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# AN ANALOG OF THE GRAM-SCHMIDT ALGORITHM FOR COMPLEX BILINEAR FORMS

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**Abstract.** By analogy with the usual notions of orthogonal and orthonormal sets, we say that a set of vectors $\{x_i\} \in \mathbb{C}^n$ is rectangular if $x_i^T x_j = 0$ whenever $i \neq j$; it is rectanormal if it is rectangular and $x_i^T x_i = 1$ for all $i$. We show that every independent set is equivalent to a rectangular (but not necessarily rectanormal) set, and give necessary and sufficient conditions for the equivalence to be achieved by a triangular transformation, as with the classical Gram-Schmidt process. We show that any rectanormal set $\{x_i\} \in \mathbb{C}^n$ can be extended to a rectanormal basis, and that any rectangular set can be extended to a basis with a canonical pattern to the bilinear products $x_i^T x_j$. We also give necessary and sufficient conditions for one given set of vectors to be an orthogonal transform of another given set of vectors.

**1. Introduction.** We denote by $M_{m,n}$ the set of m-by-n complex matrices and set $M_n \equiv M_{n,n}$. We shall use $Q$ to denote a complex orthogonal matrix ($Q \in M_n, Q^T = Q^{-1}$). A complex matrix $A \in M_n$ is symmetric if $A = [a_{ij}] = A^T$, where $A^T = [a_{ji}]$ is the ordinary transpose of $A$.

If $b(\bullet, \bullet) : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}^n$ is a symmetric bilinear form [$b(x,y) = b(y,x)$ and $b(\alpha x + \beta y, z) = \alpha b(x,z) + \beta(y,z)$ for all $x, y \in \mathbb{C}^n$ and all $\alpha, \beta \in \mathbb{C}^n$], there is a unique complex symmetric matrix $S = [s_{ij}] \in M_n$ such that $b(x,y) = x^T S y$ for all $x, y \in \mathbb{C}^n$; the entries of $S$ are $s_{ij} = b(e_i, e_j)$, where $\{e_1, e_2, \ldots, e_n\}$ is the standard orthonormal basis of $\mathbb{C}^n$. Since one may

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factorize any symmetric matrix as $S = A^TA$ for some $A \in M_n$ with the same rank as $S$ [7, corollary (4.4.6)], any symmetric bilinear form on $\mathbb{C}^n \times \mathbb{C}^n$ can be written as $b(x,y) = (Ax)^T(Ay)$ for some $A \in M_n$. The matrix $S$, and hence the matrix $A$, will be nonsingular if $y = 0$ is the only $y \in \mathbb{C}^n$ for which $b(x,y) \equiv 0$ for all $x \in \mathbb{C}^n$. Thus, the study of such bilinear forms and the geometry induced by them may, by a nonsingular change of variables, be reduced to the study of the basic bilinear form $b(x,y) = x^Ty$.

Just as the Hermitian form $h(x,y) = x^*y$ is invariant under unitary transformations, the bilinear form $x^Ty$ is invariant under complex orthogonal transformations: $(Qx)^T(Qy) = x^T(Q^TQ)y = x^Ty$ if $Q^TQ = I$. We shall be interested in geometric and algebraic results involving complex orthogonal matrices that are analogues of familiar results about Euclidean geometry and unitary matrices.

A fundamental difference between the bilinear form $b(x,y) = x^Ty$ on $\mathbb{C}^n \times \mathbb{C}^n$ and the Hermition form $h(x,y) = x^*y$ is that, if $n \geq 2$, there are always nonzero vectors $x$ such that $x^Tx = 0$; as an example consider $x = [1, i, 0, \ldots, 0]^T$.

**Definition 1.1:** A vector $x \in \mathbb{C}^n$ is said to be isotropic if $x^Tx = 0$ and nonisotropic if $x^Tx \neq 0$.

**Definition 1.2:** A set of vectors $\{x_i : i = 1, 2, \ldots, k\} \subset \mathbb{C}^n$ is said to be rectangular if $x_i^Tx_j = 0$ whenever $i \neq j$; it is said to be rectanormal if it is rectangular and $x_i^Tx_i = 1$ for all $i = 1, 2, \ldots, k$.

**Observation 1.3:** Let $X \in M_{n,k}$ with $k \leq n$. Then $X^TX = I \in M_k$ if and only if the columns of $X$ form a rectanormal set.

**Lemma 1.4:** *A rectanormal set is linearly independent.*

*Proof.* The assertion follows from observation (1.3) since $X^TX = I \in M_k$ implies that rank $X \geq k$. One may also give a proof that parallels the classical argument in the orthogonal case: Let $\{x_1, x_2, \ldots, x_k\} \subset \mathbb{C}^n$ be a rectanormal set. If $\alpha_1x_1 + \alpha_2x_2 + \ldots + \alpha_kx_k = 0$ for some choice of scalars

$\alpha_i \in \mathbb{C}$, then $0 = 0(x_i) = (\alpha_1 x_1 + \alpha_2 x_2 + \ldots + \alpha_k x_k)^T x_i = \alpha_i x_i^T x_i = \alpha_i = 0$ for $i = 1, 2, \ldots, k$.

**2. Analogues of the Gram-Schmidt algorithm and some of their consequences.** The Gram-Schmidt algorithm shows that any independent set of vectors is triangularly equivalent to an orthonormal set in the following sense, and this fact has many important consequences for Euclidean geometry and the linear algebra of unitary transformations.

**Definition 2.1:** Two set of vectors $\{x_1, x_2, \ldots, x_k\} \subset \mathbb{C}^n$ and $\{y_1, y_2, \ldots, y_k\} \subset \mathbb{C}^n$ are said to be equivalent if $Span\{x_1, x_2, \ldots, x_k\} = Span \{y_1, y_2, \ldots, y_k\}$. They are said to be triangularly equivalent if $Span\{x_1, x_2, \ldots, x_i\} = Span \{y_1, y_2, \ldots, y_i\}$ for $i = 1, 2, \ldots, k$.

If $X \equiv [x_1 x_2 \ldots x_k] \in M_{n,k}$ and $Y \equiv [y_1 y_2 \ldots y_k] \in M_{n,k}$, then the sets of columns of these two matrices are equivalent if and only if $X = YA$ for some (not necessarily unique) nonsingular $A \in M_k$. They are triangularly equivalent if and only if $X = YB$ for some nonsingular upper triangular $B \in M_k$. The classical Gram-Schmidt algorithm shows that any independent set is triangularly equivalent to an orthonormal set. It is not true that every independent set is triangularly equivalent to a rectanormal set, as the example $\{x\} = \{[1\ i]^T\}$ shows, but any set of vectors is equivalent to a rectangular set. The key insight into proving this is the following theorem of Takagi [12], for which independent proofs were given later by Jacobson [9], Siegel [11], Hua [8], and Schur [10] (for a discussion and yet another proof see [7, section (4.4)]).

**Theorem 2.2.** (*Takagi*) *Let $A \in M_n$ be symmetric. There exists a unitary $U \in M_n$ and a unique nonnegative diagonal matrix $\Sigma \equiv \mathrm{diag}(\sigma_1, \sigma_2, \ldots, \sigma_n) \in M_n$ with $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_n \geq 0$ such that $A = U\Sigma U^T$. The entries $\sigma_i$ are the singular values of $A$ and hence the number of nonzero entries $\sigma_i$ is equal to the rank of $A$.*

*It follows immediately from Takagi's theorem that any complex symmetric $A \in M_n$ can be written as $A = B^T B$ for some $B \in M_n$ with the same* *rank as A; if* $A = U\Sigma U^T$, *take* $B \in (U\Sigma^{1/2})^T$.

**Theorem 2.3.** *Let* $\{x_1, x_2, \dots, x_k\} \subset \mathbb{C}^n$ *be given with* $1 \le k \le n$ *and let* $X \equiv [x_1 x_2 \dots x_k] \in M_{n,k}$. *There exists a rectangular set* $\{y_1, y_2, \dots y_k\} \subset \mathbb{C}^n$ *that is equivalent to* $\{x_1, x_2, \dots, x_k\}$ *and is such that:*
a)

$$Y^T Y = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}.$$

*where* $Y \equiv [y_1 y_2 \cdots y_k] \in M_{n,k}, r = \text{rank} X^T X$, *and* $I_r \in M_r$ *is an identity matrix. The product* $X^T X$ *is nonsingular if and only if there is an equivalent set* $\{y_1, y_2, \dots, y_k\}$ *that is rectanormal.*
b) *The matrices* $X$ *and* $Y$ *are related by* $X = YB$, *where* $B \in M_k$ *is nonsingular and depends on* $X$ *only via the product* $X^T X$, *i.e., if* $Z \in M_{n,k}$ *is given and* $Z^T Z = X^T X$, *then also* $Z = \tilde{Y} B$ *for some* $\tilde{Y} \in M_{n,k}$ *such that*

$$\tilde{Y}^T \tilde{Y} = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} = Y^T Y.$$

*Proof:* a) Since $X^T X$ is symmetric and rank $X^T X = r \le k$, by Takagi's theorem (2.2) there exists a unitary matrix $U \in M_k$ and a diagonal matrix $\Sigma = \begin{bmatrix} \Lambda & 0 \\ 0 & 0 \end{bmatrix} \in M_k$ (where $\Lambda \in M_r$ is a nonsingular positive diagonal matrix) such that

$$\begin{aligned} X^T X = U \Sigma U^T &= U \begin{bmatrix} \Lambda^{1/2} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \Lambda^{1/2} & 0 \\ 0 & 0 \end{bmatrix} U^T \\ &= U \begin{bmatrix} \Lambda^{1/2} & 0 \\ 0 & I_{k-r} \end{bmatrix} \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \Lambda^{1/2} & 0 \\ 0 & I_{k-r} \end{bmatrix} U^T, \end{aligned} \tag{2.2a}$$

where $I_r \in M_r$ and $I_{k-r} \in M_{k-r}$ are identity matrices. The diagonal entries of $\Sigma$ are the singular values of $X^T X$. It follows from (2.2*a*) that

$$W X^T X W^T = (X W^T)(X W^T) = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} \tag{2.2b}$$

where

$$W \equiv \begin{bmatrix} \Lambda^{1/2} & 0 \\ 0 & I_{k-r} \end{bmatrix}^{-1} U^* \in M_k$$

is nonsingular and depends on $X$ only via the product $X^TX$. If we set $Y = XW^T$, we have

$$Y^TY = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix},$$

and hence the columns of $Y$ comprise a rectangular set that is equivalent to the set of columns of $X$. The last assertion follows from the observation that $X^TX$ is nonsingular if and only if $r = k$ and $Y^TY = I_k$, i.e., $\{y_1, y_2, \ldots, y_k\}$ is a rectanormal set.

b) Let $B \equiv (W^T)^{-1}$, so that $X = YB$ and $B$ depends on $X$ only via the product $X^TX$. Thus if $Z \in M_{n,k}$ is such that $Z^TZ = X^TX = U\Sigma U^T$, then the same $W$ as in a) produces a matrix $\tilde{Y} = ZW^T$ such that

$$\tilde{Y}^T\tilde{Y} = WZ^TZW^T = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$$

Hence $Z = \tilde{Y}(W^T)^{-1} = \tilde{Y}B$, where $\tilde{Y}^T\tilde{Y}$ has the same form as in (2.2b).

The crucial parameter in the preceding lemma is $r = X^TX$, where $X = [x_1x_2 \ldots x_k] \in M_{n,k}$. Although $r > \text{rank}X$ is impossible, it can happen that $r < \text{rank}X$ even if $X$ has full rank. This is in sharp contrast to the classical Euclidean situation in which $X^*X = \text{rank}X$ always.

Geometrically, theorem (2.3) says that and $k$-dimensional subspace $V$ of $\mathbb{C}^n$ has a rectangular basis in which some basis vectors may be isotropic and some are not. If $\{x_1, x_2, \ldots, x_k\}$ is a basis of $V$ and we set $X = \{x_1x_2 \ldots x_k\} \in M_{n,k}$, then any basis of $V$ form the columns of a matrix $Y = XC$ for some nonsigular $C \in M_k$. Since rank $Y^TY = \text{rank}(XC)^T(XC) = \text{rank}\, C^T(X^TX)C = \text{rank}\, X^TX$, we see that every rectangular basis of $V$ has the same number of nonisotropic vectors in it.

**Corollary 2.4.** *Let $1 \le k \le n$ and let $V$ be a $k$-dimensional subspace of $\mathbb{C}^n$. Then $V$ can be written as a rectangular direct sum $V = V_1 \oplus V_2$ of two subspaces $V_1$ and $V_2$, in which*

(a) $x^T y = 0$ *for all* $x \in V_1$, $y \in V_2$.

(b) $r = \dim\ V_1 = \text{rank}\ X^T X$ *for any* $X = [x_1 x_2 \ldots, x_k] \in M_{n,k}$ *whose columns form a basis for* $V$, *and* $0 \leq r \leq k$.

(c) *If* $r \geq 1$, $V_1$ *has a rectanormal basis.*

(d) *If* $r < k$, $V_2$ *has a rectangular basis of isotropic vectors, and every vector in* $V_2$ *is isotropic.*

(e) $r = k$ *if and only if* $V$ *has a rectanormal basis, and* $r = 0$ *if and only if* $V$ *has a rectangular basis of isotropic vectors.*

If $Y = \{y_1, y_2, \ldots, y^p\}$ is rectangular set of isotropic vectors and $y = \alpha_1 y_1 + \alpha_2 y_2, \ldots, \alpha_p y^p \in Span\ Y$,then

$$y^T y = \sum_{ij=1}^{n} \alpha_i \alpha_j y_i^T y_j = 0$$

because the terms in this sum with $i = j$ vanish by isotropy and the terms with $i \neq j$ vanish by rectangularity. The assertion (c) does not imply that there are no isotropic vectors in $V_1$ when $r \geq 2$, for if $\{x, y\}$ is a rectanormal set in $V_1$, then $\{x + iy, x - iy\}$ is an independent (but not rectangular) set of isotropic vectors in $V_1$. Thus, $V_1$ can even have a basis that is isotropic, but such a basis could not be rectangular as well because every rectangular basis of a subspace contains the same number of nonisotropic vectors.

**Definition 2.5.** *A subspace* $V \subset \mathbb{C}^n$ *is said to be*

(a) *Isotropic if* $V$ *has a rectangular basis of isotropic vectors, and hence every vector in* $V$ *is isotropic.*

(b) *Singular if* $V$ *has a rectangular basis containing at least one isotropic vector.*

(c) *Nonsingular if* $V$ *has a rectanormal basis.*

If $\{x_1, x_2, \ldots, x_k\}$ is a basis for a subspace $V \subset \mathbb{C}^n$, if $X = [x_1 x_2 \ldots x_k] \in M_{n,k}$, and if $r = \text{rank}\ X^T X$, we have seen that $V$ is isotropic if $r = 0$, singular if $r < k$, and nonsingular if $r = k$. A subspace $V \subset \mathbb{C}^n$ is singular if and only if there is a nonzero vector $z \in V$ such that $z^T y = 0$

for all $y \in V$, for if $\{v_1, v_2, \ldots, v_k\}$ is a rectanormal basis of $V$ and we write $z = z_1\nu_1 + z_2\nu_2 + \ldots + z_k\nu_k$ and $y = y_1v_1 + y_2v_2 + \ldots + y_kv_k$, then $z^Ty = z_1v_1 + z_2v_2 + \ldots + z_kv_k = 0$ for all $y \in V$ (in particular, for $y = v_i$) if and only if all $z_i = 0$.

**Lemma 2.6.** *Let $\{x_1, x_2, \ldots, x_k\} \subset \mathbb{C}^n$ be a given set of linearly independent vectors and let $X_i \equiv [x_1x_2\ldots x_i] \in M_{n,i}$ for $i = 1, 2, \ldots, k$. There exists a rectanormal set $\{y_1, y_2, \ldots, y_k\} \subset \mathbb{C}^n$ that is triangularly equivalent to $\{x_1, x_2, \ldots, x_k\}$ if and only if each $X_i^TX_i$ has full* rank *$i$, i.e.,* $\det[X_i^TX_i] \neq 0$ *for $i = 1, 2, \ldots, k$.*

*Proof.* Define $Y_i \equiv [y_1y_2\ldots y_i\}$ for $i = 1, 2, \ldots, k$. If $\{x_i\}$ and $\{y_i\}$ are triangularly equivalent and $\{y_i\}$ is a rectanormal set, then $X_i = Y_iB_i$ for some nonsingular upper triangular matrix $B_i \in M_i$ for $i = 1, 2, \ldots, k$. Then $\det[X_i^TX_i] = \det[B_i^TY_i^TY_iB_i] = \det[B_i^TB_i] = [\det B_i]^2 \neq 0$.

Conversely, suppose $\det[X_i^TX_i] \neq 0$ for $i = 1, 2, \ldots, k$. If $k = 1$, pick $y_1 = x_1/(x_1^Tx_1)^{1/2}$ and we are done since $x_1^Tx_1 \neq 0$. If $k > 1$, make the induction hypothesis that the lemma has been proved for $k = 1, 2, \ldots, p-1$. Consider the linearly independent vectors $\{x_1, x_2, \ldots, x_p\}$. By the induction hypothesis, $Y_{p-1} = X_{p-1}B_{p-1}$ for some nonsingular upper triangular matrix $B_{p-1} \in M_{p-1}$, where $Y_{p-1}^TY_{p-1} = I \in M_{p-1}$. Define

$$z_p = x_p \sum_{i=1}^{p-1}(x_p^Ty_i)y_i.$$

Then

$$z_p^TY_i - x_p^Ty_i = 0 \quad \text{for} \quad i = 1, 2, \ldots, p-1$$

and

$$z_p^Tz_p = x_p^Tx_p - \sum_{i=1}^{p-1}(x_p^Ty_i)^2 \neq 0.$$

Thus, $z_p$ is nonisotropic and we may set

$$y_p = z_p/(z_p^T z_p)^{1/2} \equiv b_p z_p = b_p x_p - \sum_{i=1}^{p-1} b_p (x_p^T y_i) Y_i.$$

The matrix

$$B_p = \begin{bmatrix} B_{p-1} & * \\ 0 & 0 \end{bmatrix}$$

is a nonsingular upper triangular matrix and $Y_p = X_p B_p$, as desired.

Every orthonormal set can be extended to an orthonormal basis of $\mathbb{C}^n$; we now prove that every rectanormal set can be extended to a rectanormal basis of $\mathbb{C}^n$.

**Theorem 2.7.** *Let $k$ be a given integer, $1 \le k < n$, and let $\{x_1, x_2, \ldots, x_k\} \subset \mathbb{C}^n$ be a given rectanormal set. There exists a vector $x_{k+1} \in \mathbb{C}^n$ such that $\{x_1, x_2, \ldots, x_{k+1}\} \subset \mathbb{C}^n$ is a rectanormal set.*

*Proof. Let $y = [y_i] \in \mathbb{C}^n$ be a vector whose coordinates are to be determined. If we set*

$$\nu = y - \sum_{i=1}^{k} (y^T x_i) x_i$$

*then*

$$\nu^T x_j = y^T x_j - \sum_{i=1}^{k} (y^T x_i) x_i^T x_j = y^T x_j - y^T x_j = 0 \text{ for } j = 1, 2, \ldots, k.$$

*Thus $\{x_1, x_2, \ldots, x_k, v\}$ is automatically a rectangular set for any $y \in \mathbb{C}^n$. If we can choose $y$ so that $v^T v \neq 0$, we can set $x_{k+1} = v/(v^T v)^{1/2}$ and $\{x_1, x_2, \ldots, x_k, x_{k+1}\}$ will be a rectanormal set.*

*Write $x_i^T = [x_{i1} x_{i2} \ldots x_{in}]^T$ for $i = 1, 2, \ldots, k$ and compute*

$$\nu^T \nu = y^T y - \sum_{i=1}^{k} (y^T x_i)^2$$

$$= \sum_{i=1}^{n} (1 - x_{1i}^2 - x_{2i}^2 - \ldots - x_{ki}^2) y_i^2 - 2 \sum_{m=1}^{k} \sum_{ij=1, i<j}^{n} x_{mi} x_{mj} y_i y_j$$

*If all the coefficients of the $y_i^2$ terms were zero, then*

$$0 = \sum_{i=1}^{n}(1 - x_{1i}^2 - x_{2i}^2 - \ldots - x_{ki}^2) = n - \sum_{m=1}^{k}\sum_{i=1}^{n} x_{mi}^2 = n - \sum_{m=1}^{k} x_m^T x_m = n - k$$

*which is a contradiction, since $k < n$ by assumption. Let $i^\circ$ be the least value of the index $i = 1, 2, \ldots, n$ for which $1 - x_{1i}^2 - x_{2i}^2 - \ldots - x_{ki}^2 \neq 0$. If we set*

$$y_i^\circ \equiv 1, y_i \equiv 0 \text{ if } i \neq i^\circ, \text{ then}$$

$$\nu^T \nu = (1 - x_{1i^\circ}^2 - x_{2i^\circ}^3 - \ldots - x_{ki^\circ}^2) \neq 0.$$

*as desired.*

**Corollary 2.8.** *Any rectanormal set in $\mathbb{C}^n$ can be extended to a rectanormal basis of $\mathbb{C}^n$.*

It is not possible to extend a given rectangular set $\{x_1, x_2, \ldots, x_k\} \subset \mathbb{C}^n$ to a rectangular basis $B = \{x_1, x_2, \ldots, x_n\}$ of $\mathbb{C}^n$ if the given set contains any isotropic vectors, for then the necessarily nonsingular matrix $X_n^T X_n$ would have a zero row, where $X_n \equiv [x_1 x_2 \ldots x_n] \in M_n$. However, we show next that it is always possible to extend a given independent rectangular set to a larger rectangular set in which the bilinear products $x_i^T x_j$ have a canonical pattern. The given set spans a singular subspace, while the extended set spans a larger nonsingular subspace, which therefore has a rectanormal basis.

**Theorem 2.9.** *Let $W = [E\ F] \in M_{n,r+p}$ have full* rank $r + p < n$ *and suppose*

$$W^T W = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$$

*where $E \in M_{n,r}$, $F \in M_{n,p}$, $p \geq 1$, $r \geq 0$ and $I_r \in M_r$ is an identity matrix. Then*

1) $n \geq r + 2p$ *and*

2) *there exists* $G \in M_{n,p}$ *such that the matrix* $H = [E\ F\ G] \in M_{n,r+2p}$ *has full* rank rank $r + 2p \leq n$ *and satisfies*

$$(2.10) \qquad H^T H = \begin{bmatrix} I_r & 0 & 0 \\ 0 & 0 & I_p \\ 0 & I_p & 0 \end{bmatrix} \in M_{r+2p}$$

*where* $I_r \in M_r, I_p \in M_p$ *are identity matrices. In particular,* $H^T H$ *is nonsingular, and hence the column space of* $H$ *is a nonsingular subspace of* $\mathbb{C}^n$.

*Proof.* 1) Let $V$ denote the span of the columns of $W$. Then $\bar{V} = \dim V = r+p$, where $\bar{V}$ denotes the set of complex conjugates of the vectors in $V$. By hypothesis, the $p$ independent columns of $F$ are orthogonal to $\bar{V}$. Therefore, *dim* $\bar{V}^{\perp} \geq p$, $r+p = \dim V =$ *dim* $\bar{V} \leq n - p$, and hence $n \geq r + 2p$.

2) Let $V'$ denote the orthogonal complement of the span of the columns of $\bar{E}$. The $p$ independent column of $F$ are in $V'$, and they may be augmented by $n-r-p$ additional independent vectors $x_1, x_2, \ldots, x_{n-r-p}$ in $V'$ to form a basis for $V'$. Let $F \equiv [f_1 f_2 \ldots f_p] \in M_{n,p}$ and $X \equiv [x_1 x_2 \ldots x_{n-r-p}] \in M_{n,n-r-p}$. Then $X^T E = 0$ by construction and the matrix $L \equiv [E\ F\ X] \in M_n$ is nonsingular. Since $L$ is nonsingular, the product

$$L^T L = \begin{bmatrix} E^T E & E^T F & E^T X \\ F^T E & F^T F & F^T X \\ X^T E & X^T F & X^T X \end{bmatrix} = \begin{bmatrix} I_r & 0 & 0 \\ 0 & 0 & F^T X \\ 0 & F^T X & X^T X \end{bmatrix}$$

is nonsingular. Consider the first row of $F^T X$, which has entries $f_1^T x_i$ for $i = 1, 2, \ldots, n - r - p$. Not all of these entries are zero since $L^T L$ is nonsingular. Suppose $\hat{i}$ is the least index for which $f_1^T x_{\hat{i}} \neq 0$. Let $\tilde{x}_{\hat{i}}, = c x_{\hat{i}}$, where $c \in \mathbb{C}^n$ is such that $f_1^T \tilde{x}_{\hat{i}} = 1$. Define $\tilde{x}_j = x_j + \alpha_j \tilde{x}_{\hat{i}}$, where the sclar $\alpha_j$ is such that $f_1^T \tilde{x}_j = 0$ for $j = 1, 2, \ldots, n - r - p, j \neq \hat{i}$. Notice taht $\tilde{x}_j^T E = 0$ for $j = 1, 2, \ldots, n - r - p$. Replace each column $x_i$ by $\tilde{x}_i$ without disturbing the rank and permute the columns so that $\tilde{x}_{\hat{i}}$ becomes that first column of $X$. After these manipulations, $F^T X$ has the form

$$F^T X = \begin{bmatrix} 1 & 0 \\ * & * \end{bmatrix}.$$

Consider the second row of $F^T X$, which has $f_2^T x_i, i = 1, 2, \ldots, n-r-p$. This whole row cannot be zero, and it cannot be that its only nonzero entry is $f_2^T x_i$ because then the first and second rows of $F^T X$ would be proportional; both possibilities are excluded because $L^T L$ is nonsingular. Suppose $i$' is the least index greater than 1 for which $f_2^T x_{i'} \neq 0$. Scale $x_{i'}$ as before to obtain $\hat{x}_{i'}$ with $f_2^T \hat{x}_{i'} = 1$, and use $\hat{x}_{i'}$ as before to zero out all of the other elements in the second row, even the first. This manipulation will not disturb the entries of the first row since $f_1^T \hat{x}_{i'} = 0$.

One can proceed in this fashion down the first $p$ rows of $F^T X$ to obtain a final $X$ such that

$$\begin{bmatrix} F^T \\ X^T \end{bmatrix} [F \ X] = \begin{bmatrix} 0 & I_p & 0 \\ I_p & X_1^T X_1 & X_1^T X_2 \\ 0 & X_2^T X_1 & X_2^T X_2 \end{bmatrix}$$

where $X = [X_1 \ X_2]$ and $X_1 \in M_{n,p}$. Notice that $X_1^T E = 0$ and $F^T X_1 = I_p$. Let

$$G = X_1 - \frac{1}{2} F(X_1^T X_1).$$

Then

$$G^T E = X_1^T E - \frac{1}{2}(X_1^T X_1) F^T E = 0$$

$$F^T G = F^T X_1 - \frac{1}{2} F^T F (X_1^T X_1) = I_p$$

$$\begin{aligned} G^T G =& X_1^T X_1 - \frac{1}{2} X_1^T F (X_1^T X_1) - \frac{1}{2}(X_1^T X_1) F^T X_1 \\ &+ \frac{1}{4}(X_1^T X_1)(F^T F)(X_1^T X_1) \\ =& X_1^T X_1 (I_p - \frac{1}{2} I_p - \frac{1}{2} I_p + 0) = 0. \end{aligned}$$

Let $H = [E \ F \ G]$. Then

$$H^TH = \begin{bmatrix} E^TE & E^TF & E^TG \\ F^TE & F^TF & F^TG \\ G^TE & G^TF & G^TG \end{bmatrix} = \begin{bmatrix} I_r & 0 & 0 \\ 0 & 0 & I_p \\ 0 & I_p & 0 \end{bmatrix}$$

has the desired form and $H^TH$ is nonsingular.

Theorem (2.9) implies a generalization of corollary (2.8) to the problem of extending a given independent rectangular (but not necessarily rectanormal) set to a basis $B = \{z_i\}$ such that the set of products $z_i^Tz_j$ has a canonical structure. If a given independent rectangular set of vectors $\{e_1, e_2, \ldots, e_r, f_1, f_2, \ldots, f_p\} \subset \mathbb{C}^n$ is such that each $f_i$ is isotropic and each $e_i$ is normalized so that $e_i^Te_i = 1$, then the theorem (2.9) says that there are $p$ additional independent isotropic vectors $g_1, g_2, \ldots, g_p$ such that the matrix $H = [E\ F\ G] \in M_{n,r+2p}$ satisfies the identity (2.10), where $E \equiv [e_1e_2 \ldots e_r] \in M_{n,r}$, $F \equiv [f_1f_2 \ldots f_p] \in M_{n,p}$ and $G \equiv [g_1g_2 \ldots g_p] \in M_{n,p}$. Since $H^TH$ is nonsingular, theorem (2.3) ensures that the column space of $H$ has a rectanormal basis, and corollary (2.8) says that there are $n-r-2p$ vectors $x_1, x_2, \ldots, x_{n-r-2p}$ that extend it to a rectanormal basis of $\mathbb{C}^n$. Let $X = [x_1x_2 \ldots x_{n-r-2p}] \in M_{n,n-r-2p}$ and set $L \equiv [E\ F\ G\ X] \in M_n$. We conclude that the given set of vectors $\{e_i\} \sqcup \{f_i\}$ may be augmented by additional vectors $\{g_i\} \sqcup \{x_i\}$ to form a basis with the following canonical bilinear product structure:

(2.11)
$$L^TL = \begin{bmatrix} E^TE & E^TF & E^TG & E^TX \\ F^TE & F^TF & F^TG & F^TX \\ G^TE & G^TF & G^TG & G^TX \\ X^TE & X^TF & X^TG & X^TX \end{bmatrix} = \begin{bmatrix} I_r & 0 & 0 & 0 \\ 0 & 0 & I_p & 0 \\ 0 & I_p & 0 & 0 \\ 0 & 0 & 0 & I_{n-r-2p} \end{bmatrix}$$

where $I_k \in M_k$ denotes an identity matrix of size $k$. If $p = 0$, this reduces to theorem (2.7). Notice that $L^TL$ is a symmetric permutation matrix, and hence $I = (L^TL)^2 = (L^TLL^T)L$. Thus $L^{-1}$ is easily computed as $L^{-1} = (L^TL)L^T$, which is a permutation of the rows of $L^T$. Such matrices may be thought of as a generalization of complex orthogonal matrices. We formalize these observations as

**Corollary 2.12** *Let* $\{e_1, e_2, \ldots, e_r, f_1, f_2, \ldots, f_p\} \subset \mathbb{C}^n$ *be a given rect-*

*angular set with* $e_i^T e_i = 1$ *for* $i = 1, 2, \ldots, r$ *and* $f_i^T f_i = 0$ *for* $i = 1, 2, \ldots, p$. *Then* $n \geq r + 2p$ *and there are* $p$ *additional isotropic vectors* $g_1, g_2, \ldots, g_p$ *and, if* $n > r + 2p$, $n - r - 2p$ *rectanormal vectors* $x_1, x_2, \ldots, x_{n-r-2p}$ *so that the set* $B = \{e_i\} \sqcup \{f_i\} \sqcup \{g_i\} \sqcup \{x_i\}$ *is a rectangular basis of* $\mathbb{C}^n$ *whose bilinear product structure is given by* (2.11).

One consequence of the inequality in (1) of theorem (2.9) is that any isotropic subspace of $\mathbb{C}^n$ has dimension at most $[n/2]$. Subject to this bound, there exist isotropic subspaces of all possible dimensions $k = 1, 2, \ldots, [n/2]$, as exemplified by the spans of the first $k$ vectors in the sequence of standard rectangular isotropic basis vectors

$$\varepsilon_1 = [1 \ i \ 0 \ 0 \ \ldots \ 0]^T, \varepsilon = [0 \ 0 \ 1 \ i \ 0 \ 0 \ \ldots \ 0]^T, \ldots$$

In general, $\varepsilon_{2k-1} \equiv e_{2k-1} + ie_{2k}$, where the standard unit basis vector $e_j = [0 \ \ldots \ 0 \ 1 0 \ \ldots \ 0]^T$ has $+1$ in the $j_{th}$ position and zero entries elsewhere.

In ordinary Euclidean geometry, any $k$-dimensional subspace can be transformed unitarily onto any other $k$-dimensional subspace. In particular, it can be transformed unitarily into a standard $k$-dimensional subspace that is the span of $e_1, e_2, \ldots, e_k$. This is a consequence of the polar decomposition [7, section (7.3)] that permits us to write any $X \in M_{n,k}$, $k \leq n$, as $X = UB$, where $U \in M_n$ is unitary and $B = M_{n,k}$ has the form $B = [(X^*X)^{1/2} \ 0]^*$, where $(X^*X)^{1/2}$ denotes the unique positive semidefinite square root of $X^*X$. Thus if $Y \in M_{n,k}$ satisfies $Y^*Y = X^*X$, we can write $Y = VB$ for some unitary $V \in M_m$ (and the same $B \in M_{n,k}$) and hence $X = UB = UV^*X = WX$, where $W \equiv UV^* \in M_n$ is unitary. The geometrical consequence of this argument is that a given set of column vectors $[x_1 x_2 \ldots x_k] = X \in M_{n,k}$ can be transformed unitarily into another given set $[y_1 y_2 \ldots y_k] \in M_{n,k}$, i.e., $X = UY$ for some unitary $U \in M_n$ , if and only if $X^*X = Y^*Y$, i.e., all the respective inner products are the same.

In the geometry associated with the bilinear form $x^T y$, things are not so simple. There are as many as $k + 1$ essentially different (i.e., not orthogonally equivalent) $k$-dimensional subspaces of $\mathbb{C}^n$ in this case, depending on whether the subspace is nonsingular, isotropic, or singular (as many $k-1$ different isotropic possibilities); one must have $k \leq [n/2]$ to realize $k+1$ different possibilities, of course. In order to show that any $k$-dimensional subspace of $\mathbb{C}^n$ can be transformed orthogonally into one of the standard $k$-dimensional subspace spanned by appropriate combinations of standard unit basis vectors $e_i$ and standard rectangular basis vectors $\varepsilon_{2j-1} \equiv e_{2j-1} + ie_{2j}$, we need to know when two matrices $X_1, X_2 \in M_{n,k}$ are of the form $X_1 = QX_2$, where $Q \in M_n$ is complex orthogonal. It is certainly necessary that $X_1^T X_1 = (QX_2)^T(QX_2) = X_2^T Q^T Q X_2 = X_2^T X_2$, but this is not sufficient, as shown by the example

$$X_1 = \begin{bmatrix} 1 & 0 \\ i & 0 \end{bmatrix}, X_1^T X_1 = 0 \quad X_2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, X_2^T X_2 = 0.$$

Even if we add the necessary condition that rank $X_1 =$ rank $X_2$, the example

$$X_1 = \begin{bmatrix} 1 & 0 \\ i & 0 \end{bmatrix}, X_1^T X_1 = 0 \quad X_2 = \begin{bmatrix} 0 & 1 \\ 0 & i \end{bmatrix}, X_2^T X_2 = 0.$$

shows that there still may be no orthogonal $Q$ such that $X_1 = QX_2$, since the zero vector cannot be the orthogonal image of a nonzero vector.

If $X_1 \in M_{n,k}$ with $k \leq n$ and if $X_1$ has rank $q < k$, there is some permutation matrix $P \in M_k$ for which $X_1 P = [\hat{X}_1,\ \tilde{X}_1]$ and $\hat{X}_1 \in M_{n,q}$ has full rank $q$; then $\tilde{X}_1 = \hat{X}_1 C \in M_{n,k-q}$ for some $C \in M_{q,k-q}$. If in addition, $X_1 = QX_2$ for some orthogonal $Q \in M_n$ and $X_2 \in M_{n,k}$ partition $X_2 P = [\hat{X}_2\ \tilde{X}_2]$ conformally with the given partition $X_1 P = [\hat{X}_1\ \tilde{X}_1]$ and observe that $X_1 P = [\hat{X}_1\ \tilde{X}_1] = [\hat{X}_1\ \hat{X}_1 C] = QX_2 P = [Q\hat{X}_2\ Q\tilde{X}_2]$. Thus $\tilde{X}_2 = Q^T(Q\tilde{X}_2) = Q^T \hat{X}_1 C = Q^T(Q\hat{X}_2)C = \hat{X}_2 C$. It is therefore necessary that if $X_1 P = ]\hat{X}_1\ \hat{X}_1 C]$ for some $\hat{X}_1 \in M_{n,q}$ with full rank and some permutation matrix $P \in M_k$, then $X_2 P = [\hat{X}_2\ \hat{X}_2 C]$ and $\hat{X}_2 \in M_{n,q}$ has full rank $q$. These necessary conditions are also sufficient.

**Theorem 2.13.** *Let $X_1, X_2 \in M_{n,q}$ with $1 \leq k \leq n$ and let* rank $X_1 = q$. *There is a complex orthogonal $Q \in M_n$ such that $X_1 = QX_2$ if and only if*

(a) $X_1^T X_1 = X_2^T X_2$.

(b) rank $X_1 =$ rank $X_2$, *and, in addition*

(c) *If* $1 \le q < k$ *and* $P \in M_k$ *is a permutation matrix such that* $X_1 P = [\hat{X}_1\ \tilde{X}_1]$. $\hat{X}_1 \in M_{n,q}$ *has full* rank $q$, *and* $\tilde{X}_1 = \hat{X}_1 C$ *for some* $C \in M_{n,k-q}$, *then* $X_2 P = [\hat{X}_2\ \hat{X}_2 C]$, *where* $\hat{X}_2 \in M_{n,q}$.

*Proof.* The necessity of these three conditions has already been shown. Suppose $X_1$ and $X_2$ satisfy (a), (b), and (c). If $q = 0$, then $X_1 = X_2 = 0$ and there is nothing to prove. If $1 \le q < k$, compute the two quantities

$$P^T(X_i^T X_i)P = (X_i P)^T(X_i P) = \begin{bmatrix} \hat{X}_i^T \hat{X}_i & \hat{X}_i^T \hat{X}_i \\ \tilde{X}_i^T \tilde{X}_i & \tilde{X}_i^T \tilde{X}_i \end{bmatrix}, \ \tilde{X}_i = \hat{X}_i C,\ i = 1, 2,$$

which are equal by (a). In particular, $\hat{X}_1^T \hat{X}_1 = \hat{X}_2^T \hat{X}_2$. There are two cases to consider:

Case 1: Suppose rank $\hat{X}_1 = \hat{X}_1^T \hat{X}_1 = q$, and hence also rank $\hat{X}_2^T \hat{X}_2 = q$. Theorem (2.3) guarantees that there is a nonsingular $B \in M_q$ such that $\hat{X}_1 = \hat{Y}_1 B$, $\hat{X}_2 = \hat{Y}_2 B$, where $\hat{Y}_1, \hat{Y}_2 \in M_{n,,q}$ have rectanormal columns. By corollary (2.8) there are matrices $\tilde{Y}_i \in M_{n,n-q}$ with rectanormal columns such that $Q_i \equiv [\hat{Y}_i\ \tilde{Y}_i] \in M_n$ is complex orthogonal, $i = 1, 2$. Now write

$$X_i P = [\hat{X}_i\ \ \hat{X}_i C] = [\hat{Y}_i B\ \ \hat{Y}_i BC] = [\hat{Y}_i\ \ \tilde{Y}_i] \begin{bmatrix} B & BC \\ 0 & 0 \end{bmatrix} \equiv Q_i R,\ i = 1, 2,$$

where $R \equiv \begin{bmatrix} B & BC \\ 0 & 0 \end{bmatrix} \in M_n$. Then $X_2 P = Q_2 R$ and $R = Q_2^T X_2 P$, so $X_1 P = Q_1 R = Q_1 Q_2^T X_2 P$ and hence $X_1 = Q X_2$, where $Q \equiv Q_1 Q_2^T$ is complex orthogonal.

Case 2: Suppose $\hat{X}_1^T \hat{X}_1$, and hence also $\hat{X}_2^T \hat{X}_2$, has rank $r < q$, and use theorem (2.3) as in the previous case to write $\hat{X}_i = \hat{Y}_i B$, $i = 1, 2$, for some nonsingular $B \in M_q$, but now $\hat{Y}_1, \hat{Y}_2 \in M_{n,q}$ satisfy

$$\hat{Y}_i^T \hat{Y}_i = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}, \in M_r.$$

By theorem (2.9), $2q - r \le n$ and there are $Z_1, Z_2 \in M_{n,q-r}$ such that the matrices $H_i \equiv [\hat{Y}_i\ Z_i] \in M_{n,2q-r}$ satisfy

$$H_i^T H_i = \begin{bmatrix} I_r & 0 & 0 \\ 0 & 0 & I_{q-r} \\ 0 & I_{q-r} & 0 \end{bmatrix} \in M_{2q-r},\ i = 1, 2;$$

in particular, both products are nonsingular. By theorem (2.3) there is one nonsingular $G \in M_{2q-r}$ such that $H_i = W_i G$, $i = 1, 2$, where $W_1, W_2 \in M_{n,2q-r}$ have rectanormal columns. Use corollary (2.8) to extend the column sets of $W_1$ and $W_2$ to rectanormal bases of $\mathbb{C}^n$ and let $Q_i = [W_i\ \tilde{W}_i] \in M_n$, $i = 1, 2$, denote the resulting complex orthogonal matrices whose first $2q - r$ columns are the columns of $W_i$. Then

$$\begin{aligned} X_i P =& [\hat{X}_i\ \ \hat{X}_i C] = [\hat{Y}_i\ \ \hat{Y}_i BC] = [\hat{Y}_i\ \ Z_i] \begin{bmatrix} B & BC \\ 0 & 0 \end{bmatrix} = H_i \tilde{R} \\ =& W_i G \tilde{R} = [W_1\ \ \tilde{W}_1] \begin{bmatrix} G\tilde{R} \\ 0 \end{bmatrix} = QR,\ i = 1, 2 \end{aligned}$$

where $\tilde{R} \equiv= \begin{bmatrix} B & BC \\ 0 & 0 \end{bmatrix} \in M_{2q-r}$ and $R \equiv \begin{bmatrix} G\tilde{R} \\ 0 \end{bmatrix} \in M_n$. Finally, $R = Q_2^T X_2 P$ and $X_1 R = Q_1 P = Q_1 Q_2^T X_2 P$, so $X_1 = Q X_2$, where $Q \equiv Q_1 Q_2^T \in M_n$ is complex orthogonal.

The preceding argument also covers the remaining possibility that $q = k$ if we simply omit all terms involving $C$ and take $\hat{X}_i = X_i$.

It follows immediately that any subspace of $\mathbb{C}^n$ can be mapped orthogonally onto a subspace of standard type with a canonical rectangular basis.

**Corollary 2.14.** *Let $V \subset \mathbb{C}^n$ be a $k$-dimensional subspace with basis $\{x_1, x_2, \ldots, x_k\}$, and let $X = [x_1 x_2 \ldots x_k] \in M_{n,q}$. If $\operatorname{rank} X^T X = r$, then $2k - r \leq n$ and there is a complex orthogonal matrix $Q \in M_n$ such that $V = Q \operatorname{span}\{e_1, e_2, \ldots, e_r\} \cup \{\varepsilon_{r+1}, \varepsilon_{r+3}, \ldots, \varepsilon_{r+2(k-r)-1}\}$, where we agree that the second set of vectors in this union is present only if $k > r$, i.e., $\{Qe_1, Qe_2, \ldots, Qe_r\} \cup \{Q\varepsilon_{r+1}, Q\varepsilon_{r+3}, \ldots, Q\varepsilon_{r+2(k-r)-1}\}$ is a rectangular basis of $V$.*

*Proof.* The subspace $V$ has a rectangular basis containing $k-r$ isotropic vectors and $r$ rectanormal vectors. The set $\{e_1, e_2, \ldots, e_r, \varepsilon_{r+1}, \varepsilon_{r+3}, \ldots, \varepsilon_{r+2(k-r)-1}\}$ is independent, rectangular, and contains $k-r$ isotropic vectors and $r$ rectanormal vectors. Existence of the required complex orthogonal matrix $Q$ that maps this standard set onto the given rectangular basis of $V$ follows from the sufficiency of conditions (a) and (b) of theorem (2.13) in the full rank case.

Given any unit vector $x \in \mathbb{C}^n$, one can always construct a unitary matrix whose first column is $x$. Analogously, given any vector $x \in \mathbb{C}^n$ such that $x^T x = 1$, one can always construct an orthogonal matrix whose first column is $x$. Let $x = [x_1\ y^T]^T$, where $x_1 \in \mathbb{C}$ and $y \in \mathbb{C}^{n-1}$. If $x_1 \neq -1$, then

$$Q = \begin{bmatrix} x_i & y^T \\ y & -I + \frac{1}{(1+x_1)} yy^r \end{bmatrix} \in M_n$$

is orthogonal. If $x_1 = -1$ then $y^T y = 0$ and

$$Q = \begin{bmatrix} -1 & y^r \\ y & I - \frac{1}{2} yy^T \end{bmatrix} \in M_n$$

is orthogonal.

Given any two vectors $x$ and $y$ with the same Euclidean length, one can always construct a unitary matrix $U$ such that $Ux = y$; $U$ may be taken to be a Householder transformation. Analogously, given two nonzero vectors $x$ and $y$ such that $x^T x = y^T y$, theorem (2.13) says that one can always find an orthogonal matrix $Q$ such that $Qx = y$. There are two cases to consider.

Case 1: If $x^T x \neq 0$, there is no loss of generality in scaling $x$ and $y$ by the same factor so that $x^T x = y^T y = 1$. If $x^T y \neq 1$, let $v = x + y$ and consider the matrix $Q = -I + 2vv^T / v^T v$. If $x^T y = -1$, let $v = x - y$ and consider the matrix

$$Q = I - 2\nu\nu^T / \nu^T \nu = I - \frac{1}{2}\nu\nu^T.$$

Case 2: Suppose $x^T x = y^T y = 0$. If $x^T y \neq 0$, let $v = x + y$ and consider the matrix $Q = -I + 2vv^T / v^T v$. If $x^T y = 0$, we can find isotropic vectors $z_1, z_2 \in \mathbb{C}^n$ such that $x^T z_1 \neq 0$, $y^T z_2 \neq 0$, and $x^T z_2 = y^T z_1 = z_1^T z_2 = 0$. To do so, one applies theorem (2.9) to $W = [x]$ if the set $\{x, y\}$ is dependent or to $W = [x\ y]$ if $x$ and $y$ are independent. Let $z = z_1 + z_2$, so that $z^T z = 0$, $x^T z \neq 0$ and $y^T z \neq 0$. We have already shown that there are orthogonal matrices $Q_1$ and $Q_2$ such that $Q_1x = z$ and $Q_2y = z$. Hence $Qx = y$ where $Q = Q_2^TQ_1$ is complex orthogonal.

## References

1. E. Artin, *Geometric Algebra*, Interscience, New York, 1957.

2. D. Choudhury and R. A. Horn, *An analog of the singular value decomposition for complex orthogonal Equivalence*, Linear and Multilinear Algebra **21** (1987), 149-162.

3. D. Choudhury and R. A. Horn, *A Complex Orthogonal-Symmetric Analog of the Polar Decomposition*, SIAM J. Algebraic Discrete Methods **8** (1987), 149-162.

4. B. D. Craven, *Complex Symmetric Matrices*, J. Austral. Math. Soc. **10** (1964), 341-354.

5. F. R Gantmacher, *The Theory of Matrices*, Vols. I and II, Chelsea Publishing company, New York, 1960.

6. Y. P. Hong and R. A. Horn, *On the reduction of a matrix to Triangular or Diagonal Form by Consimilarity*, SIAM J. Algebraic Discrete Methods **7** (1986), 80-88.

7. R. A. Horn and C. R. Johnson, *Matrix Analysis*, Cambridge University Press, New York, 1985.

8. L. K. Hua, *On the Theory of Automorphic Functions Of a Matrix Variable I-Geometric Basis*, Amer. J. Math. **66** (1944), 470-488.

9. N. Jacobson, *Normal Semi-Linear Transformations*, Amer. J. Math. **61** (1939), 45-58.

10. I. Schur, *Ein Satz über Quadratische Formen Mit Komplexen Koeffizienten*, Amer. J. Math. **67** (1945), 472-480.

11. C. L. Siegel, *Symplectic Geometry*, Amer. J. Math. **65** (1943), lemma 1,pp. 12, 14-15.

12. T. Takagi, *On an Algebraic Problem Related to an Analytic Theorem of Carathèodory and Fejèr and on an Allied Theorem of Landau*, Japan J. Math. **1** (1925), 83-93.

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