpha v_5(x)$ on $\partial\Omega$ if $\alpha\neq 0$; and with $v_1(x)=0$, $n(x)$ orthogonal to $(v_2(x), v_3(x), v_4(x))$, and any v_5 if $\alpha=0$. This is a nontrivial example for condition (4.3).

For the free space problem, $S=\{0, \pm a\}$, which consists of only one element 0 if $a=0$. This is just the uncertainty situation for the usual wave equation. If $|\lambda|\leq a$ and $\lambda\neq 0$ with $a>0$, the determinant of $\sum A_{0j}y_j+A_{04}-\lambda I$ is $\lambda^3[a^2-\lambda^2+|x|^2]\neq 0$. Then $[-a, 0)\cup(0, a]$ is a subset of the resolvent set of the free space problem for L_0. Assume now that λ is real and $|\lambda|>a$. For the radiation conditions, $m=3$ with $r_1(y;\lambda)=-r_3(y;\lambda)=|y|^{-1}(\lambda^2-a^2)^{1/2}$ of multiplicity 1 and $r_2(y;\lambda)\equiv 0$ of multiplicity 3. Let $d=(\lambda^2-a^2)^{1/2}$. Then

$$I-P_1(\tilde{y};\lambda)=\begin{bmatrix} 2^{-1} & -\tilde{x}_1 d(2\lambda)^{-1} & -\tilde{x}_2 d(2\lambda)^{-1} & -\tilde{x}_3 d(2\lambda)^{-1} & 0 \\ -\lambda\tilde{x}_1(2d)^{-1} & 1-\tilde{x}_1^2/2 & -\tilde{x}_1\tilde{x}_2/2 & -\tilde{x}_1\tilde{x}_3/2 & -a\tilde{x}_1(2d)^{-1} \\ -\lambda\tilde{x}_2(2d)^{-1} & -\tilde{x}_1\tilde{x}_2/2 & 1-\tilde{x}_2^2/2 & -\tilde{x}_2\tilde{x}_3/2 & -a\tilde{x}_2(2d)^{-1} \\ -\lambda\tilde{x}_3(2d)^{-1} & -\tilde{x}_1\tilde{x}_3/2 & -\tilde{x}_2\tilde{x}_3/2 & 1-\tilde{x}_3^2/2 & -a\tilde{x}_3(2d)^{-1} \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix};$$

$$I-P_3(\tilde{y};\lambda)=\begin{bmatrix} 2^{-1} & \tilde{x}_1 d(2\lambda)^{-1} & \tilde{x}_2 d(2\lambda)^{-1} & \tilde{x}_3 d(2\lambda)^{-1} & 0 \\ \lambda\tilde{x}_1(2d)^{-1} & 1-\tilde{x}_1^2/2 & -\tilde{x}_1\tilde{x}_2/2 & -\tilde{x}_1\tilde{x}_3/2 & a\tilde{x}_1(2d)^{-1} \\ \lambda\tilde{x}_2(2d)^{-1} & -\tilde{x}_1\tilde{x}_2/2 & 1-\tilde{x}_2^2/2 & -\tilde{x}_2\tilde{x}_3/2 & a\tilde{x}_2(2d)^{-1} \\ \lambda\tilde{x}_3(2d)^{-1} & -\tilde{x}_1\tilde{x}_3/2 & -\tilde{x}_2\tilde{x}_3/2 & 1-\tilde{x}_3^2/2 & a\tilde{x}_3(2d)^{-1} \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

When $a=0$, this is just a system form of the Sommerfeld radiation condition in the usual sense for the wave equation.

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