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UNIQUENESS OF MEROMORPHIC FUNCTIONS

BY

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Abstract. We prove a uniqueness theorem for meromorphic functions.

1. Introduction and definitions. Let f and g be two nonconstant meromorphic functions defined in the open complex plane C. In the paper we deal with the problem of finding out relations between f and g on the basis of their a-points. We do not explain the standard notations and definitions because they are available in [2].

Definition 1[1]. We denote by $E(a,k;f)$ the set of distinct zeros of $f-a$ $(a \in C)$ whose multiplicities are less than or equal to k, where k is a positive integer or infinity.

Definition 2. If k is a nonnegative integer or infinity, we denote by $n_k(r,a;f)$ the number of zeros of $f-a$ in $|z| \leq r$, where a zero of multiplicity p is counted p times if $p \leq k$ and $1+k$ times if $p > k$; $N_k(r,a;f)$ is defined in terms of $n_k(r,a;f)$ in the usual way.

Definition 3. We define $\delta_k(a;f) = 1\text{-}\limsup\limits_{r\to\infty} \frac{N_k(r,a;f)}{T(r,f)}$. Then $0 \leq \delta(a;f) \leq \delta_k(a;f) \leq \delta_{k-1}(a;f) \leq \cdots \leq \delta_0(a;f) = \Theta(a;f) \leq 1$ and $\delta_\infty(a;f) = \delta(a;f)$.

Definition 4. We denote by $\overline{n}(r,a;f,\leq k)$ and $\overline{N}(r,a;f,\leq k)$ the counting functions for distinct zeros of $f-a$ of multiplicities not greater

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than k. If $k \le 0$, we take $\overline{n}(r,a;f,\le k) \equiv 0$ and so $\overline{N}(r,a;f,\le 0) \equiv 0$. Also we define $\overline{n}(r,a;f,\ge k)$, $\overline{N}(r,a;f,\ge k)$, $\overline{n}(r,a;f,>k)$ etc. likewise and we take $\overline{n}(r,a;f,\ge k) \equiv 0$ if $k=\infty$.

Definition 5. We denote by E the exceptional set that appears in the second fundamental theorem (p.34, [2]) and by $S(r;f_1,f_2,\dots,f_n)$ a function of r such that $S(r;f_1,f_2,\dots,f_n) = o\{\sum_{i=1}^{n} T(r,f_i)\}$ as $r\to\infty$ ($r \notin E$) where f_i's are meromorphic functions defined on C.

Gopalakrishna and Bhoosnurmath [1] proved the following theorem.

Theorem A. *If* (i) *for some* $a \in C\cup\{\infty\}$ $\overline{N}(r,a;f) = S(r;f,g)$, $\overline{N}(r,a;g) = S(r;f,g)$, (ii) *there exist distinct complex numbers* $a_1,a_2,\dots,a_m$ *in* $C\cup\{\infty\}\setminus\{a\}$ *for which* $E(a_i,k_i;f)=E(a_i,k_i;g)$ $(i=1,2,\dots,m)$ *where each* k_i *is a positive integer or infinity with* $k_1\ge k_2\ge\cdots\ge k_m$ *and* (iii) $\sum_{i=2}^{m}\frac{k_i}{1+k_i}-\frac{k_1}{1+k_1}>1$ *then* $f\equiv g$.

Now one may ask is it possible to replace condition (iii) by $\sum_{i=2}^{m}\frac{k_i}{1+k_i}-\frac{k_1}{1+k_1}\le 1$ and if possible under which situation? The purpose of the paper is to answer this question.

2. Lemma and theorem. First we prove a lemma which is necessary for the theorem.

Lemma. *Let k be a nonnegative integer or infinity. Then for* $a\in C\cup\{\infty\}$

$$\overline{N}(r,a;f) \le \frac{k}{1+k}\overline{N}(r,a;f,\le k) + \frac{1}{1+k}N_k(r,a;f).$$

Proof. If $k=\infty$, the lemma is obvious. If $k<\infty$ then

$$\begin{aligned}(1+k)\overline{n}(r,a;f) &= (1+k)\overline{n}(r,a;f,>k) + (1+k)\overline{n}(r,a;f,\le k)\\ &= n_k(r,a;f) - n(r,a;f,\le k) + (1+k)\overline{n}(r,a;f,\le k)\\ &\le n_k(r,a;f) + k\overline{n}(r,a;f,\le k),\end{aligned}$$

from which the lemma follows.

Theorem. *If* (i) $\overline{N}(r,a;f)=S(r;f,g)$, $\overline{N}(r,a;g)=S(r;f,g)$ *for some* $a\in C\cup\{\infty\}$ (ii) *there exist distinct elements* $a_1,a_2,\dots,a_m$ *in* $C\cup\{\infty\}\setminus\{a\}$ *for which* $E(a_i,k_i;f)=E(a_i,k_i;g)$ $(i=1,2,\dots,m)$ *where* k_i *is a positive integer or infinity with* $k_1\geq k_2\geq\cdots\geq k_m$ *and* (iii) $\sum_{i=2}^{m}\frac{k_i}{1+k_i}-\frac{k_1}{1+k_1}\leq 1$, (iv) $\sum_{i=1}^{m}\min\{\delta_{k_i}(a_i;f),\delta_{k_i}(a_i;g)\}>\{1+\frac{k_1}{1+k_1}-\sum_{i=2}^{m}\frac{k_i}{1+k_i}\}(1+k_1)$, *then* $f\equiv g$. *In particular, if the right hand side of* (iv) *is equal to zero, "min" in the condition* (iv) *can be replaced by "max".*

Proof. By the second fundamental theorem we get because $\overline{N}(r,a;f)=S(r;f,g)$.

$$\begin{aligned}(m-1)T(r,f)&\leq \overline{N}(r,a;f)+\sum_{i=1}^{m}\overline{N}(r,a_i;f)+S(r,f)\\&=\sum_{i=1}^{m}\overline{N}(r,a_i;f)+S(r;f,g).\end{aligned}$$

We note that this inequality is true for g also because $\overline{N}(r,a;g)=S(r;f,g)$.

First we suppose that $a=\infty$. Then $a_1,a_2,\dots,a_m$ are all finite and by the Lemma we get from above

$$\begin{aligned}(m-1)T(r,f)\leq&\sum_{i=1}^{m}\frac{k_i}{1+k_i}\overline{N}(r,a_i;f\leq k_i)\\&+\sum_{i=1}^{m}\frac{1}{1+k_i}N_{k_i}(r,a_i;f)+S(r;f,g).\end{aligned}\tag{1}$$

Applying (1) to g and adding to (1) we get

$$\begin{aligned}&(m-1)\{T(r,f)+T(r,g)\}\\&\leq\sum_{i=1}^{m}\frac{k_i}{1+k_i}\{\overline{N}(r,a_i;f,\leq k_i)+\overline{N}(r,a_i;g,\leq k_i)\}\\&\quad+\sum_{i=1}^{m}\frac{1}{1+k_i}\{N_{k_i}(r,a_i;f)+N_{k_i}(r,a_i;g)\}+S(r;f,g)\\&\leq\frac{2k_1}{1+k_1}\sum_{i=1}^{m}\overline{N}(r,a_i;f,g,\leq k_i)\\&\quad+\sum_{i=1}^{m}\frac{1}{1+k_i}\{N_{k_i}(r,a_i;f)+N_{k_i}(r,a_i;g)\}+S(r;f,g),\end{aligned}\tag{2}$$

because $\frac{k_1}{1+k_1} \geq \frac{k_2}{1+k_2} \geq \cdots \geq \frac{k_m}{1+k_m}$ where $\overline{N}(r, a_i; f, g, \leq k_i)$ is the counting function for common distinct zeros of $f - a_i$ and $g - a_i$ of multiplicities not greater than k_i.

If $f \not\equiv g$, each common zero of $f - a_i$ and $g - a_i$ is a zero of $f - g$. Since $a_1, a_2, \ldots, a_m$ are all distinct, we have

$$\sum_{i=1}^{m} \overline{N}(r, a_i; f, g, \leq k_i) \leq N(r, 0; f - g) \leq T(r, f) + T(r, g) + O(1).$$

So from (2) we get

$$(3) \qquad \Big(m - 1 - \frac{2k_1}{1+k_1}\Big)\{T(r, f) + T(r, g)\} \leq \sum_{i=1}^{m} \frac{1}{1+k_i}\{N_{k_i}(r, a_i; f) + N_{k_i}(r, a_i; g)\} + S(r; f, g).$$

Now for given ϵ (>) there exists r_0 (> 0) such that for $r \geq r_0$

$$N_{k_i}(r, a_i; f) < \{1 - \delta_{k_i}(a_i; f) + \epsilon\} \cdot T(r, f) \quad \text{and}$$
$$N_{k_i}(r, a_i; g) < \{1 - \delta_{k_i}(a_i; g) + \epsilon\} \cdot T(r, g).$$

Hence from (3) we get

$$(4) \qquad \Big(\sum_{i=2}^{m} \frac{k_i}{1+k_i} - 1 - \frac{k_1}{1+k_1}\Big)\{T(r, f) + T(r, g)\} + \frac{1}{1+k_1}\sum_{i=1}^{m}\{(\delta_{k_i}(a_i, f) - \epsilon)T(r, f) + (\delta_{k_i}(a_i, g) - \epsilon)T(r, g)\} \leq S(r; f, g).$$

Since the second term of the left hand is not less than

$$\frac{1}{1+k_1}\Big[\sum_{i=1}^{m} \min\{\delta_{k_i}(a_i; f), \delta_{k_i}(a; g)\} - m\epsilon\Big]\{T(r, f) + T(r, g)\}$$

and $\sum_{i=2}^{m} \frac{k_i}{1+k_i} - 1 - \frac{k_1}{1+k_1} \leq 0$, it follows from (4) that

$$\sum_{i=2}^{m} \frac{k_i}{1+k_i} - 1 - \frac{k_1}{1+k_1} + \frac{1}{1+k_1}\Big[\sum_{i=1}^{m} \min\{\delta_{k_i}(a_i; f), \delta_{k_i}(a_i; g)\} - m\epsilon\Big] \leq 0$$

and this implies a contradiction to the condition (iv). Hence $f \equiv g$.

Now we suppose that $1+\frac{k_1}{1+k_1}-\sum_{i=2}^{m}\frac{k_i}{1+k_i}=0$. Then from (4) we get

$$\sum_{i=1}^{m}[\{\delta_{k_i}(a_i,f)-\epsilon\}\cdot T(r,f)+\{\delta_{k_i}(a_i,g)-\epsilon\}\cdot T(r,g)]\le S(r;f,g). \tag{5}$$

Further we suppose that $\sum_{i=1}^{m}\max\{\delta_{k_i}(a_i;f),\delta_{k_i}(a;g)\}>0$. Then there exists a positive integer p, $1\le p\le m$, such that at least one of $\delta_{k_p}(a_p;f)$, $\delta_{k_p}(a_p;g)$ is positive. We consider only the case $\delta_{k_p}(a_p;f)>0$ because the other case is similar. If possible, let $f\not\equiv g$. Then from (5) we get

$$\delta_{k_p}(a_p;f)T(r;f)\le m\epsilon\{T(r,f)+T(r,g)\}+S(r;f,g). \tag{6}$$

Now we show that

$$\sum_{i=1}^{m}\overline{N}(r,a_i;f,g,\le k_i)>\frac{1}{2}\{T(r,f)+T(r,g)\} \tag{7}$$

for all sufficiently large values of $r(r\notin E)$. If possible, let

$$\sum_{i=1}^{m}\overline{N}(r,a_i;f,g,\le k_i)\le\frac{1}{2}\{T(r,f)+T(r,g)\}$$

for a sequence of values of $r(r\notin E)$ tending to infinity. Then from (2) we get for a sequence of values of r tending to infinity $(r\notin E)$

$$\begin{aligned}(m-1)\{T(r,f)+T(r,g)\}&\le\frac{k_1}{1+k_1}\{T(r,f)+T(r,g)\}\\&\quad+\sum_{i=1}^{m}\frac{1}{1+k_i}\{N_{k_i}(r,a_i;f)+N_{k_i}(r,a_i;g)\}+S(r;f,g)\\&\le\left\{\frac{k_1}{1+k_1}+\sum_{i=1}^{m}\frac{1}{1+k_i}\right\}\{T(r,f)+T(r,g)\}+S(r;f,g)\end{aligned}$$

i.e. $$\left\{m-1-\frac{k_1}{1+k_1}-\sum_{i=1}^{m}\frac{1}{1+k_i}\right\}\cdot\{T(r,f)+T(r,g)\}\le S(r;f,g)$$

i.e. $$\left\{\sum_{i=1}^{m}\frac{k_i}{1+k_i}-1-\frac{k_1}{1+k_1}\right\}\cdot\{T(r,f)+T(r,g)\}\le S(r;f,g)$$

i.e. $$\left\{\sum_{i=2}^{m}\frac{k_i}{1+k_i}-1\right\}\cdot\{T(r,f)+T(r,g)\}\le S(r;f,g)$$

i.e. $$\frac{k_1}{1+k_1}\cdot\{T(r,f)+T(r,g)\}\le S(r;f,g),$$

which is a contradiction. So (7) is true.

Since $mT(r,f)\ge\sum_{i=1}^{m}\overline{N}(r,a_i;f,g,\le k_i)$, it follows from (6) and (7) that

$$\left\{\frac{\delta_{k_p}(a_p;f)}{2m}-m\epsilon\right\}\cdot\{T(r,f)+T(r,g)\}\le S(r;f,g)$$

which is again a contradiction for sufficiently small ϵ (>0). Hence $f\equiv g$.

Next we suppose that $a\ne\infty$. Then $\frac{1}{a_i-a}$ $(i=1,2,\ldots,m)$ are distinct elements of C. Let $F=(f-a)^{-1}$ and $G=(g-a)^{-1}$. Then $\overline{N}(r,\infty;F)=\overline{N}(r,a;f)=S(r;F,G)$ and $\overline{N}(r,\infty;G)=\overline{N}(r,a;g)=S(r;F,G)$. Also $E((a_i-a)^{-1},k_i;F)=E((a_i-a)^{-1},k_i;G)$ for $i=1,2,\ldots,m$. Finally $\delta_{k_i}((a_i-a)^{-1};F)=\delta_{k_i}(a_i;f)$ and $\delta_{k_i}((a_i-a)^{-1};G)=\delta_{k_i}(a_i;g)$ for $i=1,2,\ldots,m$. Now by applying what we have already proved to the functions F and G with $(a_1-a)^{-1},(a_2-a)^{-1},\ldots,(a_m-a)^{-1}$ we see that $F\equiv G$ and so $f\equiv g$. This proves the theorem.

Remark 1. Consider $f=\exp(z)$, $g=\exp(-z)$ we see that $\overline{N}(r,\infty;f)=\overline{N}(r,\infty;g)=S(r;f,g)$, $a_1=0$, $a_2=1$, $a_3=-1$, $k_1=k_2=k_3=1$ and $\sum_{i=1}^{3}\min\{\delta_1(a_i;f),\delta_1(a_i,g)\}=1$. So the condition (iv) of the theorem is necessary.

Corollary 1. *If* (i) *there exists* $a\in C\cup\{\infty\}$ *such that* $\overline{N}(r,a;f)=S(r;f,g)$, $\overline{N}(r,a;g)=S(r;f,g)$, (ii) $E(a_i,1;f)=E(a_i,1;g)$ *for* $a_i\in C\cup\{\infty\}\backslash\{a\}$ $(i=1,2,3)$ *and* (iii) $\sum_{i=1}^{3}\min\{\delta_1(a_i;f),\delta_1(a_i;g)\}>1$ *then* $f\equiv g$.

Corollary 2. *If* (i) *there exists* $a\in C\cup\{\infty\}$ *such that* $\overline{N}(r,a;f)=S(r;f,g)$, $\overline{N}(r,a;g)=S(r;f,g)$, (ii) $E(a_i,1;f)=E(a_i,1;g)$ *for* $a_i\in C\cup\{\infty\}\setminus\{a\}$ $(i=1,2,3,4)$ *and* (iii) $\sum_{i=1}^{4}\max\{\delta_1(a_i;f),\delta_1(a_i;g)\}>0$ *then* $f\equiv g$.

References

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