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# THE DECOMPOSITION METHOD FOR APPROXIMATE SOLUTION OF A BURGERS EQUATION

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**Abstract.** The Adomian decomposition method is used to investigate a linearised Burgers equation. The analytic solution of the problem is calculated in the form of a series with easily computable components. The nonhomogeneous problem is effectively solved by employing the self-canceling "noise" terms of the phennomean where sum of components vanishes in the limit. Comparing the methodology with some known techniques shows that the present approach is powerful and reliable.

**1. Introduction.** The present paper deals with Burgers equation differently by utilizing the Adomian decomposition method [1-3]. Our objective is to obtain an analytic solution which is obtained in a rapidly convergent series with easily computable components and then to have numerical results in order to compare the accuracy.

In this paper, we are concerned with a linearised Burgers equation. The problem, posed as a model test problem by Sincovec [4], the equation, in $u(x,t)$, is

$$\frac{\partial u}{\partial t} = \Psi(x,t) + \frac{\partial^2 u}{\partial x^2} - c\frac{\partial u}{\partial x}, \ 0 \leq x \leq 1, \ t \geq 0, \tag{1.1}$$

where $c$ is a constant and $\Psi(x,t)$ is given function of $x$ and $t$. The boundary

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conditions and initial condition posed are

$$u(x,0) = g(x), \ (0 < x < 1),$$
$$u(0,t) = f_1(t), \tag{1.2}$$
$$\frac{\partial u}{\partial x}(1,t) = f_2(t), \ (t \geq 0).$$

Physically, this problem can represent a simple model of a fluid flow, heat flow, or other phenomenon, in which an initially discontinuous profile is propagated by diffusion and convection, the latter with a speed of $c$ [5].

The numerical solution of the Burgers equation has been studied before in [7,8]. The studies of motivated by the desire to obtain analytic solutions and numerical approximations to (1.1) with high accuracy level. Forrington [7] and Sanugi and Evans [8] used a Fourier series technique to transform the original partial differential equation (1.1) to a system of ordinary differential equations with initial conditions. In our approach, we will apply the Adomian's decomposition method and the noise-terms phenomena.

**2. Analysis of the method.** To apply the decomposition method, we write (taking $c = 1$) equation (1.1) in an operator form

$$L_t u = \Psi(x,t) - u_x + L_x u, \tag{2.1}$$

where $L_t$ and $L_x$ are the differential operators

$$L_t = \frac{\partial}{\partial t}, \ L_x = \frac{\partial^2}{\partial x^2}. \tag{2.2}$$

It is clear that $L_t$ is invertible and $L_t^{-1}$ is the one-fold integration from 0 to $t$.

Applying the inverse operator $L_t^{-1}$ to (2.1) yields

$$L_t^{-1} L_t u(x,t) = L_t^{-1}\Psi(x,t) - L_t^{-1}(u_x) + L_t^{-1} L_x u,$$

from which it follows that

$$u(x,t) = g(x) + L_t^{-1}\Psi(x,t) - L_t^{-1}(u_x) + L_t^{-1} L_x u. \tag{2.3}$$

The decomposition method [2] consists of decomposing the unknown function $u(x,t)$ into a sum of components defined by the decomposition series

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t), \tag{2.4}$$

substituting (2.4) into (2.3) leads to the recursive relationship

$$\begin{aligned} u_0 &= g(x) + L_t^{-1}\Psi(x,t), \\ u_{n+1} &= -L_t^{-1}(u_n)_x + L_t^{-1}L_x(u_n), \quad \text{for } n \geq 0. \end{aligned} \tag{2.5}$$

It is useful to note that the recursive relationship is constructed on the basis that the zeroth component $u_0(x,t)$ is defined by all terms that arise from the initial condition and from integrating the source term. The remaining components $u_n(x,t)$, $n \geq 1$, can be completely determined such that each term is computed by using the previous term. Accordingly, considering few terms only, the relation Eq. (2.5) gives

$$\begin{aligned} u_0 &= g(x) + L_t^{-1}(\Psi(x,t)), \\ u_1 &= -L_t^{-1}(u_0)_x + L_t^{-1}L_x(u_0), \\ u_2 &= -L_t^{-1}(u_1)_x + L_t^{-1}L_x(u_1), \\ &\vdots \end{aligned}$$

and so on. As a result, the components $u_0, u_1, u_2, \ldots$ are identified and the series solution thus entirely determined. However, in many cases the exact solution in a closed form may be obtained. For numerical purposes, the approximation

$$u(x,t) = \lim_{n\to\infty} \phi_n, \tag{2.6}$$

where

$$\phi_n = \sum_{k=0}^{n-1} u_k(x,t), \tag{2.7}$$

can be used. It is clear that better approximations can be obtained by evaluating more components of $u(x,t)$. We note here that the convergence question of this technique has been formally proved and justified by [1,3].

Adomian and Rach [2] and Wazwaz [6] have investigated the phenomena of the self-canceling "noise" terms where sum of components vanishes in the limit. An important observation was made that "noise" terms appear for nonhomogenous cases only. Further, it was formally justified that if terms in $u_0$ are canceled by terms in $u_1$, even though $u_1$ includes further terms, then the remaining non canceled terms in $u_0$ constitute the exact solution of the equation only after justification. To give a clear overview of the methodology, the following examples will be discussed.

## 3. Examples.

**Example 1.** We consider a homogeneous linearised Burgers equation in order to illustrate the technique discussed above. The problem is of the form

$$u_t - u_{xx} = -u_x, \quad 0 \leq x \leq 1, \ t > 0, \tag{3.1}$$

with initial condition

$$u(x,0) = -e^{-x}. \tag{3.2}$$

Using (2.5) to determine the individual terms of the decomposition, we find

$$u_0 = -e^{-x}, \tag{3.3}$$

and

$$\begin{aligned} u_1 &= -L_t^{-1}(u_0)_x + L_t^{-1}L_x u_0 \\ &= -2te^{-x}, \end{aligned} \tag{3.4}$$

$$\begin{aligned} u_2 &= -L_t^{-1}(u_1)_x + L_t^{-1}L_x u_1 \\ &= -\frac{4t^2}{2!}e^{-x}, \end{aligned} \tag{3.5}$$

$$u_3 = -L_t^{-1}(u_2)_x + L_t^{-1}L_x u_2 \tag{3.6}$$
$$= -\frac{8t^3}{3!}e^{-x},$$

and so on for other components.

Substituting (3.3)-(3.6) into (2.4), the solution $u(x,t)$ of (3.1) in a series form

$$u(x,t) = -e^{-x} - 2te^{-x} - \frac{4t^2}{2!}e^{-x} - \frac{8t^3}{3!}e^{-x} - \ldots, \tag{3.7}$$

follows immediately. After some tedious algebra factoring, (3.7) can be rewritten as

$$u(x,t) = -e^{-x}(1 + 2t + \frac{4t^2}{2!} + \frac{8t^3}{3!} + \ldots). \tag{3.8}$$

It can be easily observed that (3.8) is equivalent to the exact solution

$$u(x,t) = -e^{-x+2t}. \tag{3.9}$$

This can be verified through substitution.

In order to verify numerically whether the proposed methodology leads to higher accuracy, we can evaluate the appproximate solution using the $n$-term approximation to $u(x,t)$ by $\phi_n$, such that $\phi_n = \sum_{k=0}^{n-1} u_k(x,t)$ or

$$\phi_1 = -e^{-x} = u_0,$$
$$\phi_2 = -e^{-x} - 2te^{-x} = u_0 + u_1,$$
$$\phi_2 = -e^{-x} - 2te^{-x} - \frac{4t^2}{2!}e^{-x} = u_0 + u_1 + u_2,$$
$$\phi_3 = -e^{-x} - 2te^{-x} - \frac{4t^2}{2!}e^{-x} - \frac{8t^3}{3!}e^{-x} = u_0 + u_1 + u_2 + u_3,$$

Table 1 below illustrates the errors obtained by using the procedure outlined above. We achieved a very good approximation with the actual solution of the equations by using 10 terms only of the decomposition derived above. It is evident that the overall errors can be made smaller by adding new terms of the decomposition.

Table 1 Absolute errors obtained for Example 1

| | t | | | |
|---|---|---|---|---|
| x | 0.5 | 1.0 | 1.5 | 2.0 |
| 0.5 | -0.165660D-07 | -0.100478D-07 | -0.609428D-08 | -0.369637D-08 |
| 1.0 | -0.100478D-07 | -0.225841D-07 | -0.136979D-04 | -0.830822D-05 |
| 1.5 | -0.356139D-02 | -0.216009D-02 | -0.131016D-02 | -0.794654D-03 |
| 2.0 | -0.940401D-01 | -0.570382D-01 | -0.345954D-01 | -0.209832D-01 |

The Adomian decomposition method avoids the difficulties and massive computational work by determining the analytic solution. We compare the approximation solution of (3.1) with the exact solution of the equations in Table 1. Numerical approximations shows a high degree of accuracy and in most cases $\phi_n$, the $n$-term approximation is accurate for quite low values of $n$. The numerical results we obtained justify the advantage of this methodology, even in the case of few terms approximation.

**Example 2.** We next consider the nonhomogeneous Burgers equation of the form

$$u_t + u_x - u_{xx} = e^{2x} + \cos t - 2te^{2x},\ \ 0 \le x \le 1, t > 0, \tag{3.10}$$

with initial condition

$$u(x,0) = 0. \tag{3.11}$$

Using (2.5) to determine the individual terms of the decomposition, we find

$$u_0 = te^{2x} + \sin t - t^2 e^{2x}, \tag{3.12}$$

and

$$\begin{aligned} u_1 &= -L_t^{-1}(u_0)_x + L_t^{-1} L_x u_0 \\ &= t^2 e^{2x} - \frac{2}{3} t^3 e^{2x}, \end{aligned} \tag{3.13}$$

$$u_2 = -L_t^{-1}(u_1)_x + L_t^{-1}L_x u_1 = \frac{2}{3}t^3 e^{2x} - \frac{1}{3}t^4 e^{2x}, \tag{3.14}$$

$$u_3 = -L_t^{-1}(u_2)_x + L_t^{-1}L_x u_2 = \frac{1}{3}t^4 e^{2x} - \frac{2}{5}t^5 e^{2x}, \tag{3.15}$$

and so on for other components. It can be easily observed that the self-canceling "noise" terms appear between various components. Canceling the third term in $u_0$ and the first term in $u_1$, the second term in $u_1$ and the first term in $u_2$, and so on keeping the non canceled terms in $u_0$ yields the exact solution of (3.10) given by

$$u(x,t) = te^{2x} + \sin t, \tag{3.16}$$

which can be easily verified.

In cloosing, the methods avoid the difficulties and massive computational work by determining the analytic solution. The solution is very rapidly convergent by utilizing the Adomian's decomposition method. The decomposition method provides a reliable technique that requires less work and highly accurate results if compared with the traditional techniques.

It is worth noting that the Adomian methodology is very powerful and efficient in finding exact solutions for wide classes of problems. The convergence can made faster if the noise terms appear as discussed in [2,6].

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