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UNIFORM STABILITY FOR SOLUTIONS OF n-DIMENSIONAL NAVIER-STOKES EQUATIONS

BY

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Abstract. Motivated by recent work of H. Beirao da Veiga, P. Secchi [20] and Wiegner [22], we are concerned with uniform stability estimates for solutions of the Cauchy problem for the $n(\geq 2)$-dimensional Navier-Stokes equations $u_t + u \cdot \nabla u - \Delta u + \nabla p = 0$, $\nabla \cdot u = 0$. Our main result demonstrates uniform stability for the solutions. The next result is concerned with the temporal asymptotic behavior of the solutions for the cases $u_0 \in H^2$ or $u_0 \in L^1 \cap H^2$. Finally for $n(\geq 3)$-dimensional problems, we establish some regularity results by iteration. Our primary motivation is to see whether or not the solutions to n-dimensional problem are uniform stable. Given different initial velocity $u_0(x)$ and $v_0(x)$, with $\nabla \cdot u_0 = \nabla \cdot v_0 = 0$, the corresponding solutions (u,p) and (v,q) will be different. We are interested in the bound and asymptotic behavior of the difference $(u - v, p - q)$. The ideas to establish various uniform stability in different spaces are related to those to obtain the asymptotic behaviors, but differ in detail. When studying the uniform stability and the asymptotic behaviors of the solutions, we need the constants to be independent of t, but we do not care how they depend on the initial data. We establish these results by various delicate integral estimates.

1. Introduction and main results. The Navier-Stokes equations occupy a central position in the study of nonlinear partial differential equations and dynamical systems. In this paper we study uniform stability with respect to perturbations of the initial velocity for solutions of the Cauchy

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problem for the $n(\geq 2)$-dimensional incompressible Navier-Stokes equations

$$u_t + u \cdot \nabla u - \Delta u + \nabla p = 0, \quad \nabla \cdot u = 0, \tag{1}$$

$$u(x,0) = u_0(x), \quad \nabla \cdot u_0 = 0, \tag{2}$$

where $x = (x_1, \ldots, x_n) \in R^n$, $t > 0$. $u = (u_1(x,t), \ldots, u_n(x,t)) \in R^n$ and $p = p(x,t)$ denote the unknown velocity and pressure respectively.

We also study the asymptotic behavior of the solutions for 2-dimensional problem and regularity for $n(\geq 3)$-dimensional problem. In the entire paper, we assume that u_0 and (u,p) of problem (1–2) satisfy $\nabla \cdot u_0 = \nabla \cdot u = 0$, and for all $t > 0$,

$$\lim_{|x|\to\infty} \frac{\partial^{\alpha_1+\cdots+\alpha_n} u(x,t)}{\partial x_1^{\alpha_1} \ldots \partial x_n^{\alpha_n}} = \lim_{|x|\to\infty} \frac{\partial^{\alpha_1+\cdots+\alpha_n} p(x,t)}{\partial x_1^{\alpha_1} \ldots \partial x_n^{\alpha_n}} = 0,$$

where $\alpha_1 \geq 0, \ldots, \alpha_n \geq 0$ are integers with $\alpha_1 + \cdots + \alpha_n \leq 2$.

Lemma 1.1. *Let $n = 2$ and $u_0 \in H^2$. Then problem (1–2) enjoys a unique global solution $(u,p) \in L^\infty(0,\infty; H^2) \cap W^{1,\infty}(0,\infty; L^2) \cap W^{1,2}(0,T; L^2)$, where $T > 0$ is any constant.*

Lemma 1.1 can be proved by the fixed point principle together with some of the estimates displayed in Lemmas 3.1, 3.2 and 2.2.

Many mathematicians have studied the asymptotic behavior of solutions to (1–2) and have made significant progress, cf [7–12, 19–24]. See also [27] for the latest and interesting results. They proved that solutions to $n(\geq 2)$-dimensional problem have the same rate of decay as that of the same dimensional heat equations, provided the initial velocity is in the same function space. The 2-dimensional problem is much more difficult than the $n(\geq 3)$-dimensional problem, because it is somewhat near the critical case [12, 21]. The author established the optimal rate of decay for 2-dimensional problem by a good use of Gronwall's inequality, which simplified other proofs [24]. Roughly speaking, the more rapidly the initial velocity u_0 decays as $|x| \to \infty$, the more rapidly the norm $\|(u,p)(t)\|$ decays as $t \to \infty$. So we must study respectively the best rate of decay of the solutions to problem (1–2) if u_0 is in different Sobolev spaces.

Let (u,p) and (v,q) be the solutions of problem (1–2) corresponding to u_0 and v_0 respectively. Let $(w,\pi) = (u-v, p-q)$. Then they satisfy the equations

$$w_t + w\cdot\nabla u + v\cdot\nabla w - \Delta w + \nabla\pi = 0, \quad \nabla\cdot w = 0, \tag{3}$$

$$w(x,0) = w_0(x) = u_0(x) - v_0(x), \quad \nabla\cdot w_0 = 0. \tag{4}$$

To deal with the uniform stability, we will study the optimal rates of decay of (w,π) if $w_0 \in H^2$ or $L^1\cap H^2$. We employ the method of global energy estimate. We also employ various inequalities. The comprehensive use of the Hölder's inequality, the Gagliardo-Nirenberg's inequality, the Gronwall's inequality and the relation $\nabla\cdot u = 0$ are very important and are used almost everywhere to establish the L^2 and L^∞ uniform stability.

The decay estimates do not automatically lead to the uniform stability of the solutions to (1–2). In fact, if $n=2$, the proof of the uniform stability $\|w(t)\| \le \exp(C\|u_0\|^4)\|w_0\|$ relys only on the elaborate energy estimates: $\|u(t)\| \le \|u_0\|$ and $\|\nabla u(t)\|^2 \in L^1(0,\infty)$. If $n\ge 3$, the proof depends only on the assumptions $u_0 \in L^2\cap L^p$ and $u \in L^q(0,\infty;L^p)$, where $p > n \ge 3$ and $n/p + 2/q = 1$. It can be clearly seen that the establishment of this elementary stability is quite independent of all the asymptotic behaviors. We do not necessarily require that any norm of the solution of problem (1–2) tends to zero as $t\to\infty$ to justify this basic uniform stability. On the other hand, the algebraic rates of decay suggest that certain norms of w have some algebraic rates of decay. To motivate the definition of the uniform stability, let us look at the decay estimates. For 2-dimensional problem (1–2), if $u_0 \in L^1\cap H^2$, the optimal rates of decay of the solution and its derivatives are as follows, cf [12, 21]

$$(1+t)\|u(t)\|^2 + (1+t)^2\|\nabla u(t)\|^2 + (1+t)^3\|\Delta u(t)\|^2 \le C.$$

If $u_0 \in L^1\cap L^2$ and $\int_{R^2} u_0 dx \ne 0$, then

$$0 < C_1 \le (1+t)\|u(t)\|^2 \le C_2 < \infty.$$

For 2-dimensional problem (3–4), we have similar results.

Since w and π decay at different rates, and different norms decay at different rates, we will investigate w and π in an unusual functional space. To get a concrete idea about the uniform stability, we need the least upper bound of the difference of the solutions depend on the initial difference as explicitly as possible. The estimate in the following definition is optimal.

Definition. Let X and Y be Banach spaces, let $\phi : X \to Y$ be the solution operator induced by problem (1–2). If there are constants $C > 0$ and $\alpha \geq 0$, which are independent of t, such that

$$\sup_{0 \leq t < \infty} [(1+t)^{\alpha} \|(w, \pi)(t)\|_Y] \leq C \|w_0\|_X,$$

then the solutions of problem (1–2) are uniformly stable.

It is very interesting and important that the least upper bound of certain norms of (w, π) depends explicitly on w_0, but does not depend on t. Although it is very difficult to study the uniform stability of the solutions to problem (1–2), we can at least establish the global estimates in different Sobolev spaces for the 2-dimensional problem, and the $n(\geq 3)$-dimensional problem under some additional restrictions on the initial function and the solution. These uniform stability results are very interesting, because they give us general ideas how they depend on w_0 and how fast they decay. They illustrate that if $v_0 \to u_0$, in some space, then the corresponding solutions $(v, q) \to (u, p)$ in another space, for all $t > 0$. They also illustrate that the solutions depend continuously on the initial velocity. It is very easy to obtain some of the decay results for the solution (u, p) of 2-dimensional problem (1–2). This can be done by letting $v_0 = 0$, which implies that $(v, q) = (0, 0)$.

Before stating the main theorems, let us give the notations. Denote by C any positive time-independent constant, which may be different from line to line, and which may depend on u_0 and v_0. Moreover, denote by $L^p = L^p(R^n)$ and $H^m = H^m(R^n)$. Let

$$\int_{R^n} f(x)dx = \int_{R^n} f(x_1, \ldots, x_n)dx_1 \cdots dx_n,$$

$$|f|^2 = \sum_{i=1}^{n} f_i^2,$$

$$|\nabla f|^2 = \sum_{i=1}^{n}\sum_{j=1}^{n} \left|\frac{\partial f_i}{\partial x_j}\right|^2,$$

$$\nabla f \cdot \nabla g = \sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\partial f_i}{\partial x_j}\frac{\partial g_i}{\partial x_j},$$

$$\|f(t)\| = \|f(t)\|_{L^2},$$

$$\|f(t)\|_\infty = \|f(t)\|_{L^\infty},$$

$$\|f(t)\|_m = \|f(t)\|_{H^m},$$

$$\|f\|_{L^1\cap L^2} = \|f\|_{L^1} + \|f\|_{L^2},$$

$$\|f\|_{L^1\cap H^2} = \|f\|_{L^1} + \|f\|_{H^2}.$$

Define the weighted Sobolev space M to be

$$\left\{f \in L^1 \cap H^2 \middle| \int_{R^n} f(x)dx = 0, \quad \|f\|_M = \int_{R^n} (1+|x|)|f|dx + \|f\|_{H^2} < \infty\right\}.$$

For $f(x) \in L^1 \cap L^2$, define its Fourier and inverse Fourier transformations by

$$F[f](\xi) = \widehat{f}(\xi) = \int_{R^n} f(x)\exp(-ix\cdot\xi)dx,$$

$$F^{-1}[f](x) = \check{f}(x) = \frac{1}{(2\pi)^n}\int_{R^n} f(\xi)\exp(ix\cdot\xi)d\xi.$$

The definition can be extended to the Hilbert space L^2 by continuity. Obviously if f and $g \in L^2$, then $\|\widehat{fg}\|_\infty \le \|f\|\,\|g\|$.

Let us state our main uniform stability theorems. To do this, define

$$\begin{aligned} S(f(t),g(t)) = {} & \|f(t)\|^2 + (1+t)\|\nabla f(t)\|^2 + (1+t)^2\|\Delta f(t)\|^2 \\ & + (1+t)\|f(t)\|_\infty^2 + (1+t)^2\|f_t(t)\|^2 \\ & + (1+t)\|g(t)\|^2 + (1+t)^2\|\nabla g(t)\|^2 + (1+t)^3\|\Delta g(t)\|^2 \\ & + (1+t)^2\|g(t)\|_\infty^2 + (1+t)^3\|g_t(t)\|^2. \end{aligned}$$

Theorem 1. *Let $n = 2$.*

(1) *Let u_0 and $v_0 \in H^2$, then we have the uniform stability estimate*

$$\sup_{1\le t<\infty} S(w(t),\pi(t)) \le C\|w_0\|_2^2.$$

(2) *Let u_0 and $v_0 \in L^1 \cap H^2$, then we have the uniform stability estimate*

$$\sup_{1\le t<\infty} [(1+t)S(w(t),\pi(t))] \le C\|w_0\|^2_{L^1\cap H^2}.$$

(3) *Let u_0 and $v_0 \in M$, then*

$$\sup_{1\le t<\infty} [(1+t)^2 S(w(t),\pi(t))] \le C\|w_0\|^2_M.$$

(4) *Let u_0 and $v_0 \in L^1 \cap H^2$ and $\int_{R^2} w_0 dx \ne 0$, then*

$$C_1(w_0)\Big|\int_{R^2} w_0 dx\Big| - \frac{C_2\ln(1+t)}{\sqrt{1+t}}\|w_0\|_{L^1\cap L^2} \le (1+t)^{1/2}\|w(t)\| \le C_3\|w_0\|_{L^1\cap L^2}.$$

We make the critical assumption for the weak solutions of $n(\ge 3)$-dimensional Navier-Stokes equations (1–2):

(H) There are constant $p > n \ge 3$ and $q > 2$ satisfying $n/p + 2/q = 1$, such that $u_0 \in L^2 \cap L^p$ and $u \in L^q(0,\infty; L^p)$.

Theorem 2. *Let $n \ge 3$ and let* (H) *hold.*

(1) *Let $v_0 \in L^2$, then we have the uniform stability estimate*

$$\|w(t)\| \le C\|w_0\|, \quad \int_0^\infty \|\nabla w(t)\|^2 dt \le C\|w_0\|, \quad \|\widehat{\pi}(t)\|_\infty \le C\|w_0\|,$$

where the constant C depends only on the $L^q(0,\infty;L^p)$-norm of u.

(2) *There is a constant $\delta > 0$, such that if $v_0 \in L^2 \cap L^p$ and $\|u_0 - v_0\|_{L^p} < \delta$, then a strong solution $v \in L^r(0,\infty;L^p)$ to problem* (1–2) *exists, where $4p/n(p-2) < r \le \infty$, such that*

$$\sup_{v_0:\|u_0-v_0\|_{L^p}<\delta} \int_0^\infty \|v(t)\|^q_{L^p} dt \le C < \infty,$$

$$\|w(t)\|_{L^p} \le C\|w_0\|_{L^p},$$

$$\frac{p}{2}\int_0^\infty\int_{R^n} |w|^{p-2}|\nabla(w)|^2 dxdt + \frac{2(p-2)}{p}\int_0^\infty\int_{R^n} |\nabla(|w|^{p/2})|^2 dxdt \le C\|w_0\|^p_{L^p},$$

where C depends only on the $L^q(0,\infty;L^p)$-norm of u.

(3) *Let $u_0 \in L^2 \cap L^\infty$, then for all $r \geq p$, there is a constant $\delta(r,n) > 0$, such that if $v_0 \in L^2 \cap L^\infty$ and $\|u_0 - v_0\|_{L^r} < \delta(r,n)$, then*

$$\|w(t)\|_{L^r} \leq C(n,r)\|w_0\|_{L^r},$$
$$\int_0^\infty \int_{R^n} |w|^{r-2}|\nabla(w)|^2 dxdt + \int_0^\infty \int_{R^n} |\nabla(|w|^{r/2})|^2 dxdt \leq C\|w_0\|_{L^r}^r,$$

where $C(n,r)$ depends only on $L^{2r/(r-n)}(0,\infty;L^r)$-norm of u.

(4) *Let u_0 and $v_0 \in M \cap L^p$, then*

$$\sup_{1\leq t<\infty} [t^{n+1}\|\pi(t)\|^2] \leq C\|w_0\|^2,$$

where $C(n,r)$ depends only on $L^{2r/(n-r)}(0,\infty;L^r)$-norm of u.

We hope that if u_0 and $v_0 \in H^2$ satisfy either of the following hypotheses

$$\sup_{x\in R^n} (1+|x|^\rho)[|u_0(x)| + |v_0(x)|] \leq C,$$
$$\int_{R^n} (1+|x|^\rho)[|u_0(x)|^2 + |v_0(x)|^2]dx \leq C,$$

then w satisfy the corresponding estimates

$$\sup_{x\in R^n, 0\leq t<\infty} (1+t)^\tau (1+|x|^\sigma)|w(x,t)| \leq C \sup_{x\in R^n} (1+|x|^\rho)|w_0(x)|,$$
$$\sup_{0\leq t<\infty} (1+t)^\tau \int_{R^n} (1+|x|^\sigma)|w(x,t)|^2 dx \leq C \int_{R^n} (1+|x|^\rho)|w_0(x)|^2 dx,$$

where C, ρ, σ and τ are positive constants.

Theorem 3. *Let $n = 2$, We have the asymptotic behaviors for the solutions of problem* (1–2)

$$\lim_{t\to\infty} [\|u(t)\|^2 + (1+t)\|\nabla u(t)\|^2] = 0,$$
$$\lim_{t\to\infty} [(1+t)\|p(t)\|^2 + (1+t)^2\|\nabla p(t)\|^2] = 0,$$
$$\sup_{1\leq t<\infty} [(1+t)^\beta S(w(t),\pi(t))] \leq C.$$

where $\beta = 0$ if $u_0 \in H^2$, $\beta = 1$ if $u_0 \in L^1 \cap H^2$, $\beta = 2$ if $u_0 \in M$.

Theorem 4. *Let* (H) *hold and let* $u_0 \in L^2 \cap L^\infty$, *then*

$$u \in \Big(\bigcap_{p \le r < \infty} L^{2r/(r-n)}(0, \infty; L^r) \Big) \bigcap \Big(\bigcap_{2 \le s < \infty} L^\infty(0, \infty; L^s) \Big)$$

Therefore u *is almost in the space* $L^\infty(R^n \times R^+)$. *Let* $n = 3$ *and let* $u_0 \in H^2 \cap L^\infty$, *then* $u \in L^\infty(0, \infty; H^2) \cap L^2(0, \infty; H^3)$.

Remark 1. If $n = 2$ and $u_0 \in L^1 \cap H^2$, various decay estimates, except for p and u_t, of the solutions of n-dimensional problem (1–2) have been well established in [12, 21]. We verified the decay estimates of the solutions (u, p) for all the cases $u_0 \in H^2$, $L^1 \cap H^2$ or M.

Remark 2. If we consider (1–2) with a forcing term, i.e.

$$u_t + u \cdot \nabla u - \Delta u + \nabla p = f, \quad \nabla \cdot u = \nabla \cdot f = 0,$$

we can get similar results. Even if the forcing term is not divergence free, we can still do this. In fact, let g be the solution of the equation $\nabla \cdot f = \Delta g$, and let $h = f - \nabla g$, then h is divergence free. Substituting $p_1 = p - g$ for p, we get

$$u_t + u \cdot \nabla u - \Delta u + \nabla p_1 = h, \quad \nabla \cdot u = \nabla \cdot h = 0.$$

The difference of different solutions of this problem corresponding to different data also satisfies equation (3). That is why the uniform stability and the long time behavior are similar.

In Section 2, we present a series of elementary estimates. Some inequalities are also listed here. Section 3 is dedicated to temporal asymptotic behavior of solutions of 2-dimensional problem (1–2) for the case $u_0 \in H^2$. After having established all the necessary asymptotic behaviors with $u_0 \in H^2$, we prove in Section 4 our uniform stability for 2-dimensional problem (1–2) with u_0, $v_0 \in H^2$ or $L^1 \cap H^2$. One can directly justify the asymptotic behavior of the solutions to problem (1–2) with $u_0 \in L^1 \cap H^2$, but it would be better to do this by letting $v_0 = 0$ in the uniform stability estimates. This can avoid a lot of unnecessary work. In Section 5, we deal with $n(\geq 3)$-dimensional problems

2. Elementary estimates. In this section we present some elementary estimates which will be very useful in demonstrating our uniform stability.

Lemma 2.1. *The following preliminary estimates for the nonlinear effects hold.*

$$
\begin{gathered}
|w\cdot\nabla u|^2 \leq |w|^2|\nabla u|^2,\\
|F[w\cdot\nabla u]|^2 \leq |\xi|^2\|w(t)\|^2\|u(t)\|^2,\\
|\nabla\cdot(w\cdot\nabla u)|^2 \leq |\nabla w|^2|\nabla u|^2,\\
|F[\nabla\cdot(w\cdot\nabla u)]|^2 \leq \|\nabla w(t)\|^2\|\nabla u(t)\|^2,\\
\|\nabla\cdot(w\cdot\nabla u)(t)\|^2 \leq 2\|\nabla u(t)\|_\infty^2\|\nabla w(t)\|^2 + 2\|w(t)\|_\infty^2\|\Delta u(t)\|^2,\\
\|\widehat{\pi}(t)\|_\infty \leq [\|u(t)\| + \|v(t)\|]\ \|w(t)\|,\\
\|\widehat{\Delta\pi}(t)\|_\infty \leq [\|\nabla u(t)\| + \|\nabla v(t)\|]\ \|\nabla w(t)\|,\\
|\widehat{\pi}|^2 \leq \sum_{i=1}^n\sum_{j=1}^n |\widehat{w_i u_j} + \widehat{v_i w_j}|^2,\\
\|\pi(t)\|^2 \leq 2[\|u(t)\|_\infty^2 + \|v(t)\|_\infty^2]\ \|w(t)\|^2,\\
\|\nabla\pi(t)\|^2 \leq 4[\|\nabla u(t)\|^2 + \|\nabla v(t)\|^2]\|w(t)\|_\infty^2 + 4[\|u(t)\|_\infty^2 + \|v(t)\|_\infty^2]\|\nabla w(t)\|^2,\\
\|\Delta\pi(t)\|^2 \leq 2[\|\nabla u(t)\|_{L^4}^2 + \|\nabla v(t)\|_{L^4}^2]\ \|\nabla w(t)\|_{L^4}^2,\\
\|w_t(t)\|^2 \leq 4\|w(t)\|_\infty^2\|\nabla u(t)\|^2 + 4\|v(t)\|_\infty^2\|\nabla w(t)\|^2 + 4\|\Delta w(t)\|^2 + 4\|\nabla\pi(t)\|^2,\\
\|\pi_t(t)\|^2 \leq 4[\|u(t)\|_\infty^2 + \|v(t)\|_\infty^2]\|w_t(t)\|^2 + 4[\|u_t(t)\|^2 + \|v_t(t)\|^2]\|w_t(t)\|_\infty^2,\\
\int_{R^n} [f\cdot(g\cdot\nabla h) + h\cdot(g\cdot\nabla f)]dx = 0,\\
\int_{R^n} f\cdot(g\cdot\nabla f)dx = 0,\\
\left|\int_{R^n} [f\cdot(g\cdot\nabla h)dx\right| \leq \int_{R^n} |g||h||\nabla f|dx,
\end{gathered}
$$

where f, g and $h \in H^1$ with $\nabla\cdot g = 0$ are arbitrary vector valued functions.

Proof. Since

$$w \cdot \nabla u = \Big(\sum_{j=1}^{n} w_j \frac{\partial}{\partial x_j} u_i \Big)_{1 \le i \le n},$$

one has the estimate by Cauchy-Schwartz inequality

$$\begin{aligned} |w \cdot \nabla u|^2 &= \sum_{i=1}^{n} \Big| \sum_{j=1}^{n} w_j \frac{\partial}{\partial x_j} u_i \Big|^2 \\ &\le \sum_{i=1}^{n} \sum_{j=1}^{n} w_j^2 \sum_{j=1}^{n} \Big| \frac{\partial}{\partial x_j} u_i \Big|^2 = |w|^2 |\nabla u|^2. \end{aligned}$$

It is easy to get the following identities by using $\nabla \cdot w = 0$,

$$w \cdot \nabla u = \Big(\sum_{j=1}^{n} \frac{\partial}{\partial x_j} (u_i w_j) \Big)_{1 \le i \le n},$$

$$F[w \cdot \nabla u] = \sqrt{-1} \Big(\sum_{j=1}^{n} \xi_j \widehat{u_i w_j} \Big)_{1 \le i \le n}.$$

Thus we get

$$\begin{aligned} |F[w \cdot \nabla u]|^2 &= \sum_{i=1}^{n} \Big| \sum_{j=1}^{n} \xi_j \widehat{u_i w_j} \Big|^2 \le \sum_{i=1}^{n} \sum_{j=1}^{n} \xi_j^2 \sum_{j=1}^{n} |\widehat{u_i w_j}|^2 \\ &\le |\xi|^2 \sum_{i=1}^{n} \sum_{j=1}^{n} \|u_i(t)\|^2 \|w_j(t)\|^2 = |\xi|^2 \|u(t)\|^2 \|w(t)\|^2, \end{aligned}$$

where we have used $|\widehat{fg}| \le \|f(t)\| \; \|g(t)\|$.

Now $\nabla \cdot u = 0$ yields

$$\nabla \cdot (w \cdot \nabla u) = \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial w_j}{\partial x_i} \frac{\partial u_i}{\partial x_j}.$$

Hence one obtains

$$|\nabla \cdot (w \cdot \nabla u)|^2 \le \sum_{i=1}^{n} \sum_{j=1}^{n} \Big| \frac{\partial w_j}{\partial x_i} \Big|^2 \sum_{i=1}^{n} \sum_{j=1}^{n} \Big| \frac{\partial u_i}{\partial x_j} \Big|^2 = |\nabla w|^2 |\nabla u|^2.$$

Clearly

$$F[\nabla \cdot (w \cdot \nabla u)] = \sum_{i=1}^{n} \sum_{j=1}^{n} F\Big[\frac{\partial w_j}{\partial x_i} \frac{\partial u_i}{\partial x_j} \Big].$$

We now arrive at the estimate

$$|F[\nabla\cdot(w\cdot\nabla u)]|^2 \le \Big[\sum_{i=1}^n\sum_{j=1}^n \Big\|\frac{\partial w_j}{\partial x_i}(t)\Big\|\Big\|\frac{\partial u_i}{\partial x_j}(t)\Big\|\Big]^2 \\ \le \|\nabla w(t)\|^2\|\nabla u(t)\|^2.$$

Because

$$\begin{aligned}|\nabla(w\cdot\nabla u)|^2 &= \sum_{i=1}^n\sum_{j=1}^n\Big|\frac{\partial}{\partial x_j}\Big(\sum_{k=1}^n w_k\frac{\partial u_i}{\partial x_k}\Big)\Big|^2\\ &= \sum_{i=1}^n\sum_{j=1}^n\Big|\sum_{k=1}^n \frac{\partial w_k}{\partial x_j}\frac{\partial u_i}{\partial x_k} + \sum_{k=1}^n w_k\frac{\partial^2 u_i}{\partial x_j\partial x_k}\Big|^2\\ &\le 2\sum_{i=1}^n\sum_{j=1}^n\Big[\Big|\sum_{k=1}^n \frac{\partial w_k}{\partial x_j}\frac{\partial u_i}{\partial x_k}\Big|^2 + \Big|\sum_{k=1}^n w_k\frac{\partial^2 u_i}{\partial x_j\partial x_k}\Big|^2\Big]\\ &\le 2\sum_{i=1}^n\sum_{j=1}^n\Big[\sum_{k=1}^n\Big|\frac{\partial w_k}{\partial x_j}\Big|^2\sum_{k=1}^n\Big|\frac{\partial u_i}{\partial x_k}\Big|^2 + \sum_{k=1}^n w_k^2\sum_{k=1}^n\Big|\frac{\partial^2 u_i}{\partial x_j\partial x_k}\Big|^2\Big]\\ &= 2|\nabla u|^2|\nabla w|^2 + 2|w|^2\sum_{i=1}^n\sum_{j=1}^n\sum_{k=1}^n\Big|\frac{\partial^2 u_i}{\partial x_j\partial x_k}\Big|^2,\end{aligned}$$

and

$$\int_{R^n}\Big|\frac{\partial^2 u_i}{\partial x_j\partial x_k}\Big|^2 dx = \int_{R^n}\frac{\partial^2 u_i}{\partial x_j^2}\frac{\partial^2 u_i}{\partial x_k^2}dx,$$

$$\int_{R^n}\sum_{i=1}^n\sum_{j=1}^n\sum_{k=1}^n\Big|\frac{\partial^2 u_i}{\partial x_j\partial x_k}\Big|^2 dx = \|\Delta u(t)\|^2,$$

one obtains

$$\|\nabla(w\cdot\nabla u)(t)\|^2 \le 2\|\nabla u(t)\|_\infty^2\|\nabla w(t)\|^2 + 2\|\nabla w(t)\|_\infty^2\|\Delta u(t)\|^2.$$

The following identity follows by taking the divergence of equations (1)

$$\sum_{i=1}^n\sum_{j=1}^n\frac{\partial^2}{\partial x_i\partial x_j}(u_iu_j) + \Delta p = 0.$$

The Fourier transformation yields

$$\sum_{i=1}^n\sum_{j=1}^n \xi_i\xi_j\widehat{u_iu_j} + |\xi|^2\widehat{p} = 0.$$

Similarly one has

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\xi_i\xi_j\widehat{v_iv_j}+|\xi|^2\widehat{q}=0.$$

Thus we get the identity by subtracting the two equations

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\xi_i\xi_j(\widehat{w_iu_j}+\widehat{v_iw_j})+|\xi|^2\widehat{\pi}=0. \tag{11}$$

Triangle inequality and Cauchy-Schwartz inequality give the estimates

$$\begin{aligned}|\xi|^2|\widehat{\pi}| &\le \sum_{i=1}^{n}\sum_{j=1}^{n}|\xi_i||\xi_j|[\|w_i(t)\|\|u_j(t)\|+\|v_i(t)\|\|w_j(t)\|]\\ &=\sum_{i=1}^{n}|\xi_i|\|w_i(t)\|\sum_{j=1}^{n}|\xi_j|\|u_j(t)\|+\sum_{i=1}^{n}|\xi_i|\|v_i(t)\|\sum_{j=1}^{n}|\xi_j|\|w_j(t)\|\\ &\le |\xi|^2[\|u(t)\|+\|v(t)\|]\|w(t)\|.\end{aligned}$$

Therefore we get

$$|\widehat{\pi}|\le[\|u(t)\|+\|v(t)\|]\|w(t)\|.$$

On the other hand, we have the relation

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{\partial u_j}{\partial x_i}\frac{\partial u_i}{\partial x_j}+\Delta p=0,$$

which leads to

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\Big(\frac{\partial w_j}{\partial x_i}\frac{\partial u_i}{\partial x_j}+\frac{\partial v_j}{\partial x_i}\frac{\partial w_i}{\partial x_j}\Big)+\Delta\pi=0.$$

Applying the Fourier transformation gives

$$\sum_{i=1}^{n}\sum_{j=1}^{n}F\Big[\frac{\partial w_j}{\partial x_i}\frac{\partial u_i}{\partial x_j}+\frac{\partial v_j}{\partial x_i}\frac{\partial w_i}{\partial x_j}\Big]+\widehat{\Delta\pi}=0.$$

Therefore we get

$$\begin{aligned}
\|\widehat{\Delta\pi}(t)\|_\infty &\le \sum_{i=1}^n\sum_{j=1}^n \Big\|\frac{\partial w_j}{\partial x_i}(t)\Big\|\Big\|\frac{\partial u_i}{\partial x_j}(t)\Big\| + \sum_{i=1}^n\sum_{j=1}^n \Big\|\frac{\partial v_j}{\partial x_i}(t)\Big\|\Big\|\frac{\partial w_i}{\partial x_j}(t)\Big\| \\
&\le \Big(\sum_{i=1}^n\sum_{j=1}^n \Big\|\frac{\partial w_j}{\partial x_i}(t)\Big\|^2\Big)^{1/2}\Big(\sum_{i=1}^n\sum_{j=1}^n \Big\|\frac{\partial u_i}{\partial x_j}(t)\Big\|^2\Big)^{1/2} \\
&\quad + \Big(\sum_{i=1}^n\sum_{j=1}^n \Big\|\frac{\partial v_j}{\partial x_i}(t)\Big\|^2\Big)^{1/2}\Big(\sum_{i=1}^n\sum_{j=1}^n \Big\|\frac{\partial w_i}{\partial x_j}(t)\Big\|^2\Big)^{1/2} \\
&= [\|\nabla u(t)\| + \|\nabla v(t)\|]\|\nabla w(t)\|.
\end{aligned}$$

We can easily get the estimate by squaring both sides of (11)

$$\begin{aligned}
|\xi|^4|\widehat{\pi}|^2 &= \Big|\sum_{i=1}^n\sum_{j=1}^n \xi_i\xi_j(\widehat{w_iu_j} + \widehat{v_iw_j})\Big|^2 \\
&\le \Big|\sum_{i=1}^n\sum_{j=1}^n |\xi_i\xi_j|^2 \sum_{i=1}^n\sum_{j=1}^n |\widehat{w_iu_j} + \widehat{v_iw_j}|^2 \\
&\le |\xi|^4 \sum_{i=1}^n\sum_{j=1}^n |\widehat{w_iu_j} + \widehat{v_iw_j}|^2,
\end{aligned}$$

or

$$|\widehat{\pi}|^2 \le \sum_{i=1}^n\sum_{j=1}^n |\widehat{w_iu_j} + \widehat{v_iw_j}|^2.$$

Integrating this inequality with respect to ξ, we obtain

$$\begin{aligned}
\|\pi(t)\|^2 &= \frac{1}{(2\pi)^n}\|\widehat{\pi}(t)\|^2 \\
&\le \sum_{i=1}^n\sum_{j=1}^n \|(w_iu_j + v_iw_j)(t)\|^2 \\
&\le 2\sum_{i=1}^n\sum_{j=1}^n [\|u_j(t)\|_\infty^2\|w_i(t)\|^2 + \|v_i(t)\|_\infty^2\|w_j(t)\|^2] \\
&\le 2\|u(t)\|_\infty^2\|w(t)\|^2 + 2\|v(t)\|_\infty^2\|w(t)\|^2 \\
&\le 2[\|u(t)\|_\infty^2 + \|v(t)\|_\infty^2]\|w(t)\|^2,
\end{aligned}$$

and multiplying that inequality by $|\xi|^2$ and then integrating with respect to

ξ, we obtain

$$\begin{aligned}\|\nabla\pi(t)\|^2 &= \frac{1}{(2\pi)^n}\|\widehat{\nabla\pi}(t)\|^2 \\ &\leq \sum_{i=1}^n\sum_{j=1}^n \|\nabla(w_iu_j+v_iw_j)(t)\|^2 \\ &\leq 4\sum_{i=1}^n\sum_{j=1}^n [\|\nabla u_j(t)\|^2\|w_i(t)\|_\infty^2 + \|u_j(t)\|_\infty^2\|\nabla w_i(t)\|^2 \\ &\quad + \|v_i(t)\|_\infty^2\|\nabla w_j(t)\|^2 + \|\nabla v_i(t)\|^2\|w_j(t)\|_\infty^2] \\ &\leq 4[\|\nabla u(t)\|^2 + \|\nabla v(t)\|^2]\|w(t)\|_\infty^2 \\ &\quad + 4[\|u(t)\|_\infty^2 + \|v(t)\|_\infty^2]\|\nabla w(t)\|^2.\end{aligned}$$

Taking the divergence of equations (3) to get

$$\nabla\cdot(w\cdot\nabla u + v\cdot\nabla w) + \Delta\pi = 0.$$

By using this identity we get the following

$$\begin{aligned}\|\Delta\pi(t)\|^2 &= \|\nabla\cdot(w\cdot\nabla u + v\cdot\nabla w)(t)\|^2 \\ &\leq 2\int_{R^2}|\nabla u|^2|\nabla w|^2dx + 2\int_{R^2}|\nabla v|^2|\nabla w|^2dx \\ &\leq 2[\|\nabla u(t)\|_{L^4}^2 + \|\nabla v(t)\|_{L^4}^2]\|\nabla w(t)\|_{L^4}^2.\end{aligned}$$

The following estimates follow directly from equations (3)

$$\begin{aligned}\|w_t(t)\|^2 &\leq 4\|(w\cdot\nabla u)(t)\|^2 + 4\|(v\cdot\nabla w)(t)\|^2 \\ &\quad + 4\|\Delta w(t)\|^2 + 4\|\nabla\pi(t)\|^2 \\ &\leq 4\|w(t)\|_\infty^2\|\nabla u(t)\|^2 + 4\|v(t)\|_\infty^2\|\nabla w(t)\|^2 \\ &\quad + 4\|\Delta w(t)\|^2 + 4\|\nabla\pi(t)\|^2.\end{aligned}$$

By differentiating (11) with respect to t, we have

$$\sum_{i=1}^n\sum_{j=1}^n \xi_i\xi_jF[w_{it}u_j + w_iu_{jt} + v_{it}w_j + v_iw_{jt}] + |\xi|^2\widehat{\pi}_t = 0.$$

Thus

$$
\begin{aligned}
|\xi|^4|\widehat{\pi}_t|^2 &\le \sum_{i=1}^n\sum_{j=1}^n |\xi_i\xi_j|^2 \sum_{i=1}^n\sum_{j=1}^n |F[w_{it}u_j + w_iu_{jt} + v_{it}w_j + v_iw_{jt}]|^2 \\
&= |\xi|^4 \sum_{i=1}^n\sum_{j=1}^n |F[w_{it}u_j + w_iu_{jt} + v_{it}w_j + v_iw_{jt}]|^2,
\end{aligned}
$$

or equivalently, we have

$$
|\widehat{\pi}_t|^2 \le \sum_{i=1}^n\sum_{j=1}^n |F[w_{it}u_j + w_iu_{jt} + v_{it}w_j + v_iw_{jt}]|^2.
$$

Now by integrating this inequality, we get

$$
\begin{aligned}
\|\pi_t(t)\|^2 &= \frac{1}{(2\pi)^n}\|\widehat{\pi}_t(t)\|^2 \\
&\le \frac{1}{(2\pi)^n}\sum_{i=1}^n\sum_{j=1}^n \|F[w_{it}u_j + w_iu_{jt} + v_{it}w_j + v_iw_{jt}](t)\|^2 \\
&= \sum_{i=1}^n\sum_{j=1}^n \|(w_{it}u_j + w_iu_{jt} + v_{it}w_j + v_iw_{jt})(t)\|^2 \\
&\le 4\sum_{i=1}^n\sum_{j=1}^n [\|u_j(t)\|_\infty^2\|w_{it}(t)\|^2 + \|w_i(t)\|_\infty^2\|u_{jt}(t)\|^2 \\
&\quad + \|w_j(t)\|_\infty^2\|v_{it}(t)\|^2 + \|v_i(t)\|_\infty^2\|w_{jt}(t)\|^2] \\
&\le 4\|u(t)\|_\infty^2\|w_t(t)\|^2 + 4\|w(t)\|_\infty^2\|u_t(t)\|^2 \\
&\quad + 4\|w(t)\|_\infty^2\|v_t(t)\|^2 + 4\|v(t)\|_\infty^2\|w_t(t)\|^2 \\
&\le 4[\|u(t)\|_\infty^2 + \|v(t)\|_\infty^2]\|w_t(t)\|^2 \\
&\quad + 4[\|u_t(t)\|^2 + \|v_t(t)\|^2]\|w(t)\|_\infty^2.
\end{aligned}
$$

Since

$$
\begin{aligned}
\int_{R^n} f\cdot(g\cdot\nabla h)dx &= \int_{R^n}\sum_{i=1}^n\sum_{j=1}^n f_ig_j\frac{\partial h_i}{\partial x_j}dx \\
&= \int_{R^n}\sum_{i=1}^n\sum_{j=1}^n f_i\frac{\partial}{\partial x_j}(g_jh_i)dx
\end{aligned}
$$

$$= -\int_{R^n} \sum_{i=1}^{n} \sum_{j=1}^{n} g_j h_i \frac{\partial f_i}{\partial x_j} dx$$

$$= -\int_{R^n} h \cdot (g \cdot \nabla f) dx,$$

we get

$$\int_{R^n} [f \cdot (g \cdot \nabla h) dx + h \cdot (g \cdot \nabla f)] dx = 0,$$

$$\int_{R^n} f \cdot (g \cdot \nabla f) dx = 0.$$

In addtion, we have

$$\Big| \sum_{i=1}^{n} \sum_{j=1}^{n} g_j h_i \frac{\partial f_i}{\partial x_j} \Big|^2 \leq \Big(\sum_{i=1}^{n} \sum_{j=1}^{n} |g_j h_i|^2 \Big) \Big(\sum_{i=1}^{n} \sum_{j=1}^{n} \Big| \frac{\partial f_i}{\partial x_j} \Big|^2 \Big) = |g|^2 |h|^2 |\nabla f|^2,$$

$$\Big| \int_{R^n} f \cdot (g \cdot \nabla h) dx \Big| = \Big| \int_{R^n} \sum_{i=1}^{n} \sum_{j=1}^{n} g_j h_i \frac{\partial f_i}{\partial x_j} dx \Big| \leq \int_{R^n} |g||h||\nabla f| dx.$$

Lemma 2.2. *The following preliminary estimates hold for problem* (1–2)

$$|u \cdot \nabla u|^2 \leq |u|^2 |\nabla u|^2,$$

$$|F[u \cdot \nabla u]|^2 \leq |\xi|^2 \|u(t)\|^4,$$

$$|\nabla \cdot (u \cdot \nabla u)|^2 \leq |\nabla u|^4,$$

$$|F[\nabla \cdot (u \cdot \nabla u)]|^2 \leq \|\nabla u(t)\|^4,$$

$$\|\nabla (u \cdot \nabla u)(t)\|^2 \leq 2\|\nabla u(t)\|^2 \|\nabla u(t)\|_\infty^2 + 2\|u(t)\|_\infty^2 \|\Delta u(t)\|^2,$$

$$\|\widehat{p}(t)\|_\infty \leq \|u(t)\|^2,$$

$$\|\widehat{\Delta p}(t)\|_\infty \leq \|\nabla u(t)\|^2,$$

$$\|p(t)\|^2 \leq \|u(t)\|_\infty^2 \|u(t)\|^2,$$

$$\|\nabla p(t)\|^2 \leq 4\|u(t)\|_\infty^2 \|\nabla u(t)\|^2,$$

$$\|\Delta p(t)\|^2 \leq \|\nabla u(t)\|_{L^4}^4,$$

$$\|u_t(t)\|^2 \leq 3\|u(t)\|_\infty^2 \|\nabla u(t)\|^2 + 3\|\Delta u(t)\|^2 + 3\|\nabla p(t)\|^2,$$

$$\|p_t(t)\|^2 \leq 4\|u(t)\|_\infty^2 \|u_t(t)\|^2.$$

This proof is parallel to that of Lemma 2.1 and is omitted.

Lemma 2.3. *Let u_0 and $v_0 \in H^2$, let $w_0 \in L^1$. Then*

$$\begin{aligned}|\widehat{w}| &\le |\widehat{w_0}| + 2|\xi| \int_0^t [\|u(s)\| + \|v(s)\|]\|w(s)\|ds\\ &\le \|w_0\|_{L^1} + 2|\xi| \int_0^t [\|u(s)\| + \|v(s)\|]\|w(s)\|ds.\end{aligned}$$

Proof. Applying the Fourier transformation to the equations (3) yields

$$\widehat{w_t} + |\xi|^2\widehat{w} + F[w \cdot \nabla u + v \cdot \nabla w + \nabla \pi] = 0.$$

It follows easily that

$$\left[\widehat{w}e^{|\xi|^2 t}\right]_t + F[w \cdot \nabla u + v \cdot \nabla w + \nabla \pi]e^{|\xi|^2 t} = 0.$$

Integrating in time gives

$$\widehat{w} = \widehat{w_0}e^{-|\xi|^2 t} - \int_0^t F[w \cdot \nabla u + v \cdot \nabla w + \nabla \pi](\xi, s)e^{-|\xi|^2(t-s)}ds.$$

Therefore by using the estimates in Lemma 2.1, we obtain

$$\begin{aligned}|\widehat{w}| &\le |\widehat{w_0}| + \int_0^t |F[w \cdot \nabla u + v \cdot \nabla w + \nabla \pi]|ds\\ &\le |\widehat{w_0}| + 2|\xi| \int_0^t [\|u(s)\| + \|v(s)\|]\|w(s)\|ds\\ &\le \|w_0|_{L^1} + 2|\xi| \int_0^t [\|u(s)\| + \|v(s)\|]\|w(s)\|ds.\end{aligned}$$

The following lemma is due to Schonbek [11].

Lemma 2.4. *Let $g = g(x,t)$ and $h = h(x,t) \in L^\infty(0,\infty; H^1(R^n))$ satisfy the energy inequality*

$$\begin{aligned}&\frac{d}{dt}[(1+t)^l\|g(t)\|^2] + \frac{2}{3}(1+t)^l\|\nabla g(t)\|^2\\ &\le C(1+t)^{l-1}\|g(t)\|^2 + C(1+t)^k\|h(t)\|^2,\end{aligned}$$

where $k \geq 0$ and $l \geq 0$ are real numbers. Let $B(t) = \{\xi \in R^n | (1+t)|\xi|^2 \leq 2C\}$. Then we have the estimate

$$\frac{d}{dt}[(1+t)^l \|g(t)\|^2] + (1+t)^l \|\nabla g(t)\|^2$$
$$\leq \frac{C}{(2\pi)^n}(1+t)^{l-1} \int_{B(t)} |\widehat{g}|^2 d\xi + C(1+t)^k \|h(t)\|^2.$$

Proof. Rewrite the given inequality as

$$\frac{d}{dt}\left[(1+t)^l \int_{R^n} |\widehat{g}|^2 d\xi\right] + \frac{2}{3}(1+t)^l \int_{R^n} |\xi|^2 |\widehat{g}|^2 d\xi$$
$$\leq C(1+t)^{l-1} \int_{R^n} |\widehat{g}|^2 d\xi + C(1+t)^k \int_{R^n} |\widehat{h}|^2 d\xi.$$

Since $R^n = B(t) \cup B(t)^c$, we get

$$\frac{1}{2}(1+t)^l \int_{R^n} |\xi|^2 |\widehat{g}|^2 d\xi$$
$$= \frac{1}{2}(1+t)^l \int_{B(t)} |\xi|^2 |\widehat{g}|^2 d\xi + \frac{1}{2}(1+t)^l \int_{B(t)^c} |\xi|^2 |\widehat{g}|^2 d\xi$$
$$\geq C(1+t)^{l-1} \int_{B(t)^c} |\widehat{g}|^2 d\xi$$
$$= C(1+t)^{l-1} \int_{R^n} |\widehat{g}|^2 d\xi - C(1+t)^{l-1} \int_{B(t)} |\widehat{g}|^2 d\xi.$$

Substituting this for the original one and using Parseval's identity, the lemma is proved.

Lemma 2.5. (Gagliardo-Nirenberg's inequality) *For all $1 \leq p, q, r \leq \infty$ and for all integers $n \geq 1$ and $m > k \geq 0$, there are constant $\alpha : k/m \leq \alpha \leq 1$ and $C > 0$ such that for all $u \in C_0^\infty(R^n)$,*

$$\|D^k u\|_{L^p} \leq C \|D^m u\|_{L^r}^{\alpha} \|u\|_{L^q}^{1-\alpha}, \quad \textit{where}$$
$$n/p - k = \alpha(n/r - m) + (1-\alpha)n/q,$$
$$\|D^k u\|_{L^p}^p = \sum_{\beta_1+\cdots+\beta_n=k} \left\| \frac{\partial^{\beta_1+\cdots+\beta_n} u}{\partial x_1^{\beta_1} \cdots \partial x_n^{\beta_n}} \right\|_{L^p}^p.$$

The only exception is that $\alpha \neq 1$ if $m - n/r = k$ and $1 < p < \infty$.

Because C_0^∞ is dense in $W^{m,r} \cap L^q$, the interpolation inequality holds for all functions in $W^{m,r} \cap L^q$.

Lemma 2.6. (Generalized Gronwall's inequality) *Let $g(t) \geq 0$ and $h(t) \geq 0$ satisfy the inequality*

$$g(t) \leq C + \int_0^t g(s)h(s)ds,$$

where $C \geq 0$ is a constant, and $h(t) \in L^1(0,\infty)$. Then we have the estimate

$$g(t) \leq C \exp\left[\int_0^\infty h(t)dt\right].$$

3. Decay of solutions of problem (1–2) with $n = 2$. Previous papers dealt with uniform decay estimates for the solutions of (1–2) for the case $u_0 \in L^r \cap L^2$ with $1 \leq r < 2$. If for all $1 \leq r < 2$, the velocity $u_0 \notin L^r$, but $u_0 \in L^2$, then no known decay result exists.

Let $u_0 \in L^2$ and $u_0 \notin L^r$ for all $1 \leq r < 2$, let g be the solution of the heat equation

$$g_t - \Delta g = 0, \quad g(x,0) = u_0.$$

Because

$$(2\pi)^2 \int_{R^2} |g|^2 dx = \int_{R^2} |\widehat{g}|^2 d\xi = \int_{R^2} |\widehat{u_0}|^2 e^{-2|\xi|^2 t} d\xi,$$

one obtains the following by using dominant convergence theorem

$$\lim_{t\to\infty} \|g(t)\| = 0.$$

The same asymptotic behavior also holds for the solutions of problem (1–2).

Before we begin our main work on problem (3–4), we need to establish some decay estimates for the solutions of problem (1–2) with $u_0 \in H^2$ and $n = 2$.

The proof of Theorem 3, case (1) with $\beta = 0$ and case (2), will be given by the following lemmas.

Lemma 3.1. *Let $u_0 \in H^2$. Then we have the following estimates*

$$\sup_{0\le t<\infty} \|u(t)\|^2 \le \|u_0\|^2,$$

$$2\int_0^\infty \|\nabla u(t)\|^2 dt \le \|u_0\|^2,$$

$$\sup_{0\le t<\infty} \|\nabla u(t)\|^2 \le \|\nabla u_0\|^2,$$

$$2\int_0^\infty \|\Delta u(t)\|^2 dt \le \|\nabla u_0\|^2,$$

$$\sup_{0\le t<\infty} [(1+t)\|\nabla u(t)\|^2] \le \|u_0\|_1^2,$$

$$2\int_0^\infty (1+t)\|\Delta u(t)\|^2 dt \le \|u_0\|_1^2,$$

$$\lim_{t\to\infty} [t\|\nabla u(t)\|^2] = 0.$$

Proof. Making the scalar product of (1) and $2u$ and integrating over R^2, we get

$$\frac{d}{dt}\|u(t)\|^2 + 2\|\nabla u(t)\|^2 = 0.$$

It follows from this identity that the first pair estimates hold.

Let $A = \frac{\partial u_1}{\partial x_2} - \frac{\partial u_2}{\partial x_1}$. Since $\nabla \cdot u = 0$, we get

$$A_t - \Delta A + \sum_{i=1}^{2} u_i \frac{\partial A}{\partial x_i} = 0, \quad A(x,0) = \frac{\partial u_{01}}{\partial x_2} - \frac{\partial u_{02}}{\partial x_1}.$$

Multiplying this equation by $2A$ and integrating over R^2, we get

$$\frac{d}{dt}\|A(t)\|^2 + 2\|\nabla A(t)\|^2 = 0.$$

It is not hard to see that $\|\nabla u(t)\|^2 = \|A(t)\|^2$ and $\|\Delta u(t)\|^2 = \|\nabla A(t)\|^2$. In fact

$$\begin{aligned}
\|A(t)\|^2 &= \int_{R^2} \left|\frac{\partial u_1}{\partial x_2} - \frac{\partial u_2}{\partial x_1}\right|^2 dx \\
&= \int_{R^2} \left[\left|\frac{\partial u_1}{\partial x_2}\right|^2 + \left|\frac{\partial u_2}{\partial x_1}\right|^2 - 2\frac{\partial u_1}{\partial x_1}\frac{\partial u_2}{\partial x_2}\right] ds, \quad \text{and} \\
0 &= \int_{R^2} \left|\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2}\right|^2 dx
\end{aligned}$$

$$= \int_{R^2} \Big[\Big|\frac{\partial u_1}{\partial x_1}\Big|^2 + \Big|\frac{\partial u_2}{\partial x_2}\Big|^2 + 2\frac{\partial u_1}{\partial x_1}\frac{\partial u_2}{\partial x_2} \Big] dx,$$

$$\|A(t)\|^2 = \int_{R^2} \Big[\Big|\frac{\partial u_1}{\partial x_1}\Big|^2 + \Big|\frac{\partial u_1}{\partial x_2}\Big|^2 + \Big|\frac{\partial u_2}{\partial x_1}\Big|^2 + \Big|\frac{\partial u_2}{\partial x_2}\Big|^2 \Big] dx$$

$$= \|\nabla u(t)\|^2.$$

Similarly one has

$$\|\nabla A(t)\|^2 = \int_{R^2} \Big[\Big|\frac{\partial^2 u_1}{\partial x_1 \partial x_2} - \frac{\partial^2 u_2}{\partial x_1^2}\Big|^2 + \Big|\frac{\partial^2 u_1}{\partial x_2^2} - \frac{\partial^2 u_2}{\partial x_1 \partial x_2}\Big|^2 \Big] dx$$

$$= \int_{R^2} \Big[\Big|\frac{\partial^2 u_1}{\partial x_1 \partial x_2}\Big|^2 + \Big|\frac{\partial^2 u_2}{\partial x_1^2}\Big|^2 + \Big|\frac{\partial^2 u_1}{\partial x_2^2}\Big|^2 + \Big|\frac{\partial^2 u_2}{\partial x_1 \partial x_2}\Big|^2 \Big] dx$$

$$- 2\int_{R^2} \Big[\frac{\partial^2 u_1}{\partial x_1^2}\frac{\partial^2 u_2}{\partial x_1 \partial x_2} + \frac{\partial^2 u_1}{\partial x_1 \partial x_2}\frac{\partial^2 u_2}{\partial x_2^2} \Big] dx, \quad \text{and}$$

$$0 = \int_{R^2} \Big[\Big|\frac{\partial^2 u_1}{\partial x_1^2} + \frac{\partial^2 u_2}{\partial x_1 \partial x_2}\Big|^2 + \Big|\frac{\partial^2 u_1}{\partial x_1 \partial x_2} + \frac{\partial^2 u_2}{\partial x_2^2}\Big|^2 \Big] dx$$

$$= \int_{R^2} \Big[\Big|\frac{\partial^2 u_1}{\partial x_1^2}\Big|^2 + \Big|\frac{\partial^2 u_2}{\partial x_1 \partial x_2}\Big|^2 + \Big|\frac{\partial^2 u_1}{\partial x_1 \partial x_2}\Big|^2 + \Big|\frac{\partial^2 u_2}{\partial x_2^2}\Big|^2 \Big] dx$$

$$+ 2\int_{R^2} \Big[\frac{\partial^2 u_1}{\partial x_1^2}\frac{\partial^2 u_2}{\partial x_1 \partial x_2} + \frac{\partial^2 u_1}{\partial x_1 \partial x_2}\frac{\partial^2 u_2}{\partial x_2^2} \Big] dx,$$

$$\|\nabla A(t)\|^2 = \int_{R^2} \Big[\Big|\frac{\partial^2 u_1}{\partial x_1^2}\Big|^2 + 2\Big|\frac{\partial^2 u_1}{\partial x_1 \partial x_2}\Big|^2 + \Big|\frac{\partial^2 u_1}{\partial x_2^2}\Big|^2 \Big] dx$$

$$+ \int_{R^2} \Big[\Big|\frac{\partial^2 u_2}{\partial x_1^2}\Big|^2 + 2\Big|\frac{\partial^2 u_2}{\partial x_1 \partial x_2}\Big|^2 + \Big|\frac{\partial^2 u_2}{\partial x_2^2}\Big|^2 \Big] dx$$

$$= \|\Delta u(t)\|^2.$$

Thus we have

$$\frac{d}{dt}\|\nabla u(t)\|^2 + 2\|\Delta u(t)\|^2 = 0.$$

Therefore we obtain the second pair estimates. Moreover we observe that

$$\frac{d}{dt}[(1+t)\|\nabla u(t)\|^2] + 2(1+t)\|\Delta u(t)\|^2 = \|\nabla u(t)\|^2.$$

If integrating in t, one obtains

$$(1+t)\|\nabla u(t)\|^2 + 2\int_0^t (1+s)\|\Delta u(s)\|^2 ds$$

$$= \|\nabla u_0\|^2 + \int_0^t \|\nabla u(s)\|^2 ds$$
$$\leq \|\nabla u_0\|^2 + \frac{1}{2}\|u_0\|^2 \leq \|u_0\|_1^2.$$

The third pair estimates follow now. Because $\|\nabla u(t)\|^2$ is monotonically decreasing, if $0 < s < t < \infty$, we have

$$\int_s^\infty \|\nabla u(\tau)\|^2 d\tau \geq \int_s^t \|\nabla u(\tau)\|^2 d\tau \geq (t-s)\|\nabla u(t)\|^2.$$

Thus we obtain

$$\int_s^\infty \|\nabla u(\tau)^2\| d\tau \geq \limsup_{t\to\infty}[(t-s)\|\nabla u(t)\|^2] = \limsup_{t\to\infty}[t\|\nabla u(t)\|^2].$$

If we choose s large enough, the quality in left hand side can be made as small as desired. Therefore

$$\lim_{t\to\infty} [t\|u(t)\|^2] = 0.$$

Lemma 3.2. *Let (u,p) be the solution of problem (1–2) corresponding to the initial velocity $u_0 \in H^2$. Then*

$$\sup_{0\leq t<\infty} [(1+t)^2\|\Delta u(t)\|^2] \leq \|u_0\|_2^2 \exp[16\|u_0\|^4],$$
$$\int_0^\infty (1+t)^2\|\nabla\Delta u(t)\|^2 dt \leq \|u_0\|_2^2 \exp[16\|u_0\|^4].$$

Proof. If we make the scalar product of equations (1) and $2\Delta^2 u$, integrate over R^2, we get

$$\frac{d}{dt}\|\Delta u(t)\|^2 + 2\|\nabla\Delta u(t)\|^2 = 2\int_{R^2} \nabla\Delta u \cdot \nabla(u\cdot\nabla u)dx,$$

where

$$\int_{R^2} \Delta^2 u \cdot \nabla p dx = 0.$$

We have the following estimates.

$$\begin{aligned}\|\nabla(u\cdot\nabla u)(t)\|^2 &\le 2\|\nabla u(t)\|^2\|\nabla u(t)\|_\infty^2 + 2\|u(t)\|_\infty^2\|\Delta u(t)\|^2\\ &\le 4\|u(t)\|\|\Delta u(t)\|\|\nabla u(t)\|\|\nabla\Delta u(t)\|\\ &\le \frac{1}{4}\|\nabla\Delta u(t)\|^2 + 16\|u(t)\|^2\|\nabla u(t)\|^2\|\Delta u(t)\|^2.\end{aligned}$$

Therefore

$$\begin{aligned}2\int_{R^2}\nabla\Delta u\cdot\nabla(u\cdot\nabla u)dx &\le \frac{1}{2}\|\nabla\Delta u(t)\|^2 + 2\|\nabla(u\cdot\nabla u)(t)\|^2\\ &\le \|\nabla\Delta u(t)\|^2 + 32\|u(t)\|^2\|\nabla u(t)\|^2\|\Delta u(t)\|^2.\end{aligned}$$

Therefore we get

$$\frac{d}{dt}\|\Delta u(t)\|^2 + \|\nabla\Delta u(t)\|^2 \le 32\|u(t)\|^2\|\nabla u(t)\|^2\|\Delta u(t)\|^2,$$

or

$$\begin{aligned}&\frac{d}{dt}[(1+t)^2\|\Delta u(t)\|^2] + (1+t)^2\|\nabla\Delta u(t)\|^2\\ &\le 2(1+t)\|\Delta u(t)\|^2 + 32(1+t)^2\|u(t)\|^2\|\nabla u(t)\|^2\|\Delta u(t)\|^2.\end{aligned}$$

Integrating in time yields

$$\begin{aligned}&(1+t)^2\|\Delta u(t)\|^2 + \int_0^t (1+s)^2\|\nabla\Delta u(s)\|^2 ds\\ &\le \|\Delta u_0\|^2 + 2\int_0^\infty (1+t)\|\Delta u(t)\|^2 dt\\ &\quad + 32\int_0^t \|u(s)\|^2\|\nabla u(s)\|^2(1+s)^2\|\Delta u(s)\|^2 ds\\ &\le \|\Delta u_0\|_2^2 + 32\|u_0\|^2\int_0^t \|\nabla u(s)\|^2(1+s)^2\|\Delta u(s)\|^2 ds.\end{aligned}$$

Gronwall's inequality yields

$$\begin{aligned}&(1+t)^2\|\Delta u(t)\|^2 + \int_0^t (1+s)^2\|\nabla\Delta u(s)\|^2 ds\\ &\le \|u_0\|_2^2\exp\left[32\|u_0\|^2\int_0^\infty \|\nabla u(t)\|^2 dt\right]\\ &\le \|u_0\|_2^2\exp[16\|u_0\|^4].\end{aligned}$$

Lemma 3.3. *Let (u, p) be the solutions of problem (1–2) corresponding to the initial velocity $u_0 \in H^2$. Then*

$$\lim_{t \to \infty} [\|u(t)\|^2 + (1+t)\|p(t)\|^2 + (1+t)^2 \|\nabla p(t)\|^2] = 0.$$

Proof. By the equation

$$\frac{d}{dt}\|u(t)\|^2 = -2\|\nabla u(t)\|^2 \leq 0,$$

one concludes the limit

$$\lim_{t \to \infty} \|u(t)\| = 2\lambda \geq 0,$$

exists. Suppose that $\lambda > 0$. Let

$$v_{0\varepsilon}(x) = \frac{u_0(x)}{1 + \varepsilon |x|^2},$$

where $\varepsilon > 0$ is a constant. Then $v_{0\varepsilon} \in L^1 \cap L^2$ and

$$\|v_{0\varepsilon}\| \leq \|u_0\|, \qquad \|v_{0\varepsilon}\|_{L^1} \leq \sqrt{\pi/\varepsilon}\|u_0\|.$$

Moreover, $v_{0\varepsilon}$ converges to u_0 in L^2, as $\varepsilon \to 0$, by Lebesgue's dominant convergence theorem.

By Lemma 4.2, which only assumes the condition u_0 and $v_0 \in L^2$, we have the stability estimate $\|u(t) - v_\varepsilon(t)\| \leq C\|u_0 - v_{0\varepsilon}\|$, where C is independent of $v_{0\varepsilon}$ and t. Choose $\varepsilon_0 > 0$ sufficiently small, such that for all $t > 0$,

$$\|u(t) - v_{\varepsilon_0}(t)\| \leq C\|u_0 - v_{0\varepsilon}\| \leq \frac{\lambda}{2}.$$

By Zhang [24], the solution v_{ε_0} of problem (1–2) corresponding to $v_{0\varepsilon_0}$ enjoys the basic decay estimate

$$\begin{aligned} \|v_{\varepsilon_0}(t)\| &\leq C[\ln(e+t)]^{-1}[\|v_{0\varepsilon_0}\|_{L^1} + \|v_{0\varepsilon_0}\|^2] \\ &\leq C[\ln(e+t)]^{-1}[\sqrt{\pi/\varepsilon_0} + \|u_0\|]\|u_0\|. \end{aligned}$$

Choose t_0 sufficiently large such that

$$C[\ln(e+t_0)]^{-1}[\sqrt{\pi/\varepsilon_0}+\|u_0\|]\|u_0\| \le \frac{\lambda}{2}.$$

Therefore one obtains

$$\|u(t)\| \le \|u(t)-v_{\varepsilon_0}(t)\| + \|v_{\varepsilon_0}(t)\| \le \lambda,$$

for all $t > t_0$. Letting $t \to \infty$, we obtain a contradiction. So the only possibility is $\lambda = 0$. This completes the proof of the first limit.

Other limits can be proved by using the following estimates

$$\begin{aligned}
(1+t)\|p(t)\|^2 &\le (1+t)\|u(t)\|_\infty^2\|u(t)\|^2 \\
&\le \|u(t)\|^3(1+t)\|\Delta u(t)\|, \\
(1+t)^2\|\nabla p(t)\|^2 &\le 4(1+t)^2\|u(t)\|_\infty^2\|\nabla u(t)\|^2 \\
&\le 4\|u(t)\|^2(1+t)^2\|\Delta u(t)\|^2, \\
(1+t)\|\Delta u(t)\| &\le C.
\end{aligned}$$

Lemma 3.4. *Proof of* Theorem 3, *case* (2).

Proof. If we make the scalar product of equations (1) and $2\Delta^3 u$, integrate over R^2, we get

$$\frac{d}{dt}\|\nabla\Delta u(t)\|^2 + 2\|\Delta^2 u(t)\|^2 = 2\int_{R^2}\Delta^2 u\cdot\Delta(u\cdot\nabla u)dx.$$

We have the following bounds

$$\begin{aligned}
&2\int_{R^2}\Delta^2 u\cdot\Delta(u\cdot\nabla u)dx \\
&\le \frac{1}{2}\|\Delta^2 u(t)\|^2 + 2\int_{R^2}|\Delta(u\cdot\nabla u)|^2dx,
\end{aligned}$$

$$\begin{aligned}
&|\Delta(u\cdot\nabla u)|^2 \\
&= \sum_{i=1}^n\Big|\sum_{j=1}^n\Delta u_j\frac{\partial u_i}{\partial x_j} + \sum_{j=1}^n u_j\frac{\partial\Delta u_i}{\partial x_j} + 2\sum_{j=1}^n\sum_{k=1}^n\frac{\partial u_j}{\partial x_k}\frac{\partial^2 u_i}{\partial x_j\partial x_k}\Big|^2 \\
&\le 4\sum_{i=1}^n\Big|\sum_{j=1}^n\Delta u_j\frac{\partial u_i}{\partial x_j}\Big|^2 + 4\sum_{i=1}^n\Big|\sum_{j=1}^n u_j\frac{\partial\Delta u_i}{\partial x_j}\Big|^2 + 8\sum_{i=1}^n\Big|\sum_{j=1}^n\sum_{k=1}^n\frac{\partial u_j}{\partial x_k}\frac{\partial^2 u_i}{\partial x_j\partial x_k}\Big|^2 \\
&\le 4|\nabla u|^2|\Delta u|^2 + 4|u|^2|\nabla\Delta u|^2 + 8|\nabla u|^2|\Delta u|^2, \quad \text{and}
\end{aligned}$$

$$
\begin{aligned}
&2\int_{R^2}|\Delta(u\cdot\nabla u)|^2dx\\
&\le 24\|\nabla u(t)\|_\infty^2\|\Delta u(t)\|^2+8\|u(t)\|_\infty^2\|\nabla\Delta u(t)\|^2\\
&\le 24\|\nabla u(t)\|^2\|\nabla\Delta u(t)\|^2+8\|u(t)\|\|\Delta u(t)\|^2\|\Delta^2u(t)\|\\
&\le 24\|\nabla u(t)\|^2\|\nabla\Delta u(t)\|^2+\frac{1}{2}\|\Delta^2u(t)\|^2+32\|u(t)\|^2\|\nabla u(t)\|^2\|\nabla\Delta u(t)\|^2.
\end{aligned}
$$

Therefore we get

$$\frac{d}{dt}\|\nabla\Delta u(t)\|^2+\|\Delta^2u(t)\|^2\le C\|\nabla u(t)\|^2\|\nabla\Delta u(t)\|^2,$$

or we have the inequality

$$
\begin{aligned}
&\frac{d}{dt}[(1+t)^3\|\nabla\Delta u(t)\|^2]+(1+t)^3\|\Delta^2u(t)\|^2\\
&\le 3(1+t)^2\|\nabla\Delta u(t)\|^2+C(1+t)^3\|\nabla u(t)\|^2\|\nabla\Delta u(t)\|^2\\
&\le 3(1+t)^2\|\Delta u(t)\|\|\Delta^2u(t)\|+C(1+t)^3\|\nabla u(t)\|^2\|\nabla\Delta u(t)\|^2\\
&\le 5(1+t)\|\Delta u(t)\|^2+\frac{1}{2}\|\Delta^2u(t)\|^2+C(1+t)^3\|\nabla u(t)\|^2\|\nabla\Delta u(t)\|^2.
\end{aligned}
$$

As before, one easily obtain the following estimates

$$
\begin{aligned}
&(1+t)^3\|\nabla\Delta u(t)\|^2+\frac{1}{2}\int_0^t(1+s)^3\|\Delta^2u(s)\|^2ds\\
&\le\|\nabla\Delta u_0\|^2+5\int_0^\infty(1+t)\|\Delta u(t)\|^2dt+C\int_0^t\|\nabla u(s)\|^2(1+s)^3\|\nabla\Delta u(s)\|^2ds\\
&\le\|u_0\|_3^2+C\int_0^t\|\nabla u(s)\|^2(1+s)^3\|\nabla\Delta u(s)\|^2ds,\quad\text{and}\\
&(1+t)^3\|\nabla\Delta u(t)\|^2+\int_0^t(1+s)^3\|\Delta^2u(s)\|^2ds\\
&\le\|\nabla\Delta u_0\|^2\exp\Big[C\int_0^\infty\|\nabla u(t)\|^2dt\Big].
\end{aligned}
$$

Therefore

$$
\begin{aligned}
&\sup_{0\le t<\infty}[(1+t)^3\|\nabla\Delta u(t)\|^2]\le\|\nabla\Delta u_0\|^2\exp[C\|u_0\|^2],\\
&\int_0^t(1+t)^3\|\Delta^2u(t)\|^2dt\le\|\nabla\Delta u_0\|^2\exp[C\|u_0\|^2].
\end{aligned}
$$

By using Lemma 2.5, it is very easy to get the estimates

$$(1+t)\|u(t)\|_\infty^2 \le \|u(t)\|(1+t)\|\Delta u(t)\| \le C,$$
$$(1+t)^2\|\nabla u(t)\|_\infty^2 \le \|\nabla u(t)\|(1+t)^2\|\nabla\Delta u(t)\| \le C.$$

The following follows directly from equations (1)

$$\begin{aligned}\|\nabla u_t(t)\|^2 &\le 3\|\nabla(u\cdot\nabla u)(t)\|^2 + 3\|\nabla\Delta u(t)\|^2 + 3\|\nabla^2 p(t)\|^2\\
&\le 6\|\nabla u(t)\|_\infty^2\|\nabla u(t)\|^2 + 6\|u(t)\|_\infty^2\|\Delta u(t)\|^2\\
&\quad + 3\|\nabla\Delta u(t)\|^2 + 3\|\Delta p(t)\|^2\\
&\le 6\|\nabla u(t)\|^3\|\nabla\Delta u(t)\| + 6\|u(t)\|\|\Delta u(t)\|^3\\
&\quad + 3\|\nabla\Delta u(t)\|^2 + 3\|\Delta p(t)\|^2.\end{aligned}$$

By Lemma 3.1–3.3, we get

$$(1+t)^3\|\nabla u_t(t)\|^2 \le C.$$

Similar to Lemma 2.1, it is easy to get the estimate

$$|\widehat{\nabla p_t}|^2 \le \sum_{i=1}^n\sum_{j=1}^n |F[\nabla(u_{it}u_j + u_i u_{jt})]|^2.$$

Thus one gets

$$\begin{aligned}\|\nabla p_t(t)\|^2 &= \frac{1}{(2\pi)^2}\|\widehat{\nabla p_t}(t)\|^2\\
&\le \sum_{i=1}^n\sum_{j=1}^n \|F[\nabla(u_{it}u_j + u_i u_{jt})](t)\|^2\\
&= \sum_{i=1}^n\sum_{j=1}^n \|\nabla(u_{it}u_j + u_i u_{jt})(t)\|^2\\
&\le C\sum_{i=1}^n\sum_{j=1}^n\{\|\nabla u_{it}(t)\|^2\|u_j(t)\|_\infty^2 + \|u_{it}(t)\|^2\|\nabla u_j(t)\|_\infty^2\\
&\quad + \|\nabla u_i(t)\|_\infty^2\|u_{jt}(t)\|^2 + \|u_i(t)\|_\infty^2\|\nabla u_{it}(t)\|^2\}\\
&\le C\|\nabla u_t(t)\|^2\|u(t)\|_\infty^2 + C\|u_t(t)\|^2\|\nabla u(t)\|_\infty^2\\
&\le C\|u(t)\|\|\Delta u(t)\|\|\nabla u_t(t)\|^2\\
&\quad + C\|\nabla u(t)\|\|\nabla\Delta u(t)\|\|u_t(t)\|^2 \le C(1+t)^{-4}.\end{aligned}$$

Remark. The following estimates hold if the initial data $u_0 \in H^\infty$.

$$(1+t)^{2m}\|\Delta^m u(t)\|^2 \le C, \qquad (1+t)^{1+2m}\|\nabla\Delta^m u(t)\|^2 \le C,$$

where $m \ge 0$ is any integer. In fact we have the following more precise results

$$\lim_{t\to\infty}[(1+t)^{2m}\|\Delta^m u(t)\|^2 + (1+t)^{2m+1}\|\nabla\Delta^m u(t)\|^2] = 0.$$

4. Proof of Theorem 1 and 3. With the aid of the elementary estimates, the decay estimates of the solutions, we can now develop our proof for the main theorems step by step.

Lemma 4.1. *We have the following estimates*

$$\frac{d}{dt}[(1+t)^3\|w(t)\|^2] + (1+t)^3\|\nabla w(t)\|^2 \le C(1+t)^2\int_{B(t)}|\widehat{w}|^2 d\xi,$$
$$\frac{d}{dt}[(1+t)^5\|\Delta w(t)\|^2] + (1+t)^5\|\nabla\Delta w(t)\|^2 \le C(1+t)^2\|w(t)\|^2,$$

where $B(t) = \{\xi \in R^2 \,|\, (1+t)|\xi|^2 \le C\}$.

Proof. If we make the scalar product of equations (3) and $2w$, integrate over R^2, we get

$$\frac{d}{dt}\|w(t)\|^2 + 2\|\nabla w(t)\|^2 = 2\int_{R^2} u\cdot(w\cdot\nabla w)dx,$$
$$\text{where} \quad \int_{R^2} w\cdot(v\cdot\nabla w)dx = 0, \quad \int_{R^2} w\cdot\nabla\pi dx = 0.$$

The estimate in Lemma 2.1 yields the following

$$\begin{aligned} 2\int_{R^2} u\cdot(w\cdot\nabla w)dx &\le 2\|u(t)\|_\infty\|w(t)\|\|\nabla w(t)\| \\ &\le 2\|u(t)\|_\infty^2\|w(t)\|^2 + \frac{1}{2}\|\nabla w(t)\|^2. \end{aligned}$$

Therefore we get

$$\frac{d}{dt}\|w(t)\|^2 + \frac{3}{2}\|\nabla w(t)\|^2 \le 2\|u(t)\|_\infty^2\|w(t)\|^2,$$

or

$$\begin{aligned}
&\frac{d}{dt}[(1+t)^3\|w(t)\|^2] + \frac{3}{2}(1+t)^3\|\nabla w(t)\|^2 \\
&\le 3(1+t)^2\|w(t)\|^2 + 2(1+t)^3\|u(t)\|_\infty^2\|w(t)\|^2 \\
&\le C(1+t)^2\|w(t)\|^2,
\end{aligned}$$

where by Lemma 3.1 and 3.2,

$$(1+t)\|u(t)\|_\infty^2 \le \|u(t)\|(1+t)\|\Delta u(t)\| \le C.$$

By virtue of Lemma 2.4, we get

$$\frac{d}{dt}[(1+t)^3\|w(t)\|^2] + (1+t)^3\|\nabla w(t)\|^2 \le C(1+t)^2\int_{B(t)}|\widehat{w}|^2 d\xi,$$

where C is independent of v_0, v and t. Actually $C = \|u_0\|_2^2\exp[16\|u_0\|^4]$.

If we make the scalar product of equations (3) and $2\Delta^2 w$, integrate over R^2, we get

$$\begin{aligned}
&\frac{d}{dt}\|\Delta w(t)\|^2 + 2\|\nabla\Delta w(t)\|^2 \\
&= 2\int_{R^2}\nabla\Delta w\cdot\nabla(w\cdot\nabla u + v\cdot\nabla w)dx \\
&\le \frac{1}{2}\|\nabla\Delta w(t)\|^2 + 4\|\nabla(w\cdot\nabla u)(t)\|^2 + 4\|\nabla(v\cdot\nabla w)(t)\|^2 \\
&\le \frac{1}{2}\|\nabla\Delta w(t)\|^2 + 8\int_{R^2}|\nabla u|^2|\nabla w|^2dx + 8\int_{R^2}|\nabla v|^2|\nabla w|^2dx \\
&\quad + 8\|w(t)\|_\infty^2\|\Delta u(t)\|^2 + 8\|v(t)\|_\infty^2\|\Delta w(t)\|^2.
\end{aligned}$$

We have the following estimates

$$\begin{aligned}
8\int_{R^2}|\nabla u|^2|\nabla w|^2x &\le 8\|\nabla u(t)\|_{L^4}^2\|\nabla w(t)\|_{L^4}^2 \\
&\le C\|u(t)\|^{1/2}\|\Delta u(t)\|^{3/2}\|w(t)\|^{1/2}\|\Delta w(t)\|^{3/2} \\
&\le C\|\Delta u(t)\|\|\Delta w(t)\|^2 + C\|u(t)\|^2\|\Delta u(t)\|^3\|w(t)\|^2, \\
8\int_{R^2}|\nabla v|^2|\nabla w|^2x &\le C\|\Delta v(t)\|\|\Delta w(t)\|^2 + C\|v(t)\|^2\|\Delta v(t)\|^3\|w(t)\|^2, \\
8\|w(t)\|_\infty^2\|\Delta u(t)\|^2 &\le 8\|w(t)\|\|\Delta w(t)\|\|\Delta u(t)\|^2 \\
&\le 4\|\Delta u(t)\|\|\Delta w(t)\|^2 + 4\|\Delta u(t)\|^3\|w(t)\|^2.
\end{aligned}$$

We now have the estimate

$$\begin{aligned}
&\frac{d}{dt}\|\Delta w(t)\|^2 + \frac{3}{2}\|\nabla\Delta w(t)\|^2 \\
&\le C[\|\Delta u(t)\| + \|\Delta v(t)\| + \|v(t)\|_\infty^2]\|\Delta w(t)\|^2 \\
&\quad + C[\|u(t)\|^2\|\Delta u(t)\|^3 + \|v(t)\|^2\|\Delta v(t)\|^3 + \|\Delta u(t)\|^3]\|w(t)\|^2.
\end{aligned}$$

Obviously the following hold

$$\begin{aligned}
&\frac{d}{dt}[(1+t)^5\|\Delta w(t)\|^2] + \frac{3}{2}(1+t)^5\|\nabla\Delta w(t)\|^2 \\
&\le 5(1+t)^4\|\Delta w(t)\|^2 + C(1+t)^4\|\Delta w(t)\|^2 + C(1+t)^2\|w(t)\|^2 \\
&\le C(1+t)^4\|\Delta w(t)\|^2 + C(1+t)^2\|w(t)\|^2,
\end{aligned}$$

where for all $t > 0$,

$$\begin{aligned}
(1+t)[\|\Delta u(t)\| + \|\Delta v(t)\| + \|v(t)\|_\infty^2] \le C, \\
(1+t)^3[\|u(t)\|^2\|\Delta u(t)\|^3 + \|v(t)\|^2\|\Delta v(t)\|^3 + \|\Delta u(t)\|^3] \le C.
\end{aligned}$$

By using Lemma 2.4, we now obtain

$$\begin{aligned}
&\frac{d}{dt}[(1+t)^5\|\Delta w(t)\|^2] + (1+t)^5\|\nabla\Delta w(t)\|^2 \\
&\le C(1+t)^4\int_{B(t)}|\xi|^4|\widehat{w}|^2 d\xi + C(1+t)^2\|w(t)\|^2 \\
&\le C(1+t)^2\int_{B(t)}|\widehat{w}|^2 d\xi + C(1+t)^2\|w(t)\|^2 \\
&\le C(1+t)^2\|w(t)\|^2.
\end{aligned}$$

Lemma 4.2. *Proof of Theorem* 1, *case* (1).

Proof. By using the identity

$$\frac{d}{dt}\|w(t)\|^2 + 2\|\nabla w(t)\|^2 = 2\int_{R^2} u\cdot(w\cdot\nabla w)dx,$$

we get

$$\frac{d}{dt}\|w(t)\|^2 + 2\|\nabla w(t)\|^2$$

$$\begin{aligned}
&\leq 2\int_{R^2} |u||w||\nabla w|dx \leq 2\|u(t)\|_{L^4}\|w(t)\|_{L^4}\|\nabla w(t)\| \\
&\leq C\|u(t)\|^{1/2}\|\nabla u(t)\|^{1/2}\|w(t)\|^{1/2}\|\nabla w(t)\|^{3/2} \\
&\leq \|\nabla w(t)\|^2 + C\|u(t)\|^2\|\nabla u(t)\|^2\|w(t)\|^2.
\end{aligned}$$

Integrating this inequality in time to give

$$\|w(t)\|^2 + \int_0^t \|\nabla w(s)\|^2 ds \leq \|w_0\|^2 + C\int_0^t \|u(s)\|^2\|\nabla u(s)\|^2\|w(s)\|^2 ds.$$

By using Gronwall's inequality and the estimate of Lemma 3.1, we obtain

$$\begin{aligned}
&\|w(t)\|^2 + \int_0^t \|\nabla w(s)\|^2 ds \\
&\leq \|w_0\|^2 \exp\left[C\int_0^\infty \|u(t)\|^2\|\nabla u(t)\|^2 dt\right] \leq \|w_0\|^2 \exp[C\|u_0\|^4].
\end{aligned}$$

Thus

$$\begin{aligned}
&\sup_{0\leq t<\infty} \|w(t)\|^2 \leq \|w_0\|^2 \exp[C\|u_0\|^4], \\
&\int_0^\infty \|\nabla w(t)\|^2 dt \leq \|w_0\|^2 \exp[C\|u_0\|^4].
\end{aligned}$$

The constants are independent of v_0 and v. In fact, they only depend on the constant C appearing in Lemma 2.5.

The second estimate in Lemma 4.1 yields

$$\begin{aligned}
&\frac{d}{dt}[(1+t)^5\|\Delta w(t)\|^2] + (1+t)^5\|\nabla\Delta w(t)\|^2 \\
&\leq C(1+t)^2\|w(t)\|^2 \leq C(1+t)^2\|w_0\|^2.
\end{aligned}$$

Integrating in time, we get

$$(1+t)^5\|\Delta w(t)\|^2 + \int_0^t (1+s)^5\|\nabla\Delta w(s)\|^2 ds \leq \|\Delta w_0\|^2 + C(1+t)^3\|w_0\|^2.$$

Thus one obtains

$$\sup_{0\leq t<\infty} [(1+t)^2\|\Delta w(t)\|^2] \leq C\|w_0\|_2^2.$$

Further, by Lemma 2.5, we have

$$(1+t)\|w(t)\|_\infty^2 \le \|w(t)\|(1+t)\|\Delta w(t)\| \le C\|w_0\|_2^2,$$
$$(1+t)\|\nabla w(t)\|^2 \le \|w(t)\|(1+t)\|\Delta w(t)\| \le C\|w_0\|_2^2.$$

Now we employ Lemma 2.1 to estimate others.

$$\sup_{0\le t<\infty} [(1+t)\|\pi(t)\|^2] \le C\|w_0\|_2^2,$$
$$\sup_{0\le t<\infty} [(1+t)^2\|\nabla\pi(t)\|^2] \le C\|w_0\|_2^2,$$
$$\sup_{0\le t<\infty} [(1+t)^3\|\Delta\pi(t)\|^2] \le C\|w_0\|_2^2,$$
$$\sup_{0\le t<\infty} [(1+t)^2\|\pi(t)\|_\infty^2] \le C\|w_0\|_2^2,$$
$$\sup_{0\le t<\infty} [(1+t)^2\|w_t(t)\|^2] \le C\|w_0\|_2^2,$$
$$\sup_{0\le t<\infty} [(1+t)^3\|\pi_t(t)\|^2] \le C\|w_0\|_2^2.$$

This lemma shows that if $(u_0, A_0) \in H^2$, the solutions of problem (1-2) are stable. We do not necessarily require that the initial velocity decay more rapidly, i.e. $(u_0, A_0) \in L^r \cap H^2$, for some $1 \le r < 2$.

Lemma 4.3. *Proof of Theorem* 1, *case* (2).

Remark. Let u_0 and $v_0 \in H^2$ and the corresponding solutions enjoy the slow decay estimate $\|u(t)\| + \|v(t)\| \le C/\ln(e+t)$ and let $w_0 \in L^1$. Then the same result holds. Notice this condition is weaker than the original one.

Proof. Using Lemma 4.1 and 2.3, we have

$$\begin{aligned}
&\frac{d}{dt}[(1+t)^3\|w(t)\|^2] + (1+t)^3\|\nabla w(t)\|^2\\
&\le C(1+t)^2\int_{B(t)} |\widehat{w}|^2 d\xi\\
&\le C(1+t)^2\int_{B(t)} \left\{\|w_0\|_{L^1} + 2|\xi|\int_0^t [\|u(s)\| + \|v(s)\|]\|w(s)\|ds\right\}^2 \delta\xi\\
&\le C(1+t)\|w_0\|_{L^1}^2 + C(1+t)\int_0^t [\|u(s)\|^2 + \|v(s)\|^2]\|w(s)\|^2 ds.
\end{aligned}$$

Integrating in time gives

$$\begin{aligned}&(1+t)^3\|w(t)\|^2+\int_0^t(1+s)^3\|\nabla w(s)\|^2ds\\ \le\ &\|w_0\|^2+C(1+t)^2\|w_0\|^2_{L^1}\\ &+C(1+t)^2\int_0^t[\|u(s)\|^2+\|v(s)\|^2]\|w(s)\|^2ds,\end{aligned}$$

or we get

$$\begin{aligned}&(1+t)\|w(t)\|^2+\int_0^t(1+s)\|\nabla w(s)\|^2ds\\ \le\ &C\|w_0\|^2_{L^1\cap L^2}+C\int_0^t[\|u(s)\|^2+\|v(s)\|^2]\|w(s)\|^2ds\\ \le\ &C\|w_0\|^2_{L^1\cap L^2}+C\int_0^t[\ln(e+s)]^{-2}\|w(s)\|^2ds.\end{aligned}$$

Set

$$\begin{aligned}g(t)&=(1+t)\|w(t)\|^2+\int_0^t(1+s)\|\nabla w(t)\|^2ds,\\ h(t)&=C(1+t)^{-1}[\ln(e+t)]^{-2}.\end{aligned}$$

By means of Gronwall's inequality, one obtains

$$\sup_{0\le t<\infty}[(1+t)\|w(t)\|^2]\le C\|w_0\|^2_{L^1\cap L^2}.$$

In addition, using Lemma 4.1, we have the estimate

$$\begin{aligned}&\frac{d}{dt}[(1+t)^5\|\Delta w(t)\|^2]+(1+t)^5\|\nabla\Delta w(t)\|^2\\ \le\ &C(1+t)^2\|w(t)\|^2\le C(1+t)\|w_0\|^2_{L^1\cap L^2}.\end{aligned}$$

Integrating in time, we obtain

$$\begin{aligned}&(1+t)^5\|\Delta w(t)\|^2+\int_0^t(1+s)^5\|\nabla\Delta w(s)\|^2ds\\ \le\ &\|\Delta w_0\|^2+C\|w_0\|^2_{L^1\cap L^2}(1+t)^2,\end{aligned}$$

this is just the estimate

$$\sup_{0\le t<\infty}[(1+t)^3\|\Delta w(t)\|^2]\le C\|w_0\|^2_{L^1\cap H^2}.$$

As before, using Lemma 2.5, we get

$$(1+t)^2\|w(t)\|_\infty^2 \le (1+t)^{1/2}\|w(t)\|(1+t)^{3/2}\|\Delta w(t)\| \le C\|w_0\|^2_{L^1\cap H^2},$$
$$(1+t)^2\|\nabla w(t)\|_\infty^2 \le (1+t)^{1/2}\|w(t)\|(1+t)^{3/2}\|\Delta w(t)\| \le C\|w_0\|^2_{L^1\cap H^2}.$$

Further, by Lemma 2.1, we have

$$\begin{aligned}
(1+t)^3\|\pi(t)\|^2 &\le C\|w_0\|^2_{L^1\cap H^2},\\
(1+t)^4\|\nabla\pi(t)\|^2 &\le C\|w_0\|^2_{L^1\cap H^2},\\
(1+t)^5\|\Delta\pi(t)\|^2 &\le C\|w_0\|^2_{L^1\cap H^2},\\
(1+t)^4\|\pi(t)\|_\infty^2 &\le C\|w_0\|^2_{L^1\cap H^2},\\
(1+t)^3\|w_t(t)\|^2 &\le C\|w_0\|^2_{L^1\cap H^2},\\
(1+t)^4\|\pi_t(t)\|^2 &\le C\|w_0\|^2_{L^1\cap H^2},
\end{aligned}$$

The proof of Theorem 3, case $\beta = 1$, is also completed by taking $v_0 = 0$.

Lemma 4.4. *Proof of Theorem* 1, *case* (3).

Proof. For all $\xi \in R^2$, Lagrange mean value theorem leads to

$$|\widehat{w_0}(\xi)| \le |\xi|\int_{R^2}|x||w_0|dx \le |\xi|\|w_0\|_M.$$

Using Lemma 4.1 and 2.3, we have

$$\begin{aligned}
&\frac{d}{dt}[(1+t)^3\|w(t)\|^2] + (1+t)^3\|\nabla w(t)\|^2\\
&\le C(1+t)^2\int_{B(t)}|\widehat{w}|^2 d\xi\\
&\le C(1+t)^2\int_{B(t)}\Big\{|\widehat{w_0}| + 2|\xi|\int_0^t[\|u(s)\| + \|v(s)\|]\|w(s)\|ds\Big\}^2 d\xi\\
&\le C\|w_0\|_M^2 + C\Big\{\int_0^t[\|u(s)\| + \|v(s)\|]\|w(s)\|ds\Big\}^2\\
&\le C\|w_0\|_M^2 + C[\ln(1+t)]^2\|w_0\|^2_{L^1\cap L^2}.
\end{aligned}$$

Integrating in time gives

$$\begin{aligned}
&(1+t)^3\|w(t)\|^2 + \int_0^t(1+s)^3\|\nabla w(s)\|^2 ds\\
&\le \|w_0\|^2 + Ct\|w_0\|_M^2 + Ct[\ln(1+t)]^2\|w_0\|^2_{L^1\cap L^2},
\end{aligned}$$

or

$$(1+t)^2\|w(t)\|^2 \le C[\ln(1+t)]^2\|w_0\|_M^2.$$

Iterated once, we have

$$\begin{aligned}
&\frac{d}{dt}[(1+t)^3\|w(t)\|^2] + (1+t)^3\|\nabla w(t)\|^2 \\
&\le C(1+t)^2\int_{B(t)} |\widehat{w}|^2 d\xi \\
&\le C(1+t)^2\int_{B(t)} \Big\{|\widehat{w_0}| + 2|\xi|\int_0^t [\|u(s)\| + \|v(s)\|]\|w(s)\|ds\Big\}^2 d\xi \\
&\le C\|w_0\|_M^2 + C\Big\{\int_0^t [\|u(s)\| + \|v(s)\|]\|w(s)\|ds\Big\}^2 \le C\|w_0\|_M^2.
\end{aligned}$$

Integrating in time gives

$$\begin{aligned}
&(1+t)^3\|w(t)\|^2 + \int_0^t (1+s)^3\|\nabla w(s)\|^2 ds \\
&\le \|w_0\|^2 + Ct\|w_0\|_M^2,
\end{aligned}$$

or

$$\sup_{0\le t<\infty} [(1+t)^2\|w(t)\|^2] \le C\|w_0\|_M^2.$$

Coupling this estimate and the second estimate in Lemma 4.1 yields

$$\begin{aligned}
&\frac{d}{dt}[(1+t)^5\|\Delta w(t)\|^2] + (1+t)^5\|\Delta\nabla w(t)\|^2 \\
&\le C(1+t)^2\|w(t)\|^2 \le C\|w_0\|_M^2.
\end{aligned}$$

Integrating in time gives

$$(1+t)^5\|\Delta w(t)\|^2 + \int_0^t (1+s)^5\|\nabla\Delta w(s)\|^2 ds \le \|\Delta w_0\|^2 + Ct\|w_0\|_M^2.$$

Therefore we have

$$\sup_{0\le t<\infty} [(1+t)^4\|\Delta w(t)\|^2] \le C\|w_0\|_M^2.$$

Now it is very easy to get

$$\sup_{0\le t<\infty} [(1+t)^3\|\nabla w(t)\|^2] \le C\|w_0\|_M^2,$$
$$\sup_{0\le t<\infty} [(1+t)^3\|w(t)\|_\infty^2] \le C\|w_0\|_M^2.$$

Other estimates follow lines which are by now familiar.

The proof of Theorem 3, case $\beta = 2$, is also accomplished by taking $v_0 = 0$.

Lemma 4.5. *Proof of Theorem* 1, *case* (4).

Proof. It suffices to prove the lower bound. Let g be the solution of the linear problem

$$g_t - \Delta g = 0, \qquad g(x,0) = w_0,$$

where w_0 is the initial velocity of problem (3–4).

Becuse $\nabla\cdot w_0 = 0$, $\widehat{g} = \widehat{w_0}\exp(-|\xi|^2 t)$, we get $\nabla\cdot g = 0$.

Let $f = w - g$. Then it satisfies the equations

$$f_t + w\cdot\nabla u + v\cdot\nabla w - \Delta f + \nabla\pi = 0, \quad \nabla\cdot f = 0, \quad f(x,0) = 0.$$

If we make the scalar product of this equation and $2f$, integrate over R^2, we get

$$\frac{d}{dt}\|f(t)\|^2 + 2\|\nabla f(t)\|^2 + 2\int_{R^2} f\cdot(w\cdot\nabla u + v\cdot\nabla w)dx = 0,$$

or

$$\frac{d}{dt}[(1+t)^3\|f(t)\|^2] + 2(1+t)^3\|\nabla f(t)\|^2$$
$$+ 2(1+t)^3\int_{R^2} f\cdot(w\cdot\nabla u + v\cdot\nabla w)dx = 3(1+t)^2\|f(t)\|^2.$$

To derive the bounds, we have to control the nonlinear effects.

$$2(1+t)^3\int_{R^2}|f\cdot(w\cdot\nabla u)|dx$$
$$\le 2(1+t)^3\|f(t)\|\|w(t)\|_\infty\|\nabla u(t)\|$$
$$\le (1+t)^2\|f(t)\|^2 + (1+t)^4\|w(t)\|_\infty^2\|\nabla u(t)\|^2,$$

$$
\begin{aligned}
&2(1+t)^3 \int_{R^2} |f \cdot (v \cdot \nabla w)| dx \\
&\leq 2(1+t)^3 \|f(t)\| \|v(t)\|_\infty \|\nabla w(t)\| \\
&\leq (1+t)^2 \|f(t)\|^2 + (1+t)^4 \|v(t)\|_\infty^2 \|\nabla w(t)\|^2.
\end{aligned}
$$

Therefore we get

$$
\begin{aligned}
&\frac{d}{dt}[(1+t)^3\|f(t)\|^2] + 2(1+t)^3\|\nabla f(t)\|^2 \\
&\leq 5(1+t)^2\|f(t)\|^2 + (1+t)^4\|w(t)\|_\infty^2\|\nabla u(t)\|^2 + (1+t)^4\|v(t)\|_\infty^2\|\nabla w(t)\|^2.
\end{aligned}
$$

Let $B(t) = \{\xi \in R^2 | 2(1+t)|\xi|^2 \leq 5\}$. As in Lemma 2.4, we have

$$
\begin{aligned}
&\frac{d}{dt}[(1+t)^3\|f(t)\|^2] \\
&\leq \frac{5(1+t)^2}{(2\pi)^2} \int_{B(t)} |\widehat{f}(\xi, t)|^2 d\xi \\
&\quad + (1+t)^4\|w(t)\|_\infty^2\|\nabla u(t)\|^2 + (1+t)^4\|v(t)\|_\infty^2\|\nabla w(t)\|^2 \\
&\leq C(1+t)^2 \int_{B(t)} [\ln(1+t)]^2|\xi|^2\|w_0\|_{L^1\cap L^2}^2 d\xi + C\|w_0\|_{L^1\cap L^2}^2 \\
&\leq C[\ln(1+t)]^2|\|w_0\|_{L^1\cap L^2}^2,
\end{aligned}
$$

where for all $t > 0$,

$$
\begin{aligned}
(1+t)^2\|w(t)\|_\infty^2 &\leq C\|w_0\|_{L^1\cap L^2}^2 \\
(1+t)^2\|v(t)\|_\infty^2 &\leq C\|v_0\|_{L^1\cap L^2}^2
\end{aligned}
$$

and

$$
\begin{aligned}
\widehat{f} &= -\int_0^t F[w \cdot \nabla u + v \cdot \nabla w + \nabla \pi] e^{-|\xi|^2(t-s)} ds, \\
|\widehat{f}| &\leq 2|\xi| \int_0^t [\|u(s)\| + \|v(s)\|] \|w(s)\| ds.
\end{aligned}
$$

Integrating in time gives

$$
(1+t)^3\|f(t)\|^2 \leq Ct[\ln(1+t)]^2\|w_0\|_{L^1\cap L^2}^2,
$$

or

$$
(1+t)\|f(t)\|^2 \leq C\ln(1+t)\|w_0\|_{L^1\cap L^2}^2.
$$

Let $2\alpha = |\int_{R^2} w_0 dx|$, then $\alpha > 0$, and there is a constant $\delta > 0$, such that for all $\xi \in R^2$, if $|\xi| \leq \delta$, then $|\widehat{w_0}| \geq \alpha$.

Now we have the estimate for the solution of the heat equation

$$\begin{aligned}
(2\pi)^2 \int_{R^2} |g|^2 dx &= \int_{R^2} |\widehat{g}|^2 d\xi \\
&= \int_{R^2} |\widehat{w_0}|^2 e^{-2|\xi|^2 t} d\xi \\
&\geq \int_{R^2} |\widehat{w_0}|^2 e^{-2|\xi|^2(1+t)} d\xi \\
&\geq \int_{|\xi|\leq\delta} |\widehat{w_0}|^2 e^{-2|\xi|^2(1+t)} d\xi \\
&\geq \left|\int_{R^2} w_0 dx\right|^2 \int_{|\xi|\leq\delta} e^{-2|\xi|^2(1+t)} d\xi \\
&\geq \left|\int_{R^2} w_0 dx\right|^2 \int_0^{2\pi}\int_0^{\delta} r e^{-2r^2(1+t)} dr d\theta \\
&\geq C(1+t)^{-1}\left|\int_{R^2} w_0 dx\right|^2.
\end{aligned}$$

Finally we have

$$\begin{aligned}
\|w(t)\| &\geq \|g(t)\| - \|f(t)\| \\
&\geq C(1+t)^{-1/2}\left|\int_{R^2} w_0 dx\right| \\
&\quad - C(1+t)^{-1}\ln(1+t)\|w_0\|_{L^1\cap L^2}.
\end{aligned}$$

Remark. If there is a constant $0 < C_0 \leq 1$, such that

$$\left|\int_{R^2} w_0 dx\right| \geq C_0 \int_{R^2} |w_0| dx > 0,$$

and

$$\begin{aligned}
&C(1+t)^{-1/2}\int_{R^2} |w_0| dx - C(1+t)^{-1}\ln(1+t)\|w_0\|_{L^1\cap L^2} \\
&\geq C_0(1+t)^{-1/2}\|w_0\|_{L^1\cap L^2},
\end{aligned}$$

then we get

$$C_1\|w_0\|^2_{L^1\cap L^2} \leq (1+t)\|w(t)\|^2 \leq C_2\|w_0\|^2_{L^1\cap L^2}.$$

5. $n(\geq 3)$-dimensional problems. We are concerned with L^2 and $L^p(p > n \geq 3)$-uniform stability for solutions to Caychy problem for n-dimensional incompressible Navier-Stokes equation (1–2).

Let us begin with some relevant results on existence and long time behavior of solutions of problem (1–2).

Proposition 1. *Let $n \geq 3$.*

(1) *Let $u_0 \in L^2$, then there is a weak solution $u \in L^\infty(0,\infty;L^2) \cap L^2(0,T;H^1)$, for all $T > 0$.*

(2) *Let $u_0 \in L^1 \cap L^2$, (1–2) has a weak solution $u \in L^\infty(0,\infty;L^2) \cap L^2(0,T;H^1)$, such that $(1+t)^{n/2}\|u(t)\| \leq C$.*

(3) *Let $u_0 \in M$, then $(1+t)^{1+n/2}\|u(t)\| \leq C$.*

(4) *Let $u_0 \in M \cap L^p$ for some $p > n$, then $\|u(t)\|_\infty = O(t^{-(n+1)/2})$.*

(5) *Let $u_0 \in L^2 \cap L^p$ for some $p > n$, then there is a constant $C > 0$, such that the smallness condition $\|u_0\|^{2(p-n)/p(n-2)}\|u_0\|_{L^p} \leq C$ implies problem (1–2) has a unique global solution $u \in L^2(0,\infty;H^1) \cap C(0,\infty;L^2 \cap L^p)$. See* [7–8, 12, 19–23].

Recently, there has been much attention to stability of the solutions to the $n(\geq 3)$-dimensional Navier-Stokes equations. For examples, Secchi [13] utilized energy method to prove an L^2 stability result for 3-dimensional problem. Veiga and Secchi [20], Wiegner [22] studied stability in L^p-norm with $p > n$ for strong solutions to problem (1–2). All these papers established very interesting results for $n(\geq 3)$-dimensional problems, without employing any kinds of smallness hypotheses. Let us look at their stability results.

Proposition 2. *Let $u_0 \in H^1$ with $\nabla \cdot u_0 = 0$, let $f \in L^1(0,\infty;L^2) \cap L^2(0,\infty;L^2)$ and $u \in W^{1,2}(0,\infty;L^2) \cap L^2(0,\infty;H^2)$. Then for each $v_0 \in H$, there is a weak solution corresponding to the initial velocity v_0 and the external force f, such that $\|u(t) - v(t)\| \to 0$ as $t \to \infty$. See* [13].

Proposition 3. *Let $p > 3$, let $u_0 \in L^1 \cap L^{p+2}$ and $v_0 \in L^1 \cap L^p$ with $\nabla \cdot u_0 = \nabla \cdot v_0 = 0$, let $u \in L^\infty(0,\infty;L^{p+2})$ be a strong solution corresponding* *to u_0. Then there is a constant $\delta > 0$, such that if $\|u_0 - v_0\|_{L^p} < \delta$, a strong solution of problem* (1–2) *$v \in C(0,\infty; L^p)$ exists, corresponding to the initial velocity v_0, and*

$$\|u(t) - v(t)\|_{L^p} \leq C(1+t)^{-3/4},$$

where δ and C depend on L^1 and L^p norms of u_0, v_0 and on the $L^\infty(0,\infty; L^{p+2})$-norm of u. See [20].

Proposition 4. *Let the critical assumption* (H) *hold: There are constants $p > n \geq 3$ and $q > 2$ satisfying $n/p + 2/q = 1$, such that $u_0 \in L^2 \cap L^p$ and $u \in L^q(0,\infty; L^p)$. Then $u \in L^\infty(0,\infty; L^p)$ and $(1+t)^{(p-2)n/4}\|u(t)\|_{L^p}^p \leq C$. Moreover, there is a constant $\delta = \delta(n,p) > 0$, such that if $v_0 \in L^2 \cap L^p$ and $\|u_0 - v_0\|_{L^p} < \delta$, a strong solution of problem* (1–2) *$v \in L^r(0,\infty; L^p)$ exists, where $4p/n(p-2) < r \leq \infty$. See* [22].

Wiegner skipped the assumption $u \in L^\infty(0,\infty; L^{p+2})$ and gave the proof for all $n(\geq 3)$-dimensional problems.

In [19], Veiga proved that if a global solution of n-dimensional problem satisfies $u \in L^q(0,\infty; L^p)$, then $u \in L^\infty(0,\infty; L^p)$, where $u_0 \in L^2 \cap L^p$ for some constants p, q satisfying $p > n \geq 3$ and $n/p + 2/q = 1$. Unfortunately, the existence of a solution in the class $L^q(0,\infty; L^p)$ is still an open problem for general data u_0. In [23] Wiegner prove that if $u_0 \in L^1 \cap L^p$, for some $p > n$, and if $(1+|x|)u_0(x) \in L^1$ and $\int_{R^n} u_0(x)dx = 0$, then $\|u(t)\|_\infty = O(t^{-(n+1)/2})$.

For 2-dimensional problem, it has been established that for all initial data $u_0 \in H^2$, there is a unique global strong solution. For $n(\geq 3)$-dimensional problem, it is known that for small initial velocity, there exists a global strong solution. However, we can utilize Veiga's hypothese rather than the smallness condition to prove the uniform stability. Then by utilizing Wiegner's L^∞-decay result [23], we can obtain the uniform stability for (u,p).

We use the critical assumption (H) below.

Lemma 5.1. *Proof of Theorem* 2, *case* (1).

Proof. Making the scalar product of (3) and $2w$ and integrating over R^n, we obtain

$$\frac{d}{dt}\|w(t)\|^2 + 2\|\nabla w(t)\|^2 = 2\int_{R^n} u\cdot(w\cdot\nabla w)dx.$$

Applying Lemma 2.1, gives

$$\begin{aligned}\frac{d}{dt}\|w(t)\|^2 + 2\|\nabla w(t)\|^2 &\le 2\int_{R^n} |u||w||\nabla w|dx\\ &\le 2\|u(t)\|_{L^p}\|w(t)\|_{L^l}\|\nabla w(t)\|\\ &\le C\|u(t)\|_{L^p}\|w(t)\|^{1-\alpha}\|\nabla w(t)\|^{1+\alpha}\\ &\le \|\nabla w(t)\|^2 + C\|u(t)\|_{L^p}^{2/(1-\alpha)}\|w(t)\|^2,\end{aligned}$$

where $p > n \ge 3$, $l > 2$, and

$$\begin{gathered}1/p + 1/l = 1/2,\\ 0 < \alpha = n/2 - n/l = n/p < 1,\\ n/p + 2/q = 1,\\ 2/q = 1 - n/p = 1-\alpha,\\ q = 2/(1-\alpha).\end{gathered}$$

Therefore

$$\|u(t)\|_{L^p}^{2/(1-\alpha)} = \|u(t)\|_{L^p}^q \in L^1(0,\infty).$$

Integrating the above inequality in time to give

$$\|w(t)\|^2 + \int_0^t \|\nabla w(s)\|^2 ds \le \|w_0\|^2 + C\int_0^t \|u(s)\|_{L^p}^q \|w(s)\|^2 ds.$$

By using Gronwall's inequality, we obtain

$$\|w(t)\|^2 + \int_0^t \|\nabla w(s)\|^2 ds \le \|w_0\|^2 \exp\left[C\int_0^t \|u(t)\|_{L^p}^q dt\right].$$

Thus

$$\sup_{0\le t<\infty} \|w(t)\|^2 \le C\|w_0\|^2, \quad \int_0^\infty \|\nabla w(t)\|^2 dt \le C\|w_0\|^2.$$

The last estimate follows immediately from the inequality in Lemma 2.1

$$\|\widehat{\pi}(t)\|_\infty \le [\|u(t) + \|v(t)]\|w(t)\|.$$

Lemma 5.2. *Proof of Theorem* 2, *case* (2).

Proof. The first assertion was proved by Wiegner [22]. For the estimates, let $\|u_0 - v_0\|_{L^p} < \delta$. Because $q = 2p/(p-n) > 4p/n(p-2)$ and $\|v(t)\|_{L^p}^q \le C(1+t)^{-(p-2)n/2(p-n)}$, and $(p-2)n/2(p-n) > 1$, so

$$\int_0^\infty \|v(t)\|_{L^p}^q dt \le C(n,p,\|v_0\|,\|v_0\|_{L^p}) \le C(n,p,\|u_0\|,\|u_0\|_{L^p}).$$

Hence

$$\sup_{v_0:\|u_0-v_0\|_{L^p}<\delta} \int_0^\infty \|v(t)\|_{L^p}^q dt \le C(n,p,\|u_0\|,\|u_0\|_{L^p}) < \infty.$$

If we make the scalar product of equations (3) and $p|w|^{p-2}w$, integrate over R^n, we get

$$\begin{aligned}
&\frac{d}{dt}\int_{R^n} |w|^p dx + p\int_{R^n} |w|^{p-2}|\nabla w|^2 dx + \frac{4(p-2)}{p}\int_{R^n} |\nabla(|w|^{p/2})|^2 dx \\
&= -p\int_{R^n} (w\cdot\nabla u, |w|^{p-2}w)dx - p\int_{R^n} (\nabla\pi, |w|^{p-2}w)dx,
\end{aligned}$$

where

$$\int_{R^n} (v\cdot\nabla w, |w|^{p-2}w)dx = 0.$$

To justify the uniform stability, we must control the right hand side.

$$\begin{aligned}
&-\int_{R^n} (w\cdot\nabla u, |w|^{p-2}w)dx \\
&= -\sum_{i=1}^n\sum_{j=1}^n \int_{R^n} |w|^{p-2}w_i \frac{\partial}{\partial x_j}(u_i w_j)dx \\
&= \sum_{i=1}^n\sum_{j=1}^n \int_{R^n} u_i w_j \frac{\partial}{\partial x_j}(|w|^{p-2}w_i)dx
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=1}^{n}\sum_{j=1}^{n}\int_{R^n}|w|^{p-2}u_iw_j\frac{\partial w_i}{\partial x_j}dx \\
&\quad + (p-2)\sum_{i=1}^{n}\sum_{j=1}^{n}\int_{R^n}|w|^{p-4}u_iw_iw_j\Big(w,\frac{\partial w}{\partial x_j}\Big)dx \\
&\le \int_{R^n}|w|^{p-2}\Big(\sum_{i=1}^{n}\sum_{j=1}^{n}|u_iw_j|^2\Big)^{1/2}\Big(\sum_{i=1}^{n}\sum_{j=1}^{n}\Big|\frac{\partial w_i}{\partial x_j}\Big|^2\Big)^{1/2}dx \\
&\quad + (p-2)\int_{R^n}|w|^{p-3}\Big(\sum_{i=1}^{n}|u_iw_i|\Big)\Big(\sum_{j=1}^{n}|w_j|\Big|\frac{\partial w}{\partial x_j}\Big|\Big)dx \\
&\le (p-1)\int_{R^n}|u||w|^{p-1}|\nabla w|dx \\
&\le \frac{1}{4}\int_{R^n}|w|^{p-2}|\nabla w|^2dx + (p-1)^2\int_{R^n}|u|^2|w|^pdx, \quad \text{and}
\end{aligned}
$$

$$
\begin{aligned}
&p(p-1)^2\int_{R^n}|u|^2|w|^pdx \\
&\le p(p-1)^2\|u(t)\|_{L^p}^2\|w(t)\|_{L^{p^2/(p-2)}}^p \\
&= p(p-1)^2\|u(t)\|_{L^p}^2\|\phi(t)\|_{L^{2p/(p-2)}}^p \quad (\text{where } \phi = |w|^{p/2}) \\
&\le C\|u(t)\|_{L^p}^2\|\phi(t)\|^{2(1-n/p)}\|\nabla\phi(t)\|^{2n/p} \\
&\le C\|u(t)\|_{L^p}^{2p/(p-n)}\|\phi(t)\|^2 + \frac{p-2}{p}\|\nabla\phi(t)\|^2 \\
&= C\|u(t)\|_{L^p}^{q}\|\phi(t)\|^2 + \frac{p-2}{p}\|\nabla\phi(t)\|^2 \\
&= C\|u(t)\|_{L^p}^{q}\|w(t)\|_{L^p}^p + \frac{p-2}{p}\int_{R^n}|\nabla(|w|^{p/2})|^2dx, \quad \text{and}
\end{aligned}
$$

$$
\begin{aligned}
&-\int_{R^n}(\nabla\pi, |w|^{p-2}w)dx \\
&= \sum_{i=1}^{n}\int_{R^n}\pi\frac{\partial}{\partial x_i}(|w|^{p-2}w_i)dx \\
&= (p-2)\sum_{i=1}^{n}\int_{R^n}\pi w_i|w|^{p-4}\Big(w,\frac{\partial w}{\partial x_i}\Big)dx \\
&\le (p-2)\int_{R^n}|\pi||w|^{p-2}|\nabla w|dx \\
&\le \frac{1}{4}\int_{R^n}|w|^{p-2}|\nabla w|^2dx + (p-2)^2\int_{R^n}|\pi|^2|w|^{p-2}dx.
\end{aligned}
$$

Taking the divergence of equations (3) yields

$$-\Delta\pi = \nabla\cdot(w\cdot\nabla u + v\cdot\nabla w) = \sum_{i=1}^{n}\sum_{j=1}^{n}\frac{\partial^2}{\partial x_i\partial x_j}(u_iw_j + v_iw_j).$$

Applying the Fourier transformation and Cauchy-Schwartz inequality, one obtains

$$|\widehat{\pi}|^2 \le \sum_{i=1}^{n}\sum_{j=1}^{n}|\widehat{w_iu_j} + \widehat{v_iw_j}|^2.$$

By Calderon-Zygmund's inequality [16], one has

$$\|\pi(t)\|_{L^r}^2 \le C(n,r)\sum_{i=1}^{n}\sum_{j=1}^{n}\|(u_iw_j + v_iw_j)(t)\|_{L^r}^2,$$

where $1 < r < \infty$. Thus

$$\begin{aligned}
& p(p-2)^2\int_{R^n}|\pi|^2|w|^{p-2}dx. \\
&\le p(p-2)^2\|\pi(t)\|^2_{L^{p^2/2(p-1)}}\|w\|^{p-2}_{L^{p^2/(p-2)}} \\
&\le C[\|u(t)\|^2_{L^p} + \|v(t)\|^2_{L^p}]\|w(t)\|^2_{L^{p^2/(p-2)}}\|w(t)\|^{p-2}_{L^{p^2/(p-2)}} \\
&= C[\|u(t)\|^2_{L^p} + \|v(t)\|^2_{L^p}]\|\phi(t)\|^2_{L^{p^2/(p-2)}} \quad (\text{where } \phi = |w|^{p/2}) \\
&\le C[\|u(t)\|^2_{L^p} + \|v(t)\|^2_{L^p}]\|\phi(t)\|^{2(1-n/p)}\|\nabla\phi(t)\|^{2n/p} \\
&\le C[\|u(t)\|^{2p/(p-n)}_{L^p} + \|v(t)\|^{2p/(p-n)}_{L^p}]\|\phi(t)\|^2 + \frac{p-2}{p}\|\nabla\phi(t)\|^2 \\
&= C[\|u(t)\|^q_{L^p} + \|v(t)\|^q_{L^p}]\|\phi(t)\|^2 + \frac{p-2}{p}\|\nabla\phi(t)\|^2 \\
&= C[\|u(t)\|^q_{L^p} + \|v(t)\|^q_{L^p}]\|w(t)\|^p_{L^p} + \frac{p-2}{p}\int_{R^n}|\nabla(|w(t)|^{p/2})|^2dx.
\end{aligned}$$

Now the original equation is simplified to the inequality

$$\begin{aligned}
&\frac{d}{dt}\int_{R^n}|w|^p dx + \frac{p}{2}\int_{R^n}|w|^{p-2}|\nabla w|^2 dx \\
&\quad + \frac{2(p-2)}{p}\int_{R^n}|\nabla(|w|^{p/2})|^2dx \\
&\le C[\|u(t)\|^q_{L^p} + \|v(t)\|^q_{L^p}]\|w(t)\|^p_{L^p}.
\end{aligned}$$

Gronwall's inequality yields the estimate

$$\int_{R^n}|w|^p dx + \frac{p}{2}\int_0^t\int_{R^n}|w|^{p-2}|\nabla w|^2 dxds$$

$$+\frac{2(p-2)}{p}\int_0^t\int_{R^n}|\nabla(|w|^{p/2})|^2dxds$$
$$\leq C\exp\left\{C\int_0^{\infty}[\|u(t)\|_{L^p}^q+\|v(t)\|_{L^p}^q]dt\right\}\int_{R^n}|w_0|^pdx$$
$$\leq C(n,p,\|u_0\|,\|u_0\|_{L^p})\int_{R^n}|w_0|^pdx.$$

The proof of Lemma 5.3 is finished now. The proof of Theorem 2, case (3) follows easily from case (2).

Lemma 5.3. *Proof of Theorem* 4.

Proof. Let $v_0=0$, then $v(x,t)=0$. So we get

$$\sup_{0\leq t<\infty}\|u(t)\|_{L^p}\leq C\|u_0\|_{L^p},$$
$$\frac{p}{2}\int_0^{\infty}\int_{R^n}|u|^{p-2}|\nabla u|^2dxdt$$
$$+\frac{2(p-2)}{p}\int_0^{\infty}\int_{R^n}|\nabla(|u|^{p/2})|^2dxdt\leq C\|u_0\|_{L^p}^p,$$

where C depends only on the $L^q(0,\infty;L^p)$-norm of u.

Let

$$m=\frac{n^2}{n^2-2n+4}.$$

Then

$$1<m<\frac{n}{n-2},\quad\text{and}\quad\frac{(p-2)(2m-mn+n)n}{4m(p-n)}=\frac{p-2}{p-n}>1.$$

Let $q_1=2mp/(mp-n)$, then $n/mp+2/q_1=1$, Let $f=|u|^{p/2}$, Then one obtains the estimates

$$\|u(t)\|_{L^{mp}}=\|f(t)\|_{L^{2m}}^{2/p}$$
$$\leq C\|f(t)\|^{(2m-mn+n)/mp}\|\nabla f(t)\|^{(mn-n)/mp},$$
$$\|u(t)\|_{L^{mp}}^{q_1}\leq C\|f(t)\|^{2(2m-mn+n)/(mp-n)}\|\nabla f(t)\|^{2(mn-n)/(mp-n)}$$
$$\leq C\|f(t)\|^{2(2m-mn+n)/m(p-n)}+\|\nabla f(t)\|^2$$
$$=C\|u(t)\|_{L^p}^{p(2m-mn+n)/m(p-n)}+\int_{R^n}|\nabla(|u|^{p/2})|^2dx,$$

$$\|u(t)\|_{L^p}^{p(2m-mn+n)/m(p-n)} \le C(1+t)^{-(p-2)(2m-mn+n)n/4m(p-n)}.$$

Therefore

$$\|u(t)\|_{L^p}^{p(2m-mn+n)/m(p-n)} \in L^1(0,\infty)$$

and

$$\int_{R^n} |\nabla(|u|)^{p/2}|^2 dx \in L^1(0,\infty)$$

implies that

$$\|u(t)\|_{L^{mp}}^{q_1} \in L^1(0,\infty),$$

namely

$$u \in L^{q_1}[0,\infty;L^{mp}) \text{ and } u \in L^\infty[0,\infty;L^{mp}).$$

Since $mp > n \ge 3$ and $n/mp + 2/q_1 = 1$, the initial velocity $u_0 \in L^2 \cap L^{mp}$ and the solution $u \in L^{q_1}(0,\infty;L^{mp}) \cap L^\infty(0,\infty;L^{mp})$, where $m = n^2/(n^2-2n+4) > 1$, repeat the same procedure, we obtain

$$u \in L^{q_2}(0,\infty;L^{m^2p}) \bigcap L^\infty(0,\infty;L^{m^2p}).$$

Therefore if iterated this procedure for infinitely many times, we obtain

$$u \in L^{q_k}(0,\infty;L^{m^kp}) \bigcap L^\infty(0,\infty;L^{m^kp}),$$

for all $k \ge 1$, where $n/m^kp + 2/q_k = 1$. Notice that $m > 1$, hence $m^k \to \infty$, as $k \to \infty$.

Let α, β, γ be real numbers such that $1 \le \alpha \le \beta \le \gamma < \infty$ and $\alpha < \gamma$. Let $f \in L^{2\alpha/(\alpha-n)}(0,\infty;L^\alpha) \cap L^{2\gamma/(\gamma-n)}(0,\infty;L^\gamma)$. The Hölder's inequality yields the estimate

$$\begin{aligned}
&\|f(t)\|_{L^\beta}^\beta = \int_{R^n} |f|^{\alpha(\gamma-\beta)/(\gamma-\alpha)+\gamma(\beta-\alpha)/(\gamma-\alpha)} dx \\
&\le \Big(\int_{R^n} |f|^\alpha dx\Big)^{(\gamma-\beta)/(\gamma-\alpha)} \Big(\int_{R^n} |f|^\gamma dx\Big)^{(\beta-\alpha)/(\gamma-\alpha)} \\
&\le \|f(t)\|_{L^\alpha}^{\alpha(\gamma-\beta)/(\gamma-\alpha)} \|f(t)\|_{L^\gamma}^{\gamma(\beta-\alpha)/(\gamma-\alpha)}, \quad \text{and} \\
&\int_0^\infty \|f(t)\|_{L^\beta}^{2\beta/(\beta-n)} dt
\end{aligned}$$

$$\le \int_0^\infty \|f(t)\|_{L^\alpha}^{2\alpha(\gamma-\beta)/(\gamma-\alpha)(\beta-n)} \|f(t)\|_{L^\gamma}^{2\gamma(\beta-\alpha)/(\gamma-\alpha)(\beta-n)} dt$$
$$\le \Big(\int_0^\infty \|f(t)\|_{L^\alpha}^{2\alpha/(\alpha-n)} dt\Big)^{(\gamma-\beta)(\alpha-n)/(\gamma-\alpha)(\beta-n)}$$
$$\times \Big(\int_0^\infty \|f(t)\|_{L^\gamma}^{2\gamma/(\gamma-n)} dt\Big)^{(\beta-\alpha)(\gamma-n)/(\gamma-\alpha)(\beta-n)}$$

For all r with $m^k p \le r \le m^{k+1}p$, let $\alpha = m^k p$, $\beta = r$, $\gamma = m^{k+1}p$. Let $f(x,t) = u(x,t)$. Then $f \in L^{2\alpha/(\alpha-n)}(0,\infty; L^\alpha) \cap L^{2\gamma/(\gamma-n)}(0,\infty; L^\gamma)$. Moreover,

$$\|u(t)\|_{L^r}^r = \|u(t)\|_{L^\beta}^\beta$$
$$\le \|u(t)\|_{L^\alpha}^{\alpha(\gamma-\beta)/(\gamma-\alpha)} \|u(t)\|_{L^\gamma}^{\gamma(\beta-\alpha)/(\gamma-\alpha)}$$
$$= \|u(t)\|_{L^{m^k p}}^{m^k p(m^{k+1}p-r)/(m^{k+1}p-m^k p)} \|u(t)\|_{L^{m^{k+1}p}}^{m^{k+1}p(r-m^k p)/(m^{k+1}p-m^k p)} < \infty,$$
$$\int_0^\infty \|u(t)\|_{L^r}^{2r/(r-n)} dt$$
$$= \int_0^\infty \|u(t)\|_{L^\beta}^{2\beta/(\beta-n)} dt$$
$$\le \Big(\int_0^\infty \|u(t)\|_{L^\alpha}^{2\alpha/(\alpha-n)} dt\Big)^{(\gamma-\beta)(\alpha-n)/(\gamma-\alpha)(\beta-n)}$$
$$\times \Big(\int_0^\infty \|u(t)\|_{L^\gamma}^{2\gamma/(\gamma-n)} dt\Big)^{(\beta-\alpha)(\gamma-n)/(\gamma-\alpha)(\beta-n)}$$
$$= \Big(\int_0^\infty \|u(t)\|_{L^{m^k p}}^{2m^k p/(m^k p-n)} dt\Big)^{(m^{k+1}p-r)(m^k p-n)/(m^{k+1}p-m^k p)(r-n)}$$
$$\times \Big(\int_0^\infty \|u(t)\|_{L^{m^{k+1}p}}^{2m^{k+1}p/(m^{k+1}p-n)} dt\Big)^{(r-m^k p)(m^{k+1}p-n)/(m^{k+1}p-m^k p)(r-n)} < \infty.$$

Therefore $u \in L^\infty(0,\infty; L^r)$ for all $r : p \le r < \infty$. By Lemma 3.1, $u \in L^\infty(0,\infty; L^2)$. As before, applying the Hölder's inequality yields $u \in L^\infty(0,\infty; L^s)$ for all $2 \le s < \infty$.

To get a concrete result, let us look at the case $n = 3$. If we make the scalar product of equations (1) and $2\Delta^2 u$, integrate over R^3, we get

$$\frac{d}{dt}\|\Delta u(t)\|^2 + 2\|\nabla\Delta u(t)\|^2 = 2\int_{R^3} \nabla\Delta u \cdot \nabla(u\cdot\nabla u) dx$$
$$\le \frac{1}{4}\|\nabla\Delta u(t)\|^2 + 4\|\nabla(u\cdot\nabla u)(t)\|^2$$

$$\leq \frac{1}{4}\|\nabla\Delta u(t)\|^2 + 8\|\nabla u(t)\|_{L^4}^4 + 8\|u(t)\|_\infty^2\|\Delta u(t)\|^2.$$

We have the following estimates. Since $p > 3$, we have $2(p+12)/3(p+2) < 2$.

$$\begin{aligned}
\|u(t)\|_\infty &\leq C\|u(t)\|_{L^p}^{p/(p+2)}\|\nabla\Delta u(t)\|^{2/(p+2)},\\
\|\nabla u(t)\|_{L^4} &\leq C\|u(t)\|_{L^p}^{5p/6(p+2)}\|\nabla\Delta u(t)\|^{(p+12)/6(p+2)},\\
\|\Delta u(t)\| &\leq C\|u(t)\|_{L^p}^{2p/3(p+2)}\|\nabla\Delta u(t)\|^{(p+6)/3(p+2)},\\
8\|\nabla u(t)\|_{L^4}^4 &\leq C\|u(t)\|_{L^p}^{10p/3(p+2)}\|\nabla\Delta u(t)\|^{2(p+12)/3(p+2)}\\
&\leq C\|u(t)\|_{L^p}^{5p/(p-3)} + \frac{1}{8}\|\nabla\Delta u(t)\|^2,\\
8\|u(t)\|_\infty^2\|\Delta u(t)\|^2 &\leq C\|u(t)\|_{L^p}^{10p/3(p+2)}\|\nabla\Delta u(t)\|^{2(p+12)/3(p+2)}\\
&\leq C\|u(t)\|_{L^p}^{5p/(p-3)} + \frac{1}{8}\|\nabla\Delta u(t)\|^2.
\end{aligned}$$

We now have the estimate

$$\frac{d}{dt}\|\Delta u(t)\|^2 + \frac{3}{2}\|\nabla\Delta u(t)\|^2 \leq C\|u(t)\|_{L^p}^{5p/(p-3)}.$$

Obviously the following hold

$$\begin{aligned}
&\frac{d}{dt}[(1+t)^5\|\Delta u(t)\|^2] + \frac{3}{2}(1+t)^5\|\nabla\Delta u(t)\|^2\\
&\leq 5(1+t)^4\|\Delta u(t)\|^2 + C(1+t)^5\|u(t)\|_{L^p}^{5p/(p-3)}\\
&\leq 5(1+t)^4\|\Delta u(t)\|^2 + C(1+t)^{5-15(p-2)/4(p-3)}.
\end{aligned}$$

By using Lemma 2.4, we now obtain

$$\begin{aligned}
&\frac{d}{dt}[(1+t)^5\|\Delta u(t)\|^2] + (1+t)^5\|\nabla\Delta u(t)\|^2\\
&\leq C(1+t)^4\int_{B(t)}|\xi|^4|\widehat{u}|^2 d\xi + C(1+t)^2\\
&\leq C(1+t)^2\int_{B(t)}|\widehat{u}|^2 d\xi + C(1+t)^2\\
&\leq C(1+t)^2\|u(t)\|^2 + C(1+t)^2.
\end{aligned}$$

Integrating in time yields

$$(1+t)^5\|\Delta u(t)\|^2 + \int_0^t (1+s)^5\|\nabla\Delta u(s)\|^2 ds \le \|\Delta u_0\|^2 + C(1+t)^3.$$

Therefore

$$\sup_{0\le t<\infty} [(1+t)^2\|\Delta u(t)\|^2] \le C, \quad \int_0^\infty \|\nabla\Delta u(t)\|^2 dt \le C.$$

By Lemma 2.5 and 3.1, we have

$$\sup_{0\le t<\infty} [(1+t)\|\Delta u(t)\|^2] \le C,$$
$$\int_0^\infty \|\Delta u(t)\|^2 dt \le C,$$
$$(1+t)^{3/4}\|u(t)\|_\infty \le C.$$

This completes the proof of Theorem 4.

6. Discussion. We have repeatedly used the critical assumption (H) to justify that u has more regularity and prove the uniform stability for $n(\ge 3)$-dimensional problem (1–2). A natural question is that what space u_0 should be in such that (H) holds. Wiegner [21–23] proved that if $u_0 \in M \cap L^p$ for some $p > n$, then $(1+t)^{(n+1)/2}\|u(t)\|_\infty = O(1)$ and $(1+t)^{1+n/2}\|u(t)\|^2 = O(1)$. Therefore

$$\|u(t)\|_{L^p}^q \le (\|u(t)\|_\infty^{p-2}\|u(t)\|^2)^{2/(p-n)} \le C(1+t)^{-(n+1)(p-1)/(p-n)}.$$

It turns out that $u \in L^q(0,\infty;L^p)$, for all $p > n \ge 3$ and $q > 2$ with $n/p + 2/q = 1$.

Another sufficient condition is that u_0 is small. This is well known.

Let $L = \{u_0 \in L^2 \cap L^p|$ for some $p > n \ge 3$ and some $q > 2$, satisfying $n/p+2/q = 1$, such that $u \in L^q(0,\infty;L^p)$, where u is the solution of problem (1–2)$\}$. Our analysis shows L is not empty and is actually an open set in L^2.

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