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SOME CONDITIONS FOR A COMPLEX STRUCTURE

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The purpose of this note is to give a sufficient condition and therefrom a necessary condition for an almost complex structure to be a complex structure. The results are contained in the following theorem.

Theorem. *Let J be an almost complex structure on a Riemannian $2n$-manifold M^{2n} $(n \geq 2)$. Let $J_i{}^j$ and R_{hijk} denote respectively the components of the tensor of J and the Riemann curvature tensor of M^{2n} with respect to a Riemannian metric g_{ij} and a local coordinate system, where all indices take the values $1, \ldots, 2n$, If $J_i{}^j$ satisfy*

$$\nabla_{i_1} J_{i_2}{}^j - \nabla_{i_2} J_{i_1}{}^j = 0 \tag{1}$$

for all i_1, i_2, j, where ∇ denotes the covariant derivation with respect to g_{ij}, then J is a complex structure and

$$J_{i_1}{}^i J_{i_2}{}^j R_{iji_3k} + J_{i_2}{}^i J_{i_3}{}^j R_{iji_1k} + J_{i_3}{}^i J_{i_1}{}^j R_{iji_2k} = 0 \tag{2}$$

for all i_1, i_2, i_3, k, where the repeated indices imply summation.

Proof. Let $x^1, \ldots, x^{2n}$ be a local coordinate system on the manifold M^{2n}. Then with respect to this system the torsion tensor T^h_{ij} of the almost complex structure $J_i{}^j$ is given by

$$T^h_{ij} = J_j{}^s\Big(\frac{\partial J_i{}^h}{\partial x^s} - \frac{\partial J_s{}^h}{\partial x^i}\Big) - J_i{}^s\Big(\frac{\partial J_j{}^h}{\partial x^s} - \frac{\partial J_s{}^h}{\partial x^j}\Big), \tag{3}$$

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which can be written, in term of the covariant derivation ∇ with respect to the Riemannian metric g_{ij}, as

$$T_{ij}^h = J_j{}^s(\nabla_s J_i{}^h - \nabla_i J_s{}^h) - J_i{}^s(\nabla_s J_j{}^h - \nabla_j J_s{}^h). \tag{4}$$

From (1) it follows that $T_{ij}^h = 0$, so that the almost complex structure J is integrable, and the Newlander and Nirenberg's theorem [1] shows that J is a complex structure.

On the other hand, using the Ricci identity we obtain

$$(\nabla_{i_1}\nabla_{i_2} - \nabla_{i_2}\nabla_{i_1})J_{i_3}{}^k = J_{i_3}{}^j R^k{}_{ji_2i_1} - J_j{}^k R^j{}_{i_3i_2i_1}, \tag{5}$$

$$-(\nabla_{i_3}\nabla_{i_2} - \nabla_{i_2}\nabla_{i_3})J_{i_1}{}^k = -J_{i_1}{}^j R^k{}_{ji_2i_3} + J_j{}^k R^j{}_{i_1i_2i_3}, \tag{6}$$

$$(\nabla_{i_3}\nabla_{i_1} - \nabla_{i_1}\nabla_{i_3})J_{i_2}{}^k = J_{i_2}{}^j R^k{}_{ji_1i_3} - J_j{}^k R^j{}_{i_2i_1i_3}. \tag{7}$$

Adding (5), (6), (7) together gives

$$\begin{aligned} &\nabla_{i_1}(\nabla_{i_2} J_{i_3}{}^k - \nabla_{i_3} J_{i_2}{}^k) + \nabla_{i_2}(\nabla_{i_3} J_{i_1}{}^k - \nabla_{i_1} J_{i_3}{}^k) \\ &\qquad + \nabla_{i_3}(\nabla_{i_1} J_{i_2}{}^k - \nabla_{i_2} J_{i_1}{}^k) \\ &= - J_{i_1}{}^j R^k{}_{ji_2i_3} + J_{i_2}{}^j R^k{}_{ji_1i_3} + J_{i_3}{}^j R^k{}_{ji_2i_1} \\ &\qquad + J_j{}^k (R^j{}_{i_1i_2i_3} - R^j{}_{i_2i_1i_3} - R^j{}_{i_3i_2i_1}). \end{aligned} \tag{8}$$

By (1) and

$$R^j{}_{i_1i_2i_3} + R^j{}_{i_2i_3i_1} + R^j{}_{i_3i_1i_2} = 0,$$

(8) is reduced to

$$J_{i_1}{}^j R^k{}_{ji_2i_3} + J_{i_2}{}^j R^k{}_{ji_3i_1} + J_{i_3}{}^j R^k{}_{ji_1i_2} = 0. \tag{9}$$

Multiplying (9) by g_{ki} and summing for k we have

$$J_{i_1}{}^j R_{iji_2i_3} + J_{i_2}{}^j R_{iji_3i_1} + J_{i_3}{}^j R_{iji_1i_2} = 0. \tag{10}$$

Multiplying (10) by $J_k{}^i$ and summing for i yield

$$J_{i_1}{}^j J_k{}^i R_{iji_2i_3} + J_{i_2}{}^j J_k{}^i R_{iji_3i_1} + J_{i_3}{}^j J_k{}^i R_{iji_1i_2} = 0. \tag{11}$$

By changing i_1, k, i_2 to i_2, i_1, k; k, i_2, i_3 to i_2, i_3, k; and i_1, k, i_3 to k, i_3, i_1, respectively, from (11) we obtain

$$J_{i_2}{}^j J_{i_1}{}^i R_{ijki_3} + J_k{}^j J_{i_1}{}^i R_{iji_3i_2} + J_{i_3}{}^j J_{i_1}{}^i R_{iji_2k} = 0,$$

$$J_{i_1}{}^j J_{i_2}{}^i R_{iji_3k} + J_{i_3}{}^j J_{i_2}{}^i R_{ijki_1} + J_k{}^j J_{i_2}{}^i R_{iji_1i_3} = 0,$$

$$J_k{}^j J_{i_3}{}^i R_{iji_2i_1} + J_{i_2}{}^j J_{i_3}{}^i R_{iji_1k} + J_{i_1}{}^j J_{i_3}{}^i R_{ijki_2} = 0.$$

Adding the above three equations together and making use of (1) we immediately arrive at (2), and the proof of the theorem is complete.

Referee's remark. Condition (1) is by no means necessary for an almost complex structure J to be integable; in fact, for $n = 2$ if the metric $g = (g_{ij})$ and the almost complex structure J are compatible, i.e., if $g(u, v) = g(Ju, Jv)$ for any two tangent vectors u and v, then condition (1) implies that the metric g is flat.

References

1. A. Newlander & L. Nirenberg, *Complex analytic coordinates in almost complex manifolds*, Ann. of Math. **65** (1957), 391–404.

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