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ENUMERATING PARTITIONS WITH GIVEN PART-SIZES

BY

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Abstract. This paper studies the problems of enumerating partitions of the set $\{1, 2, \ldots, n\}$ with given partition-sizes and specific properties. Six properties are considered.

1. Introduction. Consider the problem of partitioning an ordered set $N_n = \{1, 2, \ldots, n\}$ of objects into indistinguishable parts. Let $\pi_1, \ldots, \pi_p$ denote the parts and n_i the cardinality of π_i. A partition is called a p-partition if the number p of parts is specified, and an $\{n_i\}$-partition if the cardinality n_i of π_i for each $i = 1, 2, \ldots, p$ is. Let p_j denote the number of parts of size j in $\{n_i\}$. Clearly, the number of p-partition is

$$\#(n,p) = \frac{1}{p!}\sum_{k=0}^{p-1}(-1)^k\binom{p}{k}(p-k)^n,$$

which is exponential in general. Thus it is very time-consuming, in searching for an optimal partition under some cost functions, to examine all these partitions. One approach popular in the operation research literature to deal with this "size" problem is to work with cost functions such that an optimal partition exists in a specific class of polynomial size. Six such classes have been identified in the literature. We say a part A *penetrates* another

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part B if there exist x, $z \in B$ and $y \in A$ such that $x < y < z$. The six classes are:

Consecutive (C)	No part penetrates any other parts.
Nested (N)	No two parts penetrate each other.
Order-consecutive (O)	The parts can be labeled such that $\pi_i$does not penetrate $\bigcup_{j<i} \pi_i$.
Fully nested (F)	The penetration relation forms a linear order.
Almost fully nested (A)	Fully nested except for parts of size one.
Size-consecutive (S)	Consecutive plus the condition that $i < j$, $i \in \pi_h$, $j \in \pi_k$ implies $n_h \leq n_k$.

Let $\#_Q(n,p)$ and $\#_Q(\{n_i\})$ denote the number of p-partitions and $\{n_i\}$-partitions in class Q. Hwang and Mallows [2] counted $\#_C(n,p)$, $\#_N(n,p)$, $\#_O(n,p)$, and $\#_F(n,p)$. In this paper, we provide the rest of the enumeration picture. Some simple cases will be settled here immediately.

$$\#_C(\{n_i\}) = \frac{p!}{\prod_{j\geq 1} p_j!} \quad \text{and} \quad \#_S(\{n_i\}) = 1.$$

It follows that $\#_S(n,p)$ is simply the number of shapes for given n and p, or equivalently, the number of ways of partitioning n into p unordered positive integers, which is a well-known [1] unsolved problem in combinatorial theory, with $\frac{n^{p-1}}{(p-1)!p!}$ being the dominant term.

2. The main results.

Theorem 1.

$$\#_F(\{n_i\}) = \begin{cases} \dfrac{(p-1)!\prod_{i=1}^{p}(n_i-1)}{\prod_{j>1} p_j!} \sum_{k>1} \frac{p_k}{k-1} & \textit{if } \min n_i > 1; \\ \dfrac{(p-1)!\prod_{n_i>1}(n_i-1)}{\prod_{j\geq 1} p_j!} & \textit{if } \min n_i = 1. \end{cases}$$

Proof. (i) $\min n_i > 1$. Suppose that the part which penetrates all other parts is of size k. Then the other $p-1$ parts consist of p_j parts of size j for $j \neq k$ and $p_k - 1$ parts of size k. Since the parts are indistinguishable, there are

$$\frac{(p-1)!}{(p_k-1)!\prod_{j\neq k} p_j!} = \frac{(p-1)!p_k}{\prod_{j\geq 1} p_j!}$$

ways of ordering $p-1$ parts in the linear order of penetration. Note that if π_i penetrates π_j. Then there are $n_j - 1$ spaces (between the elements of π_j) that π_i can lie. Hence there are $\frac{\prod_{i=1}^{p}(n_i-1)}{k-1}$ ways of choosing the spacing for each π_j except the part which penetrates all other parts.

(ii) $\min n_i = 1$. The argument is similar to (i) except the part penetrating all other parts must be the unique part of size 1.

Theorem 2.

$$\#_A(\{n_i\}) = \binom{n}{p_1} \frac{(p-p_1-1)!\prod_{n_i>1}(n_i-1)}{\prod_{j>1} p_j!} \sum_{k>1} \frac{p_k}{k-1}.$$

Proof. There are $\binom{n}{p_1}$ ways of choosing a set of p_1 positions for parts of size one. For each such choice S, the remaining partition of $N_n \setminus S$ into $p-p_1$ parts must be fully nested. Theorem 2 follows from Theorem 1 immediately.

Theorem 3.

$$\#_A(n,p) = \sum_{i=0}^{p-1} \binom{n}{i} \binom{n-i-2}{2p-2i-2}.$$

Proof. Consider almost fully nested partitions with exactly i parts of size one. There are $\binom{n}{i}$ ways of choosing these i parts. The remaining partition of $N_n \setminus S$ into $p-i$ parts must be fully nested. However, the latter includes fully nested partitions which has a unique part of size 1 penetrating the other; these partitions should not be counted since parts of size one are already taken out. We count such partitions.

We use the same parenthesis representation of a partition as used in [2]. For each part π_i, put a left parenthesis to the space immediately left to the minimum element of π_i, and a right parenthesis to the space immediately right to the maximum element of π_i. Then the number of fully nested partition of $n-i$ elements into $p-i$ parts is the number of ways of inserting $2(p-i)$ parentheses into the $n-i+1$ spaces, except that the first space and the last space must be occupied by the parts which does not penetrate any other parts. Hence the number of fully nested partitions is $\binom{n-i-1}{2p-2i-2}$. A parts of size 1 corresponds to a pair of left-right parentheses with one space apart. Hence once the position of the left parenthesis is chosen. The position of the right parenthesis is also determined. Hence there are only $2p-2i-3$ parentheses to choose. Furthermore, the space occupied by the above-mentioned right parenthesis cannot be occupied by other parentheses. So the number of spaces should also be reduced by one. Therefore the number of fully nested partitions with a unique part of size one is $\binom{n-i-2}{2p-2i-3}$. It follows that the number of fully nested partitions without a part of size one is

$$\binom{n-i-1}{2p-2i-2} - \binom{n-i-2}{2p-2i-3} = \binom{n-i-2}{2p-2i-2}.$$

Theorem 3 follows immediately.

For a subset S of $\{n_1, \ldots, n_p\}$, let $p_j(S)$ denote the number of parts of size j in S.

Theorem 4.

$$\#o(\{n_i\}) = \sum_{S \subseteq \{n_i\}} \left(\binom{p+|S|-1}{2|S|} \frac{(p-|S|)!}{\prod_{j\geq 1} p_j(\{n_i\} \setminus S)!} \frac{|S|!}{\prod_{j\geq 1} p_j(S)!} \prod_{j\geq 1} (j-1)^{p_j(S)} \right).$$

Proof. The elements of a part in an order-consecutive partition are either consecutive or split into two consecutive subsets, called *left block* and *right block*. We call the former type of part a *solid part* and the latter a *split part*. Let S denote the set of split parts in order-consecutive partition. Consider the permutation of the $2|S|$ blocks and the $p-|S|$ solid parts. Note that in an order-consecutive partition, the split parts must be fully nested among themselves. So the first $|S|$ blocks always represent the left blocks. Also note that the two middle blocks, representing the left and right block of the same part, cannot be adjacent or the part would not be split.

Therefore there exists a one-to-one mapping between this set of constrained permutation and the set of unconstrained permutation of $2|S|$ blocks and $p-|S|-1$ solid parts, by eliminating the solid part following the $|S|$th block in a constrained permutation. It is well known [1] that the cardinality of the latter set is $\binom{p+|S|-1}{2|S|}$.

There are $(p-|S|)!/\prod_{j\geq 1} p_j(\{n_i\}\setminus S)!$ ways of ordering the $p-|S|$ solid parts, and $|S|!/\prod_{j\geq 1} p_j(S)!$ ways of ordering the $|S|$ split parts. Finally, the numbers of ways of splitting j objects into nonempty left and right blocks is simply $j-1$.

Let $\#_{N^*}(\{n_i\})$ denote the number of nested $\{n_i\}$-partitions when the n elements are arranged into a cycle. Kreweras [3] proved

$$\#_{N^*}(\{n_i\}) = \frac{n!}{(n-p+1)!\prod_{j\geq 1} p_j!}.$$

Lemma 5.

$$\#_N(\{n_i\}) = \#_{N^*}(\{n_i\}).$$

Proof. Consider the cyclic array as obtained from the linear array by connecting its first and last elements. Clearly, this connection preserves nonnestedness. So if the linear array is not a nested partition, then the cyclic array is not. On the other hand, suppose that the cyclic array is not a nested partition. Then there exist four elements w, x, y, z in that order on the cycle such that w and y belong to one part, while x and z belong to another part. No matter where we cut the cycle into a line the linear order of these four elements must be one of the following four patterns: $wxyz$, $xyzw$, $yzwx$, $zwxy$, which are all nonnested.

Corollary 6.

$$\#_N(\{n_i\}) = \frac{n!}{(n-p+1)!\prod_{j\geq 1} p_j!}.$$

Surprisingly, no direct argument on $\#_N(\{n_i\})$ is known.

3. Distinguishable parts. We call a partition an *ordered partition* if the parts are distinguishable. An ordered p-partition can obtained from its unordered counterpart by multiplying a factor of $p!$. An ordered $\{n_i\}$-partition can be obtained from its unordered counterpart by multiplying a factor of $\prod_{j\geq 1} p_j!$. This is because an ordered $\{n_i\}$-partition, the cardinality of π_i is fixed at n_i. Hence only parts of the same size can be interchanged.

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