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ON COMPACT PERTURBATIONS OF M-ACCRETIVE AND MAXIMAL MONOTONE OPERATORS IN BANACH SPACES

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Abstract. Let X be a real Banach space and G a bounded, open subset of X. The solvability of the problem $Tx+Cx \ni s$, $s \in X$ in $D(T)\cap\bar{G}$ is considered, where $T : X \supset D(T) \to 2^X$ is m-accretive and $C : D(T) \to X$ is either compact or is continuous with $(T+I)^{-1}$ being compact, under the various assumptions of boundary conditions and coercivities which the operators T and C possess. Certain eigenvalue results are given involving the solvability of $Tx + \lambda Cx \ni 0$ with respect to $(\lambda, x) \in (0,\infty) \times \partial G$. Some analogous result on maximal monotone operators are also discussed in this paper.

1. Introduction preliminaries. In this paper, the symbol X stands for a real Banach space with norm $\|\cdot\|$ and (normalized) duality mapping F. For $x \in X$ and $x^* \in X^*$, we use the symbol $\langle x, x^*\rangle$ or the symbol $\langle x^*, x\rangle$ to denote the value of x^* at x. An operator $T : X \supset D(T) \to 2^X$ is called "accretive" if for every $x, y \in D(T)$, $u \in Tx$ and $v \in Ty$, there exists $j \in F(x-y)$ such that

$$\langle u - v, j\rangle \geq 0$$

An accretive operator T is m-accretive if $R(T+\lambda I) = X$ for all $\lambda \in (0,\infty)$. An accretive operator T is called "strongly accretive" if there exists a con-

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stant $\alpha > 0$ such that : for each $x, y \in D(T)$ there exists $j \in F(x-y)$ satisfying

$$(*) \qquad \langle u - v, j\rangle \geq \alpha\|x - y\|^2$$

for all $u \in Tx, v \in Ty$. It is called strongly accretive at zero if $0 \in D(T)$ and (*) holds for all $u \in Tx, y = 0$ and $v \in T(0)$. Let a function $\phi : R_+ \to R_+$ be strictly increasing with $\phi(0) = 0$, where $R_+ = [0, \infty)$. It is called ϕ-accretive if

$$\langle u - v, j\rangle \geq \phi(\|x - y\|)\|x - y\|,$$

for all $x, y \in D(T)$, $u \in Tx$ and $v \in Ty$. It is called ϕ-expansive on $D(T)$, if $\|u - v\| \geq \phi(\|x - y\|)$ for all $x, y \in D(T)$, $u \in Tx$ and $v \in Ty$.

We denote by $B_r(0)$ the open ball with center at zero and radius $r > 0$. For an m-accretive operator T, $\lambda \in (0, \infty)$ the "resolvent" $J_\lambda : X \to D(T)$ of T are defined by $J_\lambda = (I + \frac{1}{\lambda}T)^{-1}$ and the "Yosida approximation" $T_\lambda : X \to X$ of T is defined by $T_\lambda = \lambda(I - J_\lambda)$. For $x \in X$, we define $|Tx|$ by

$$|Tx| = \lim_{\lambda\to 0} \|T_\lambda x\|.$$

if the limit exists. For a set Q, we set $|Q| = \inf\{\|y\|; y \in Q\}$.

Some well-known properties of J_λ and T_λ are given below:

1. $\|J_\lambda x - J_\lambda y\| \leq \|x - y\|$ for all $x, y \in X$.
2. $\|J_\lambda x - x\| = \frac{1}{\lambda}\|T_\lambda x\| \leq \frac{1}{\lambda}\inf\{\|y\|; y \in Tx\} = \frac{1}{\lambda}|Tx|$ for all $x \in D(T)$.
3. T_λ is m-accretive on X and $\|T_\lambda x - T_\lambda y\| \leq 2\lambda\|x - y\|$ for all $\lambda > 0$, $x, y \in X$.
4. $T_\lambda x \in TJ_\lambda x$ for all $x \in X$.

For these facts and other general results involving accretive operators, the reader may refer to Barbu [1], Browder [4], Cioranescu [5], Deimling [6], Lakshmikantham and Leela [26] and Pavel [31]. We cite the books of Lloyd [27], Rothe [33] and the paper of Nagumo [29] as references to the degree theory discussed herein. For a survey article on recent mapping theorems involving compactness and accretivity, we refer to [22]. We denote by Γ the family of all functions $\beta : R^+ \to R^+$ such that $\beta(r) \to 0$ as $r \to \infty$.

In the sequel, "continuous" means "strongly continuous" and the symbol " $\to$ " (" $\xrightarrow{w}$ ") means strong (weak) convergence. The symbol ∂G, int G, $\bar{G}$ denote the boundary, interior and closure of the set G, respectively.

An operator $T : X \supset D(T) \to Y$, Y is a Banach space, is bounded if it maps bounded subsets of $D(T)$ onto bounded set of Y. It is compact if it is continuous and maps bounded subsets of $D(T)$ onto relatively compact sets of Y. It is called demicontinuous (completely continuous) if it is strong-weak (weak-strong) continuous on $D(T)$ into Y.

2. Perturbations of M-accretive operators. Kartsatos introduced in [19, Theorem 7] a homotopy argument concerning the sum of two operators $T+C$, where T is demicontinuous, strongly accretive and C is compact. The space X^* is assumed to be uniformly convex. He showed in [19] that the equation $Tx+Cx \ni s$ can be solved under some boundary conditions on an open, bounded subset G of X. This result is based on Kartsatos' invariance of domain result for demicontinuous, ϕ-accretive or strongly accretive mappings in [18, Theorem 1]. Many authors have established new results related the involving equations $Tx + Cx \ni s$ and their applications, see [8, 12-13, 16-17, 19-25, 28, 35].

Our purpose here is to consider sums of three operators $T+A+C$, where T is m-accretive, A is demicontinous, bounded and strongly accretive, and C is compact. Results for such triplets of operators can be proved by using simple the homotopy theory for Leray-Schauder operators in connection with the methods and the results that have been developed in [8]-[25].

Theorem 1. *Let X^* be uniformly convex and $0 \in G$ be a bounded, open subset of X. Let $T : X \supset D(T) \to 2^X$ be m-accretive with $0 \in D(T)$, $A : \bar{G} \subset D(T) \to X$ bounded, demicontinuous and strongly accretive and $C : \bar{G} \to X$ compact. Assume that $Q \subset X$ and for every $q \in Q$ such that*

$$\langle Ax + Cx - q, Fx\rangle \geq 0, \quad x \in D(T) \cap \partial G. \tag{1}$$

Then $Q \subset (T+A+C)(D(T)\cap \bar{G})$.

Proof. For fixed $q \in Q$, we want to solve the problem

$$Tx + Ax + Cx \ni q.$$

First, for any positive integer n, we consider the problem

$$T_n x + Ax + Cx = q.$$

We may assume that $0 \in T(0)$, $A(0)=0$, for otherwise, we may consider the operators $\breve{T}x \equiv Tx - v$, $\breve{A}x \equiv Ax - u$, $\breve{C}x \equiv Cx + v + u$, instead of the operators T, A, C, where v is some point in $T(0)$ and $A(0)=u$. It is clear that these mappings have exactly the same properties as T, A, C, respectively. We now note that the operator $\breve{T} : x \to T_n x + Ax$ is demicontinuous and strongly accretive on $\bar{U}$, where $U \equiv D(T)\cap G$, because the Yosida approximation T_n is a Lipschitz continuous m-accretive mapping defined on all of X. Due to Theorem 1 in [18], we know that $\tilde{T}U$ is open and $\tilde{T}\bar{U}$ is closed. Moreover, $\tilde{T}^{-1}$ exists and is continous and bounded on the set $\overline{\tilde{T}U} \subset \tilde{T}\bar{U}$. Also $\partial\tilde{T}U \subset \tilde{T}(\partial U)$ and

$$\tilde{T}^{-1}(\partial\tilde{T}U) \subset \partial\tilde{T}^{-1}(\tilde{T}U) = \partial U.$$

These facts can be found in the proof of Theorem 7 [19].

It follows that the Leray-Schauder degree $d(H(t,\cdot), \tilde{T}U, 0)$ is well-defined for the mapping

$$H(t,y) = y + t(C\tilde{T}^{-1}y - q), \quad y \in \overline{\tilde{T}(U)},\ t \in [0,1].$$

We also note that $H(0,\partial\tilde{T}U) = y \in \tilde{T}(U)$ and $0 \in \tilde{T}(U)$ because $\tilde{T}(0)=0$. In order to show that $H(1,y)=0$ is solvable in $\overline{\tilde{T}(U)}$, we must show that the equation $H(t,y)=0$ has no solution on $\partial\tilde{T}(U)$ for any $t\in[0,1)$. Obviously, this is trivial for $t=0$. Assume that there exist $t\in(0,1)$ and $y_t \in \partial\tilde{T}(U)$ such that $H(t,y_t)=0$. Then, let $x_t = \tilde{T}^{-1}y_t \in \partial U$, we obtain

$$T_n x_t + Ax_t + tCx_t = tq.$$

Since $T_n(0) = 0$, we obtain that

$$\langle T_n x_t,\ F x_t\rangle \geq 0,\ \ x_t \in \partial U.$$

Using this fact, we have

$$\begin{aligned} 0 &= \langle T_n x_t + A x_t + t(C x_t - q),\ F x_t\rangle \\ &\geq \langle A x_t + t(C x_t - q),\ F x_t\rangle > 0. \end{aligned}$$

We now prove the last inequality above, that is, we show that

$$\langle A x_t + t(C x_t - q),\ F x_t\rangle > 0. \tag{2}$$

To this end, we observe that

$$\langle A x_t,\ F x_t\rangle \geq \alpha \|x_t\|^2 > 0.$$

where $\alpha > 0$ is the strong accretiveness constant of the operator A. This implies that (2) will be true if

$$\langle C x_t - q,\ F x_t\rangle \geq 0.$$

Assume that

$$\langle C x_t - q,\ F x_t\rangle < 0.$$

Then

$$\langle A x_t,\ F x_t\rangle \leq -t\langle C x_t - q,\ F x_t\rangle < -\langle C x_t - q,\ F x_t\rangle,$$

which is equivalent to

$$\langle A x_t + C x_t - q,\ F x_t\rangle < 0.$$

Since $x_t \in D(T) \cap \partial G$, we have a contradiction to the assumption (1). Consequently, (2) is true and implies that

$$\begin{aligned} 0 &= \langle T_n x_t + A x_t + t(C x_t - q),\ F x_t\rangle \\ &\geq \langle A x_t + t(C x_t - q),\ F x_t\rangle > 0. \end{aligned}$$

This contradiction implies that the equation $H(t,y)=0$ has no solution on $\partial\tilde{T}(U)$ for any $t\in[0,1)$, and hence the equation $T_nx+Ax+Cx=q$ is solvable with a solution $x_n\in\bar{U}$. Due to the boundedness of the operators A and C, we have that the sequence $\{T_nx_n\}$ is bounded and that $\{Cx_n\}$ is a precompact set. Then we have

$$\begin{aligned}
&\alpha\|x_n-x_m\|^2\\
\leq&\langle Ax_n-Ax_m,\ F(x_n-x_m)\rangle\\
=&-\langle T_nx_n-T_mx_m+Cx_n-Cx_m,\ F(x_n-x_m)\rangle\\
\leq&-\langle T_nx_n-T_mx_m,\ F(J_nx_n-J_mx_m)\rangle-\langle T_nx_n-T_mx_m,\ F(x_n-x_m)\\
&\qquad -F(J_nx_n-J_mx_m)\rangle+\|Cx_n-Cx_m\|\cdot\|x_n-x_m\|.\\
\leq&\|T_nx_n-T_mx_m\|\cdot\|F(x_n-x_m)-F(J_nx_n-J_mx_m)\|\\
&\qquad +\|Cx_n-Cx_m\|\cdot\|x_n-x_m\|.
\end{aligned}$$

At this point, we note that $\|J_nx_n-x_n\|\leq\frac{1}{n}\|T_nx_n\|$, $\|Cx_n-Cx_m\|\to 0$ as $m,n\to\infty$, and F is locally uniformly continuous. Thus, $\{x_{n_k}\}$ is a Cauchy subsequence and $x_{n_k}\to x_0\in\bar{U}$ as $k\to\infty$.

By the fact $\|J_{n_k}x_{n_k}-x_0\|\leq\|J_{n_k}x_{n_k}-x_{n_k}\|+\|x_{n_k}-x_0\|\to 0$ as $k\to\infty$, we have $J_{n_k}x_{n_k}\to x_0$. Since A is demicontinuous, $Ax_{n_k}\xrightarrow{w}Ax_0$, C is continuous, $Cx_{n_k}\to Cx_0$, hence $v_{n_k}\xrightarrow{w}-Ax_0-Cx_0+q$, where $v_n=-Ax_n-Cx_n+q\in Tx_n$. By the demiclosedness of T, we have that $x_0\in D(T)\cap\bar{G}$ and $Tx_0+Ax_0+Cx_0\ni q$ or $Q\subset(T+A+C)(D(T)\cap\bar{G})$.

As a special case of Theorem 1, we have the following result which is due to A. G. Kartsatos and X. Liu [24, Theorem 8].

Corollary 1. *Let X^* be uniformly convex. Let $T:X\supset D(T)\to 2^X$ be m-accretive with $0\in D(T)$, $A:\overline{B_b(0)}\subset D(T)\to X$ bounded, demicontinuous and strongly accretive, and $C:\overline{B_b(0)}\to X$ compact. Assume that there exist $v\in T(0)$ and $r>0$ such that*

$$\langle Ax+Cx,\ Fx\rangle\geq(r+\|v\|)b,\quad x\in\partial B_b(0).$$

Then $\overline{B_r(0)}\subset(T+A+C)(D(T)\cap\overline{B_b(0)})$.

Proof. We may set $Q = \overline{B_r(0)}$, then for every $q \in \overline{B_r(0)}$ we have

$$\begin{aligned}\langle Ax + Cx - q, Fx\rangle &\geq (r + \|v\|)b - \langle q, Fx\rangle \\ &\geq (r + \|v\|)b - r \cdot b \\ &= \|v\| \cdot b > 0,\end{aligned}$$

and Theorem 1 is applicable.

We want to study the applicability of Schauder's fixed point theorem to the problem considered herein. Kartsatos gave in [16, Lemma 1] a result in this direction to which the problem $Tx + Cx + \frac{1}{n}x = f$ is solvable for every $f \in X$, $n = 1, 2, \ldots$, provided that T is single-valued m-accretive operator with $0 \in D(T)$ and $T(0) = 0$, and $C : X \to X$ is compact such that

$$\liminf_{m\to\infty}\{(1/m) \sup_{\|x\|\leq m} (\|Cx\|)\} = 0.$$

We show below that we can extend the conclusion of this result to multi-valued operator T and obtain the sum of the operators $T + C$ is dense and surjective on X.

Theorem 2. *Let $T : X \supset D(T) \to 2^X$ be m-accretive and $C : D(T) \to X$ continuous with $(T + I)^{-1}$ being compact. Assume that*

$$\limsup_{m\to\infty, \|x\|\leq m} \{(1/m)\|Cx\|\} = 0. \tag{3}$$

Then, for each $s \in X$, the equation

$$Tx + Cx + \frac{1}{n}x \ni s, \tag{4}$$

has a solution x_n for each $n \in N$. Assume further that

$$\liminf_{\|x\|\to\infty, x\in D(T)} \frac{|Tx + Cx|}{\|x\|} > 0.$$

Then $\overline{R(T + C)} = X$. Furthermore, if one of the following conditions holds:
(a) *X is uniformly convex and $C : \overline{D(T)} \to X$ is completely continuous,*
(b) *C is bounded,*
then $R(T + C) = X$.

Proof. We may assume that $0 \in D(T)$ and $0 \in T(0)$. Otherwise, for $z \in D(T)$, $v \in Tz$, we may consider the operators $\tilde{T}(x) \equiv T(x+z) - v$, $\tilde{C}(x) \equiv C(x+z) + v$, for every $x \in D(T) - z$. it is easy to see that the operator $\tilde{T}$ is m-accretive on $D(\tilde{T}) \equiv D(T) - z$. To show the compactness of the operator $(\tilde{T}+I)^{-1}$ and hence $(\tilde{T}+\lambda I)^{-1}$, for every $\lambda > 0$. We note that the m-accretivity of the operator $\tilde{T}$, implies the continuity of $(\tilde{T}+I)^{-1}$. Let $\{y_n\}$ be a bounded sequence in X and let $x_n = (\tilde{T}+I)^{-1}y_n$. Then

$$y_n = \tilde{v}_n + x_n = v_n + x_n - v,$$

where $\tilde{v}_n \in \tilde{T}x_n$ and $v_n \in T(x_n + z)$. Thus,

$$(x_n + z) + v_n = y_n + z + v,$$

or

$$x_n = (T+I)^{-1}(y_n + z + v) - z.$$

By the boundedness of $\{y_n\}$, z and v, as well as the compactness of the operator $(T+I)^{-1}$, we concluded that $\{x_n\}$ lies in a compact set. This proves the compactness of the operator $(\tilde{T}+I)^{-1}$. To see that (3) is satisfied for the operator $\tilde{C}$, we observe that

$$\begin{aligned}
& m^{-1}\Big(\sup_{\|x\|\le m}\{\|\tilde{C}x\|\}\Big) \\
=& m^{-1}\Big(\sup_{\|x\|\le m}\{\|C(x+z)+v\|\}\Big) \\
\le& m^{-1}\Big(\sup_{\|x\|\le m}\{\|C(x+z)\|\} + \|v\|\Big) \\
=& m^{-1}\Big(\sup_{\|y-z\|\le m}\{\|Cy\|\} + \|v\|\Big) \\
\le& m^{-1}\Big(\sup_{\|y\|\le m+\|z\|}\{\|Cy\|\}\Big) + m^{-1}\|v\| \\
=& m^{-1}(m+\|z\|)\cdot\Big(\frac{1}{m+\|z\|}\Big)\Big(\sup_{\|y\|\le m+\|z\|}\{\|Cy\|\}\Big) + m^{-1}\|v\| \\
& \to 0 \text{ as } m \to \infty.
\end{aligned}$$

we have shown that it suffices to prove the theorem with $0 \in D(T)$ and $0 \in T(0)$.

Now, we fix n, and consider the equation

$$u = \left(T + \frac{1}{n}I\right)^{-1}(s - Cu),\ u \in X.$$

Now, let $u \in D(T)$ be given. Then for $j \in F(u)$, we have

$$\langle v + \frac{1}{n}u, j\rangle = \langle v, j\rangle + \frac{1}{n}\|u\|^2 \geq \frac{1}{n}\|u\|^2, \tag{5}$$

where $v \in Tu$. In order to solve (4), we apply the Schauder theorem to the compact operator $U : X \to X$ defined by

$$Uu = \left(T + \frac{1}{n}I\right)^{-1}(s - Cu),\ u \in X.$$

Now we claim that U maps some closed ball of X into itself, suppose that this is not true. Then for each $m \in N$, there exists u_m such that $u_m \in \overline{B_m(0)}$ and $\|Uu_m\| > m$. It follows from (5) that we have

$$\begin{aligned} m < \|Uu_m\| &= \left\|\left(T + \frac{1}{n}I\right)^{-1}(s - Cu_m)\right\| \\ &\leq n(\|s\| + \|Cu_m\|) \end{aligned}$$

and by (3)

$$1 < n\left[\frac{1}{m}\|s\| + \frac{1}{m}\|Cu_m\|\right] \to 0 \quad \text{as } m \to \infty.$$

This is a contradiction. Consequently, there is a $r > 0$ such that $U(\overline{B_r(0)}) \subset \overline{B_r(0)}$. The Schauder theorem implies the solvability of the equation $Ux = x$, i.e., the solvability of the inclusion (4).

Let x_n be a solution of the equation (4). We claim that $\{x_n\}$ is a bounded sequence. To see this, assume that $\|x_n\| \to \infty$ as $n \to \infty$. Then, by our hypothesis, there exists a positive number p such that

$$\liminf_{n\to\infty} \frac{|Tx_n + Cx_n|}{\|x_n\|} \geq \liminf_{\|x\|\to\infty, x\in D(T)} \frac{|Tx + Cx|}{\|x\|} \geq p > 0.$$

However, we know that for some $v_n \in Tx_n$, we have

$$\|v_n + Cx_n\| = \left\| s - \frac{1}{n}x_n \right\| \leq \frac{1}{n}\|x_n\| + \|s\|$$

which implies

$$p \leq \liminf_{n\to\infty} \frac{\|v_n + Cx_n\|}{\|x_n\|} \leq \liminf_{n\to\infty} \left[\frac{1}{n} + \frac{\|s\|}{\|x_n\|}\right] = 0.$$

This contradiction says that $\{x_n\}$ is bounded, and hence $\frac{1}{n}x_n \to 0$ as $n \to \infty$. So we have $s \in \overline{R(T+C)}$.

Now, if (a) holds, then C is completely continuous and X is uniformly convex. Since $\{x_n\}$ is a bounded sequence, we may assume that $x_n \xrightarrow{w} x_0$ for some $x_0 \in \overline{D(T)}$ and that $Cx_n \to Cx_0$ implies $v_n \in Tx_n, v_n = -Cx_n - \frac{1}{n}x_n + s \to -Cx_0 + s$ as $n \to \infty$. By the fact that every m-accretive operator on uniformly convex space is demiclosed, so we have $x_0 \in D(T)$ and $Tx_0 + Cx_0 \ni s$.

In case (b), C is bounded. Since $\{x_n\}$ is a bounded sequence and satisfies that $v_n + Cx_n + \frac{1}{n}x_n = s$, for some $v_n \in Tx_n$. We may have

$$x_n = (T+I)^{-1}\left[s - Cx_n + (1 - \frac{1}{n})x_n\right].$$

By the boundedness of the sequence $\{x_n\}$ and the operator C, as well as the compactness of $(T+I)^{-1}$ implies that $\{x_n\}$ lies in a compact set. Thus, it has a convergent subsequence $\{x_{n_k}\}$, say $x_{n_k} \to x_0 \in \overline{D(T)}$. The continuity of C implies $v_{n_k} = -Cx_{n_k} - \frac{1}{n_k}x_{n_k} + s \to -Cx_0 + s$ as $k \to \infty$, and the closedness of T implies that $x_0 \in D(T)$ and $Tx_0 + Cx_0 \ni s$.

The following result provides a "product condition" on CJ_1x for the solvability of $Tx + Cx \ni s$. He [15] studied a variant of equation, for single-valued operator T, by

(6) $$TJ_1x + CJ_1x + \frac{1}{n}x = s.$$

This equation is equivalently to the equation $x - Sx = 0$, where

(7) $$Sx = \frac{n}{1+n}(I - C)J_1x + \frac{n}{1+n}s.$$

If J_1 is a compact operator, then the operator S is also compact and (6) can be solved by using homotopy argument associated with the equation $x - Sx = 0$.

We are going to use equation (7) in order to get a new result for the range of $T + C$.

Theorem 3. *Let $T : X \supset D(T) \to 2^X$ be m-accretive with $0 \in D(T)$, $0 \in T(0)$ and $(T + I)^{-1}$ compact. Let $C : D(T) \to X$ be continuous. Let $s \in X$ and assume that there exist $K(s) > 0$ and $\beta = \beta_s \in \Gamma$ such that*

$$\langle CJ_1x - s, j\rangle \geq -K(s) - \beta(\|x\|)\|x\|,$$

for all $x \in X$ with $\|x\|$ sufficiently large and some $j \in F(x)$. Then $s \in \overline{R(T + C)}$.

Proof. We consider the approximating equation

$$TJ_1x + CJ_1x + \frac{1}{n}x = s, \ n = 1, 2, \ldots,$$

where s is a given point in X and $J_1 = (T + I)^{-1}$, which is equivalent to the equation

$$x = \frac{n}{1+n}(I - C)J_1x + \frac{n}{1+n}s. \tag{8}$$

Define the homotopy mapping $H(t, x)$ as follows: For $x \in X$,

$$H(t, x) = t\left[\frac{n}{1+n}(I - C)J_1x + \frac{n}{1+n}s\right], \ t \in (0, 1],$$

and $H(0, x) = 0$. Since J_1 is a compact operator, C is a continuous operator, therefore for each $t \in (0, 1]$, the mapping $H(t, x)$ is compact on $x \in X$. It follows that the Leray-Schauder degree $d(I + H(t, \cdot), Q, 0)$ is well-defined for any ball $Q \equiv B_r(0)$ for some $r > 0$, provided that the equation $x - H(t, x) = 0$ has no solution on $\partial B_r(0)$ for all $t \in [0, 1]$.

In order to show that there is some $r > 0$ satisfying

$$\begin{aligned} d(I-H(t,\cdot),\ Q,\ 0) &= d(I-H(1,\cdot),\ Q,\ 0) \\ &= d(I-H(0,\cdot),\ Q,\ 0) \\ &= 1, \end{aligned}$$

for all $t \in [0,1]$, we must show that all possible solutions x of the equations $x - H(t,x) = 0$ are uniformly bounded, i.e., they all lie in a ball $Q \equiv B_r(0)$ for some $r > 0$. If this is not true, there exist $\{t_m\} \subset (0,1]$ and $\{x_m\} \subset X$ such that

$$x_m = \frac{n \cdot t_m}{1+n}(I-C)J_1 x_m + \frac{n \cdot t_m}{1+n} s$$

and $\|x_m\| \to \infty$. Since $J_1 x_m \in D(T)$, $0 \in D(T)$ and $0 \in T(0)$ we have $\|J_1 x_m\| \le \|x_m\|$. Hence for some $j \in F(x_m)$ we have

$$\begin{aligned} \|x_m\|^2 = \langle x_m, j\rangle &\le \frac{n \cdot t_m}{1+n}\|x_m\|^2 - \frac{n \cdot t_m}{1+n}\langle CJ_1 x_m - s, j\rangle \\ &\le \frac{n}{1+n}\|x_m\|^2 + \frac{n}{1+n}[K(s) + \beta(\|x_m\|)\|x_m\|] \end{aligned}$$

and thus

$$\frac{1}{n}\|x_m\|^2 \le K(s) + \beta(\|x_m\|)\|x_m\|.$$

This shows the boundedness of the sequence $\{x_m\}$ and implies the solvability of eq. (8). Let u_n be a solution of eq. (8) for each $n \in N$. Then for some $j \in F(u_n)$, we have

$$\begin{aligned} \|u_n\|^2 &= \frac{n}{1+n}\langle J_1 u_n, j\rangle - \frac{n}{1+n}\langle CJ_1 u_n - s, j\rangle \\ &\le \frac{n}{1+n}\|u_n\|^2 + \frac{n}{1+n}[K(s) + \beta(\|u_n\|)\|u_n\|]. \end{aligned}$$

This implies that

$$\frac{1}{n}\|u_n\|^2 \le K(s) + \beta(\|u_n\|)\|u_n\|.$$

for all large n. We conclude that $\frac{1}{n}\|u_n\| \to 0$ as $n \to \infty$ in all possible cases. Thus, we have $TJ_1 u_n + CJ_1 u_n \to s$, as $n \to \infty$. Consequently, $s \in \overline{R(T+C)}$.

In order to prove our next theorem, we need the following theorem which can be found in Guan and Kartsatos [14].

Theorem A. *Let $G \subset X$ be open, bounded with $0 \in G$. Let $C : \bar{G} \to X$ be compact and such that $\|Cx\| \geq \alpha$, $x \in \partial G$, where α is a positive constant number. Then there exists $\lambda_0 > 0$ and $x \in \partial G$ such that $(I - \lambda_0 C)x = 0$.*

The next result provides a method for proving the existence of certain eigenvalues for the pair (T, C). A real number λ is called an eigenvalue of a pair of operators (T, C) if the equation $\lambda Tx + Cx \ni 0$ is solvable in $D(T) \cap D(C)$. The following theorem generalizes some results in Guan and Kartsatos [14]. The operator T does not have to be ϕ-expansive on ∂G. We also assume the boundedness of the operator T only on the set ∂G, for the approximate solvability of the relevant eigenvalue problems.

Theorem 4. *Let G be an open, bounded subset of X with $0 \in G$. Let $T : \bar{G} \to 2^X$ be accretive with $0 \in T(0)$, $0 \notin T(\partial G)$ and $T(\partial G)$ bounded. Let $C : \bar{G} \to X$ be continuous with $C(T+I)^{-1}$ being compact. Let the constant $\alpha > 0$ be such that $\|Cx\| \geq \alpha$, for all $x \in \partial G$, and satisfy one of the following conditions:*
(i) X^* is uniformly convex and T is demicontinuous,
(ii) T is continuous.
Then there exist $\lambda_0 > 0$ and $x_0 \in \partial G$ such that $\lambda_0 Cx_0 \in Tx_0$.

Proof. We consider the inclusion problem

$$Tx - \lambda Cx + \frac{1}{n}x \ni 0, \; n = 1, 2, \ldots, \tag{9}$$

or equivalently

$$u - \lambda C(T + \frac{1}{n}I)^{-1}u = 0 \tag{10}$$

for all $n \in N$. We want to show that (9) has at least one solution, say, $(\lambda_n, x_n) \in (0, \infty) \times \partial G$. Then we shall show that $Tx - \lambda Cx \ni 0$ is solvable, with solution $(\lambda_0, x_0) \in (0, \infty) \times \partial G$. If $(\lambda, u) \in (0, \infty) \times \partial(\tilde{T}G)$ is a solution of (10), then $(\lambda, x) \in (0, \infty) \times \partial G$ is a solution of (9), where $\tilde{T} \equiv T + \frac{1}{n}I$ and $x = (T + \frac{1}{n}I)^{-1}u$. In fact, $\tilde{T}$ is a strongly accretive and injective mapping such that $\tilde{T}G$ is open and $\tilde{T}\bar{G}$ is closed under the assumption (i)

[18, Theorem 1] or the assumption (ii) [7, Theorem 3]. Hence, under either one of these assumptions we have

$$\tilde{T}G \cup \partial(\tilde{T}G) = \overline{\tilde{T}G} \subset \overline{\tilde{T}\bar{G}} = \tilde{T}\bar{G} = \tilde{T}G \cup \tilde{T}(\partial G),$$

which implies that $\partial(\tilde{T}G) \subset \tilde{T}(\partial G)$. Consequently, the mapping $y \to y - \lambda C\tilde{T}^{-1}y$ is well-defined on $\overline{\tilde{T}G}$, and the range of the mapping $y \to \lambda C\tilde{T}^{-1}y$ on $\overline{\tilde{T}G}$ is a relatively compact subset of X. We also observe that if $u \in \partial(\tilde{T}G)$ then $\tilde{T}^{-1}u \in \tilde{T}^{-1}(\partial\tilde{T}G) \subset \partial G$ and $\|C\tilde{T}^{-1}u\| \geq \alpha$. Appling Theorem A, we obtain, for each n, a solution $(\lambda_n, u_n) \in (0,\infty) \times \partial(\tilde{T}G)$ of eq. (10). Letting $x_n = \tilde{T}^{-1}u_n$, we have the solvability of (9) with solution $(\lambda_n, x_n) \in (0,\infty) \times \partial G$, or

$$Tx_n - \lambda_n Cx_n + \frac{1}{n}x_n \ni 0, \ n = 1, 2, \ldots.$$

Since $x_n = \tilde{T}^{-1}u_n$, we have that

$$u_n = v_n + \frac{1}{n}x_n \tag{11}$$

for some $v_n \in Tx_n$. Due to the boundedness of $\{v_n\}$, $\{x_n\}$, we have that $\{u_n\}$ is bounded. From the fact that $\{u_n\}$ is bounded and $\|C(T+\frac{1}{n}I)^{-1}u_n\| \geq \alpha$, it follows that $\{\lambda_n\}$ is also bounded. Thus, we may assume that $\lambda_{n_k} \to \lambda_0$ as $n_k \to \infty$. From (11), we have

$$v_n + x_n = u_n + (1 - \frac{1}{n})x_n,$$

that is

$$x_n = (T+I)^{-1}\left[u_n + \left(1 - \frac{1}{n}\right)x_n\right],$$

which implies that

$$C(T + \frac{1}{n}I)^{-1}u_n = Cx_n = C(T+I)^{-1}\left[u_n + \left(1 - \frac{1}{n}\right)x_n\right].$$

The fact $C(T+I)^{-1}$ is compact and $\{u_n\}$, $\{x_n\}$ are bounded imply that $\{Cx_n\}$ lies in a compact set, we may assume that $Cx_{n_k} \to y \in X$. Hence we obtain that

$$v_{n_k} + \frac{1}{n_k} x_{n_k} = \lambda_{n_k} C x_{n_k} \to \lambda_0 y$$

for some $v_{n_k} \in T x_{n_k}$. Moreover, $\tilde{T} \equiv T + \frac{1}{n} I$ is a strongly accretive on ∂G, thus $x_{n_k} \to x_0$, for some $x_0 \in \bar{G}$. Under the hypothesis of (i) T is demicontinuous, $T x_{n_k} \xrightarrow{w} T x_0$. By the continuity of C, we have

$$v_{n_k} = -\frac{1}{n_k} x_{n_k} + \lambda_{n_k} C x_{n_k} \to \lambda_0 C x_0,$$

for some $v_{n_k} \in T x_{n_k}$. Due to the demiclosedness of T, we have $x_0 \in D(T)$ and $\lambda_0 C x_0 \in T x_0$.

Under the hypothesis of (ii), T is continuous, it is obvious that $\lambda_0 C x_0 \in T x_0$,

Corollary 2. *Let $G \subset X$ be open, bounded with $0 \in G$. Let $T : \bar{G} \to X$ be bounded, accretive with $T(0) = 0$ and $C : \bar{G} \to X$ compact. Assume further that $Tx \neq 0$, $x \in \partial G$ and T is ϕ-expansive on ∂G. Let the constant $\alpha > 0$ be such that $\|Cx\| \geq \alpha$, $x \in \partial G$, and satisfy one of the following conditions:*

(i) *X^* is uniformly convex and T is demicontinuous.*

(ii) *T is continuous.*

Then there exists $(\lambda_0, x_0) \in (0, \infty) \times \partial G$ such that $T x_0 - \lambda_0 C x_0 = 0$.

Proof. The fact that $C : \bar{G} \to X$ is compact, it is obviously that C is continuous and $C(T+I)^{-1}$ is compact. Since $T : \bar{G} \to X$ is bounded, it is easy to see that $T(\partial G)$ is bounded. It follows from the proof of Theorem 4, we have that this conclusion holds.

3. Perturbations of maximal monotone operators. In this section, we shall consider analogous results in [12] and [23] with some weaker conditions. The operator $T : X \to 2^{X^*}$ will be assumed to be maximal monotone. We assume that the space X is a real reflexive locally uniformly convex Banach space with locally uniformly convex dual space X^*. The duality mapping F is now singe-valued and bicontinuous.

An operator $T : X \supset D(T) \to 2^{X^*}$ is monotone if for every $x, y \in D(T)$ and $u \in Tx$, $v \in Ty$ we have

$$(**) \qquad \langle u - v, x - y \rangle \geq 0.$$

A monotone operator T is strongly monotone if 0 in the right-hand side of (**) is replaced by $\alpha\|x - y\|^2$ where $\alpha > 0$ is a fixed constant. A monotone operator T is called maximal monotone if $R(T + \lambda F) = X^*$ for all $\lambda > 0$. An operator $T : X \supset D(T) \to 2^{X^*}$ is said to be of "type (S_+)" if for every sequence $\{x_n\} \subset D(T)$ with $x_n \xrightarrow{w} x_0 \in X$, and $\limsup_{n\to\infty} < v_n, x_n - x_0 > \leq 0$, for some $v_n \in Tx_n$, we have $x_n \to x_0$. It is well-known that, under our assumptions on the space X, X^*, the duality mapping F is of type (S_+) on X.

For fundamental properties of monotone operators and other related concepts, the reader can refer to Barbu [1], Barbu and Precupanu [2], Browder [4], Cioranescu [5], Pascali and Sburlan [30], Phelps [32] and Zeidler [34].

For other recent results of this nature, we refer to the papers by Brézis, Crandall and Pazy [3], Guan [9-11], Guan and Kartsatos [12-13] and Kartsatos [23].

The next result improves Theorem 2 of Guan and Kartsatos [12]. There, it was assumed that T m-accretive and A with positively homogeneous of degree $q \in (0, 1]$ and C, positively homogeneous of degree $p \in (1, \infty)$.

Theorem 5. *Let X^* be strictly convex. Let $T : X \supset D(T) \to 2^{X^*}$ be maximal monotone and strongly monotone at 0 with constant $\alpha > 0$ and $0 \in D(T)$, $0 \in T(0)$. Let $A, C : \overline{D(T)} \to X^*$ be completely continuous. Assume further that $\langle Cu, u\rangle \geq 0$ for every $u \in \overline{D(T)}$. Moreover, $\frac{\|Ax_n\|}{\|x_n\|} \to 0$ for any sequence $\{x_n\} \subset \overline{D(T)}$ with $\|x_n\| \to \infty$. Then $R(T + A + C) = X$.*

Proof. We want to solve the inclusion

$$Tx + Ax + Cx \ni f,$$

where f is any (but fixed) point in X^*. To this end, we consider the approximate problem

(12) $$Tx + Ax + Cx + \frac{1}{m}Fx \ni f,$$

for every $m = 1, 2, \ldots$. Since T is maximal monotone, we can define $R_m = (T + \frac{1}{m}F)^{-1} : X^* \to D(T)$. Since X is reflexive locally uniformly convex and X^* is strictly convex, the duality mapping F is of type (S_+). This implies that R_m is continuous [11,Theorem 2.1]. Now, (12) is equivalent to

(13) $$x + R_m(Ax + Cx - f) = 0.$$

Since R_m is continuous and A, C are compact, $R_m(A+C-f) : X^* \to D(T)$ is compact. By the Leray-Schauder degree theory (see Lloyd [27]), (13) is solvable if we can show that there exists $b > 0$ such that $x + tR_m(Ax + Cx - f) \neq 0$ for any $t \in [0,1]$, $x \in \partial B_b(0)$. Equivalently, we only need to show that the solutions of $x + tR_m(Ax + Cx - f) = 0$, for any $t \in [0,1]$, are uniformly bounded. This is certainly true for $t = 0$. If $t = 1$, then (12) is solvable, hence we take $t \in (0,1)$. Assume that there exist $\{t_n\} \subset (0,1)$ and $\{u_n\} \subset D(T)$ such that

$$u_n + t_n R_m(Au_n + Cu_n - f) = 0$$

and $\|u_n\| \to \infty$ as $n \to \infty$. We have $\frac{1}{t_n}u_n = -R_m(Au_n + Cu_n - f) \in D(T)$, and

$$T\left(\frac{1}{t_n}u_n\right) + \frac{1}{m}F\left(\frac{1}{t_n}u_n\right) + Au_n + Cu_n \ni f,$$

or

$$\frac{1}{m}Fu_n \in -t_n T\left(\frac{1}{t_n}u_n\right) - t_n Au_n - t_n Cu_n + t_n f,$$

which implies

$$\frac{1}{m}\langle Fu_n, u_n\rangle = -t_n^2\langle v_{t,n}, \frac{1}{t_n}u_n\rangle - t_n\langle Au_n, u_n\rangle - t_n\langle Cu_n, u_n\rangle + t_n\langle f, u_n\rangle,$$

where $v_{t,n} = -\frac{1}{m}F(\frac{1}{t_n}u_n) - Au_n - Cu_n + f \in T(\frac{1}{t_n}u_n)$, and we get that

$$\frac{1}{m}\|u_n\|^2 \leq -t_n\langle Au_n, u_n\rangle + t_n\langle f, u_n\rangle,$$

thus

$$\frac{1}{m} \leq -t_n\langle \frac{1}{\|u_n\|}\cdot Au_n, \frac{1}{\|u_n\|}\cdot u_n,\rangle + t_n\langle f, \frac{1}{\|u_n\|}\cdot u_n\rangle \cdot \frac{1}{\|u_n\|} \to 0 \text{ as } n\to\infty.$$

This is a contradiction. Hence, we obtain that (12) and (13) are solvable for any integer m.

Now, we are going to show that all solutions of (12) are uniformly bounded with respect to m. If this is not true, we may assume that there exists $\{w_m\}\subset D(T)$, $Tw_m + Aw_m + Cw_m + \frac{1}{m}Fw_m - f \ni 0$, and $\|w_m\|\to\infty$ as $m\to\infty$. Since X is reflexive, we may assume that $\frac{w_m}{\|w_m\|} \xrightarrow{w} w_0$. Since

$$-Aw_m = v_m + Cw_m + \frac{1}{m}Fw_m - f,$$

for some $v_m \in Tw_m$, and thus

$$\begin{aligned}-\langle Aw_m, w_m\rangle &= \langle v_m, w_m\rangle + \langle Cw_m, w_m\rangle + \frac{1}{m}\langle Fw_m, w_m\rangle - \langle f, w_m\rangle \\ &\geq \alpha\|w_m\|^2 - \langle f, w_m\rangle,\end{aligned}$$

which implies

$$-\langle \frac{Aw_m}{\|w_m\|}, \frac{w_m}{\|w_m\|}\rangle \geq \alpha - \langle f, \frac{w_m}{\|w_m\|}\rangle \cdot \frac{1}{\|w_m\|} \to \alpha > 0,$$

as $m\to\infty$, the left-hand side of the last inequality converges to zero, and we have a contradiction. Therefore, we get that $\{w_m\}$ is bounded.

Now, since X is reflexive, we may assume that $w_m \xrightarrow{w} w_0$, for some $w_0\in X$. Then $\frac{1}{m}Fw_m \to 0$ and by our assumptions, $Cw_m \to Cw_0$, $Aw_m \to Aw_0$. So, $v_m \to -Aw_0 - Cw_0 + f$. Since T is maximal monotone and T is demiclosed, hence we have $w_0 \in D(T)$ and $Tw_0 \ni -Aw_0 - Cw_0 + f$ or $Tw_0 + Aw_0 + Cw_0 \ni f$.

We now give Theorem 6 below, which generalizes the main result of Kartsatos in [23, Theorem 7]. Here, we assume that without the assumption that $0 \notin Tx - v^*$, for some $z \in D(T)\cap G$ and some $v^*\in Tz$, and for every $x\in D(T)\cap\partial G$ and without the assumption that $(T+F)^{-1}$ is compact.

Theorem 6. *Let $T : X \supset D(T) \to 2^{X^*}$ be maximal monotone and $C : D(T) \to X^*$ with $C(T+F)^{-1}$ being compact. Assume, further that $G \subset X$ is open, bounded and such that for some $p \in X^*$, $z \in D(T) \cap G$ satisfying that*

$$\langle u + Cx - p, x - z \rangle > 0, \ (x, u) \in (D(T) \cap \partial G) \times Tx. \tag{14}$$

Then $p \in \overline{(T+C)(D(T) \cap \bar{G})}$.

Proof. We want to solve the problem

$$Tx + Cx + \frac{1}{n} Fx \ni p, \quad \forall n \in N, \tag{15}$$

or the equivalently equation

$$u + C(T + \frac{1}{n} F)^{-1} u = p. \tag{16}$$

We may assume that $z = 0 \in D(T) \cap G$ and $0 \in T(0)$. In fact, if this is not true, we consider the new operators $\tilde{T}, \tilde{C}$ defined by

$$\tilde{T}x \equiv T(x+z) - v, \tilde{C}x \equiv C(x+z) + v, x \in D(\tilde{T}) \equiv D(T) - z,$$

where $v \in Tz$. We also set $\tilde{G} \equiv G - z$. It is easy to see that the operator $\tilde{T}$ is maximal monotone on $D(\tilde{T})$. To show the compactness of $\tilde{C}(\tilde{T} + F)^{-1}$, we must show that $\{\tilde{C}(\tilde{T} + F)^{-1} u_n\}$ is a relatively compact set, for any bounded sequence $\{u_n\} \subset X^*$. To this end, let $y_n = (\tilde{T} + F)^{-1} u_n$, by the boundedness of $(\tilde{T} + F)^{-1}$ we have that $\{y_n\}$ is a bounded sequence and

$$\begin{aligned} u_n \in \tilde{T} y_n + F y_n &= T(y_n + z) - v + F y_n \\ &= T(y_n + z) + F(y_n + z) - v + F y_n - F(y_n + z), \end{aligned}$$

that is

$$T(y_n + z) + F(y_n + z) \ni u_n + v + [F(y_n + z) - F y_n],$$

which implies that

$$y_n + z = (T + F)^{-1} [u_n + v + F(y_n + z) - F y_n],$$

and

$$\tilde{C}(\tilde{T}+F)^{-1}u_n = \tilde{C}y_n = C(y_n+z)+v$$
$$= C(T+F)^{-1}[u_n+v+F(y_n+z)-Fy_n]+v.$$

By the boundedness of $\{u_n\}$, v, $\{F(y_n+z)\}$, $\{F(y_n)\}$ and the compactness of $C(T+F)^{-1}$, we have $\tilde{C}(\tilde{T}+F)^{-1}u_n$ is a relatively compact set.

If X^* is locally uniformly convex, it is well-known that F is continuous. In order to show that $\tilde{C}(\tilde{T}+F)^{-1}$ is continuous, let $\{u_n\} \subset X^*$ with $u_n \to u_0 \in X^*$. Since $\tilde{T}$ is maximal monotone and X is locally uniformly convex space, implies that $(\tilde{T}+F)^{-1}$ is continuous [13, Lemma 3.1] and we have

$$y_n \equiv (\tilde{T}+F)^{-1}u_n \to (\tilde{T}+F)^{-1}u_0 \equiv y_0.$$

Moreover, by the continuity of $C(T+F)^{-1}$, we have that

$$\begin{aligned}\tilde{C}(\tilde{T}+F)^{-1}u_n &= C(T+F)^{-1}[u_n+v+F(y_n+z)-Fy_n]+v\\ &\to C(T+F)^{-1}[u_0+v+F(y_0+z)-Fy_0]+v\\ &= \tilde{C}(\tilde{T}+F)^{-1}u_0,\end{aligned}$$

completing the proof of the compactness of $\tilde{C}(\tilde{T}+F)^{-1}$. To see that (14) is satisfied with $z=0$, it suffices to observe that

$$\langle (w-v)+C(x+z)+v-p, x\rangle > 0,$$

for every $x \in D(\tilde{T}) \cap \partial\bar{G}$ and every $w \in T(x+z)$. Thus it suffices to prove the theorem with $z=0$ and $0 \in T(0)$.

Since T is maximal monotone and X is locally uniformly convex, we have $(T+\frac{1}{n}F)^{-1}$ is a continuous mapping on all of X, if $(T+\frac{1}{n}F)$ is denoted by T_0, then T_0 is a set-valued mapping that maps relatively open (closed) sets in its domain $D(T)$ onto open sets in the space X. For the set $T_0(G\cap D(T))$ is open in X and $T_0(\bar{G}\cap D(T))$ is closed in X [18, Theorem 1]. Hence we have

$$\begin{aligned} T_0(G\cap D(T))\cup\partial(T_0(G\cap D(T))) &= \overline{T_0(G\cap D(T))} \\ &\subset \overline{T_0(\bar{G}\cap D(T))} \\ &= T_0(\bar{G}\cap D(T)) \\ &= T_0(G\cap D(T))\cup T_0(\partial G\cap D(T)), \end{aligned}$$

which implies that $\partial(T_0(G\cap D(T))\subset T_0(\partial G\cap D(T))$. Since

$$\overline{T_0(G\cap D(T))}\subset T_0(\bar{G}\cap D(T)) \text{ or } T_0^{-1}(\partial(T_0G\cap D(T)))\subset \partial G\cap D(T).$$

Now, we consider the homotopy mapping

$$H(t,u)\equiv u+t\left[C(T+\frac{1}{n}F)^{-1}u-p\right],\ (t,u)\in[0,1]\times\overline{T_0(G\cap D(T))}.$$

If we show that $0\neq H(t,\partial T_0(G\cap D(T)))$, then the Leray-Schauder degree $d(H(t,\cdot),\ T_0(G\cap D(T)),\ 0)$ is well-defined, for all $t\in[0,1]$, because $0\in T_0(0)$ and the range of the mapping $u\to C(T+\frac{1}{n}F)^{-1}u$ on $\overline{T_0(G\cap D(T))}$ is a relatively compact subset of X^*. To show that (15) or (16) is solvable, it suffices to show that $H(t,\cdot)$ has no zero on $\partial T_0(G\cap D(T))$ for any $t\in[0,1)$. This is certainly true for $t=0$. Assume that $u_t\in\partial T_0(G\cap D(T))$, for some $t\in(0,1)$ and let

$$x_t=\left(T+\frac{1}{n}F\right)^{-1}u_t\in\partial G\cap D(T).$$

Then

$$Tx_t+tCx_t+\frac{1}{n}Fx_t\ni tp. \tag{17}$$

However we shall show that (17) does not hold by showing that

$$\langle v_t+t(Cx_t-p),x_t\rangle>0, \tag{18}$$

for some $v_t\in Tx_t$, for all $t\in(0,1)$. If $\langle Cx_t-p,x_t\rangle\geq 0$, then our assertion is trivally true. Let $\langle Cx_t-p,x_t\rangle<0$, by our assumption

$$\langle v_t,x_t\rangle>-\langle Cx_t-p,x_t\rangle>-t\langle Cx_t-p,x_t\rangle,$$

and

$$\langle v_t + t(Cx_t - p), x_t\rangle > 0.$$

Combine the inequalities (17) and (18), we get the contradiction:

$$\frac{1}{n}\|x_t\|^2 \leq \frac{1}{n}\|x_t\|^2 + \langle v_t + t(Cx_t - p), x_t\rangle = 0.$$

Thus, $H(1,u) = 0$ is solvable with solution $u \in \overline{T_0(G \cap D(T))}$, i.e., $Tx + Cx + \frac{1}{n}Fx \ni p$ is solvable with solution $x_n \in \bar{G} \cap D(T)$. Since $\bar{G} \cap D(T)$ is bounded, we have $p \in \overline{(T+C)(D(T) \cap \bar{G})}$.

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