CRITICAL LENGTH FOR A DEGENERATE PARABOLIC EQUATION

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Abstract. We study the critical length for a degenerate parabolic equation. We show that the critical length exists and is finite.

1. Introduction. The aim of this work is to study the critical length of the following degenerate parabolic initial boundary value problem

$$(1.1) \qquad u_t - (x^\alpha u_x)_x = (1-u)^{-\beta}, \ (x,t) \in (0,a) \times (0,T),$$

$$(1.2) \qquad u(0,t) = u(a,t) = 0, \ t \in (0,T),$$

$$(1.3) \qquad u(x,0) = u_0(x), \ x \in [0,a],$$

where $0 < \alpha < 1$, $\beta > 0$, $0 < T \le \infty$ and $u_0 \in C^{2+\gamma}(0,a) \cap C[0,a]$ for some $\gamma \in (0,1)$ with $0 \le u_0 < 1$ and $u_0(0) = u_0(a) = 0$, and $\int_0^a x^\alpha [u_0'(x)]^2 dx < \infty$.

The problem (1.1)-(1.3) was studied by Ke and Ning [5] for the case $0 < \beta < 1$ and for the more general diffusion coefficient $p(x)$ which including the case $p(x) = x^\alpha$, $0 < \alpha < 1$. Here we restrict our attention on the typical case when $p(x) = x^\alpha$, $0 < \alpha < 1$, so that the equation becomes degenerate on the left-hand boundary $x = 0$. But, we only assume that $\beta > 0$.

Applying a method used in [5], we show in Section 2 that a unique classical solution of problem (1.1)-(1.3) exists for a small time interval. Let

T be the largest time so that the solution u of problem (1.1)-(1.3) is classical. If $T < \infty$, then we must have $\max_{x \in [0,1]} u(x,t) \to 1$ as $t \to T$. In this case we say that the solution quenches.

Quenching phenomena for various types of problems have been studied by many authors during past years. For more references, we refer to the nice survey papers of Chan [1,2].

In is well-known that the steady state solutions of the parabolic problem play an important role in studying the quenching problem. In our case, $w(x)$ is a steady state solution of the problem (1.1)-(1.3) if $w(x)$ satisfies the following boundary value problem (D_a):

$$(x^\alpha w'(x))' + (1-w)^{-\beta} = 0, \ 0 < x < a,$$

$$w(0) = w(a) = 0.$$

A number a^* is called the critical length for the problem (1.1)-(1.3) if the problem (D_a) has a solution for any $a < a^*$ and has no solution for any $a > a^*$. Applying a method used in [4], we prove in Section 3 that the critical length of the problem (1.1)-(1.3) exists and is finite. We emphasize that our method is entirely different from the one used in [5].

2. Local existence and uniqueness. In this section, we prove that there exists a $t_0 > 0$ such that a classical solution of the problem (1.1)-(1.3) exists uniquely for $0 < t < t_0$.

Let $\epsilon \in (0, a)$. Consider the following problem

$$(2.1) \qquad (u_\epsilon)_t - (x^\alpha (u_\epsilon)_x)_x = (1 - u_\epsilon)^{-\beta}, \ (x,t) \in (\epsilon, a) \times (0, T),$$

$$(2.2) \qquad\qquad u_\epsilon(\epsilon, t) = u_\epsilon(a, t) = 0, \ t \in (0, T),$$

$$(2.3) \qquad\qquad u_\epsilon(x, 0) = u_0(x), \ x \in [\epsilon, a],$$

where $0 < \alpha < 1$, $\beta > 0$, and the initial data u_0 satisfies the conditions stated in Section 1. It is well-known (cf, [3]) that a classical solution u_ϵ of (2.1)-

(2.3) exists on some time interval $(0, t_\epsilon)$ and the solution u_ϵ is continuous in $[\epsilon, a] \times [0, t_\epsilon)$ except possibly at the point $(\epsilon, 0)$.

We define a function $h(t)$ on $(0, \tilde{t})$ by

$$h(t) = 1 - \{[1 - h(0)]^{\beta+1} - (\beta+1)t\}^{\frac{1}{\beta+1}}$$

where $\tilde{t} = [1 - h(0)]^{\beta+1}/[\beta + 1]$ and $h(0) = \max_{x \in [0,1]} u_0(x)$. It is easy to check that $h(t)$ is an upper solution of (2.1)-(2.3). Hence, by taking any $t_0 \in (0, \tilde{t})$, we have the estimate

$$t_\epsilon \geq t_0, \ \forall \epsilon > 0.$$

Now, we fix a $t_0 \in (0, \tilde{t})$. By the strong maximum principle, it is easy to show that u_ϵ is monotone increasing as ϵ decreases, that is, if $0 < \epsilon_1 < \epsilon_2 < a$, then $u_{\epsilon_1} > u_{\epsilon_2}$ for all $x \in (\epsilon_2, a), t \in (0, t_0)$. Thus $\lim_{\epsilon \to 0} u_\epsilon(x, t) = u(x, t)$ exists. We will show that $u(x, t)$ is a classical solution of (1.1)-(1.3) with $T \geq t_0$.

Lemma 2.1. *There exists a constant c independent of ϵ such that*

$$\int_\epsilon^a x^\alpha (u_{\epsilon x}(x, t))^2 dx < c$$

for $t \in (0, t_0)$.

Proof. First, we note that

$$u_\epsilon(x, t) \leq h(t) \leq h(t_0) < 1, \ \forall x \in [\epsilon, a], t \in (0, t_0).$$

Following [5], we let

$$E(t; \epsilon) = \frac{1}{2} \int_\epsilon^a x^\alpha (u_{\epsilon x}(x, t))^2 dx - \int_\epsilon^a \int_0^{u_\epsilon} (1 - s)^{-\beta} ds dx$$

for $t \in (0, t_0)$. Then, using integration by parts, we can derive that

$$E'(t; \epsilon) = - \int_\epsilon^a (u_{\epsilon t})^2 dx \leq 0.$$

Hence $E(t; \epsilon) \leq E(0; \epsilon)$ for $t \in (0, t_0)$. Then

$$\frac{1}{2}\int_\epsilon^a x^\alpha (u_{\epsilon x}(x,t))^2\,dx \le \frac{1}{2}\int_\epsilon^a x^\alpha [u_0'(x)]^2\,dx - \int_\epsilon^a\int_0^{u_0}(1-s)^{-\beta}\,ds\,dx$$

$$+\int_\epsilon^a\int_0^{u_\epsilon}(1-s)^{-\beta}\,ds\,dx$$

$$<\frac{1}{2}\int_0^a x^\alpha [u_0'(x)]^2\,dx + a\int_0^{h(t_0)}(1-s)^{-\beta}\,ds.$$

Let

$$c = \int_0^a x^\alpha [u_0'(x)]^2\,dx + 2a\int_0^{h(t_0)}(1-s)^{-\beta}\,ds.$$

Then $c < \infty$ and the lemma follows.

Using Lemma 2.1, we can prove that $u(x,t)$ is the unique classical solution of the problem (1.1)-(1.3) with $T \ge t_0$. Moreover, $u(x,t)$ is positive in $(0,a) \times (0,t_0)$. The proof can be found in [5] and we omit it.

3. Critical length.

In this section, we study the critical length of the problem (1.1)-(1.3). Hence we shall study the following boundary value problem:

$$(3.1) \qquad (y^\alpha w'(y))' + (1-w)^{-\beta} = 0, \ 0 < y < a,$$

$$(3.2) \qquad w(0) = w(a) = 0.$$

First, we make the following transformations

$$\phi(x) = w(y), \ x = \frac{y}{a}, \ \lambda = a^{2-\alpha}.$$

Then $w(y)$ is a solution of the problem (3.1)-(3.2) if and only if $\phi(x)$ is a solution of the boundary value problem (D_λ):

$$(3.3) \qquad (x^\alpha \phi'(x))' + \lambda(1-\phi)^{-\beta} = 0, \ 0 < x < 1,$$

$$(3.4) \qquad \phi(0) = \phi(1) = 0.$$

Let L be the operator such that $Lu(x) = (x^\alpha u'(x))'$. It is well-known that the Green's function $G(x,y)$ for the operator $-L$ in $(0,1)$ with zero

Dirichlet boundary condition exists. Indeed, $G(x, y)$ is given by

$$G(x, y) = \begin{cases} x^{1-\alpha}(1 - y^{1-\alpha})/(1 - \alpha), & 0 < x < y < 1, \\ y^{1-\alpha}(1 - x^{1-\alpha})/(1 - \alpha), & 0 < y < x < 1. \end{cases}$$

Define

$$U(x) = \int_0^1 G(x, y)dy, \ 0 < x < 1,$$

and

$$U(0) = U(1) = 0.$$

Then $U(x)$ is continuous on $[0, 1]$ and satisfies the equation

$$(x^\alpha U'(x))' + 1 = 0, \ 0 < x < 1.$$

By the maximum principle, we see that $U > 0$ in $(0, 1)$. Set

$$M = \max_{x \in [0,1]} U(x).$$

Notice that a classical solution ϕ of (3.3)-(3.4) exists if and only if there is a $\sigma \in (0, 1)$ such that $0 \le \phi(x) \le \sigma$, for all $x \in [0, 1]$.

Now, given a fixed $\sigma \in (0, 1)$, define the set

$$X = \{\phi \in C^0([0, 1]) : 0 \le \phi \le \sigma \text{ in } [0, 1], \phi(0) = \phi(1) = 0\},$$

and define the mapping $T : X \to C^0([0, 1])$ by

$$T\phi(x) = \begin{cases} \lambda \int_0^1 G(x, y)[1 - \phi(y)]^{-\beta}dy, & 0 < x < 1, \\ 0, & x = 0, 1, \end{cases}$$

for any $\phi \in X$.

Note that X is a Banach space with the sup norm $\| \cdot \|_\infty$. Applying the contraction mapping principle, we obtain the following theorem.

Theorem 3.1. *The problem (D_λ) has a solution if λ is sufficiently small.*

Proof. Since, for any $\phi \in X$,

$$0 \leq T\phi(x) \leq \lambda(1-\sigma)^{-\beta}M, \ \forall x \in [0,1],$$

and by the mean value theorem, for $\phi_1, \phi_2 \in X$,

$$|(T\phi_1 - T\phi_2)(x)| \leq \lambda\beta(1-\sigma)^{-\beta-1}M\|\phi_1 - \phi_2\|_\infty, \ \forall x \in [0,1],$$

T is a contraction mapping from X into X if λ is chosen sufficiently small. From the contraction mapping principle, if λ is sufficiently small, then T has a unique fixed point ϕ_λ in X. Clearly, ϕ_λ is a solution of the problem (D_λ).

In fact, the solution ϕ_λ in the proof of Theorem 3.1 is unique when we restrict ourselves to the space X for a fixed σ.

We will prove the following result by the method of super-sub-solution.

Theorem 3.2. *If a solution of the problem (D_λ) exists for some $\lambda_1 > 0$, then there exists a solution of the problem (D_λ) for any $\lambda < \lambda_1$.*

Proof. Let w be a solution of (D_{λ_1}). Given a fix $\lambda < \lambda_1$. First, we know that $v \equiv 0$ is a subsolution of (D_λ). Since

$$(x^\alpha w'(x))' + \lambda(1-w)^{-\beta}$$
$$\leq (x^\alpha w'(x))' + \lambda_1(1-w)^{-\beta}$$
$$= 0, \forall x \in (0,1),$$

w is a supersolution of (D_λ). Now, define $\phi_0 \equiv 0$ and ϕ_{n+1} be the solution of the problem

$$(x^\alpha \phi'_{n+1}(x))' + \lambda(1-\phi_n)^{-\beta} = 0, \ x \in (0,1)$$
$$\phi_{n+1}(0) = \phi_{n+1}(1) = 0,$$

for any $n \geq 0$. Then $\{\phi_{n+1}\}$ is a bounded monotone sequence in $[0,1]$ such that

$$0 \leq \phi_n \leq \phi_{n+1} \leq w.$$

Hence $\{(1-\phi_n)^{-\beta}\}$ is also uniformly bounded in $[0,1]$. Therefore, there is

a function ϕ such that $\phi_n \to \phi$ uniformly and ϕ is a solution of the problem (D_λ).

Next, we define

$$\lambda^* = \sup\{\lambda : (D_\lambda) \text{has a solution}\}.$$

Theorem 3.3. $\lambda^* < \infty$.

Proof. Since (D_λ) has a solution ϕ_λ for any $\lambda < \lambda^*$, we have

$$\phi_\lambda(x) = \lambda \int_0^1 G(x,y)[1 - \phi_\lambda(y)]^{-\beta} dy.$$

It follows that

$$1 > \phi_\lambda(x) \geq \lambda U(x), \ \forall x \in (0,1).$$

Hence $\lambda M < 1$, $\forall \lambda < \lambda^*$, and so $\lambda^* \leq 1/M < \infty$.

Hence the critical length for the problem (1.1)-(1.3) exists and is finite.

References

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