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A COMPARISON THEOREM FOR ASYMPTOTICALLY MONOTONE SOLUTIONS OF NONLINEAR DIFFERENCE EQUATIONS

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Sufficient conditions as well as necessary conditions for the existence of asymptotically monotone solutions are important since such results are useful in deriving oscillation theorems of difference equations (see e.g. [2, 4, 5, 8]). One class of nonlinear difference equations which motivates this note is of the following form

$$\Delta(p_{n-1}\Delta x_{n-1}) + q_n f(x_n) = 0, \quad n = 1, 2, 3, \ldots \tag{1}$$

where $p_n > 0$ for $n = 0, 1, 2, \ldots$ and f is a real nondecreasing function defined on R such that $\operatorname{sign} f(x) = \operatorname{sign} x$. By imposing various conditions on $\{p_n\}$, $\{q_n\}$ and f, existence theorems for the asymptotically monotone solutions of (1) were derived [8]. The search for existence theorems of a different nature, however, motivates our concern in this note. More specifically, we shall consider a class of difference equations of the form

$$\Delta(p_{n-1}(\Delta x_{n-1})^{\sigma-1}) + q_n x_n^{\sigma-1} = 0, \quad n = 1, 2, 3, \ldots \tag{2}$$

where $p_n > 0$ for $n \geq 0$ and σ is a real number different from 0 or 1. When $\sigma = 2$, equation (2) reduces to the standard second order linear equation which has been studied to some extent (see e.g. [1-8]). Our main result is a comparison theorem for existence of asymptotically monotone solutions of (2). As an application, we derive a necessary condition for the existence of asymptotically monotone solutions of (1).

A solution of equations (1) or (2) is a real sequence $\{x_n\}_0^\infty$ satisfying

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(1) or (2) respectively. A solution is said to be asymptotically or eventually positive if there is an integer N such that $x_n > 0$ for $n \geq N$. It is said to be eventually increasing if there is some integer M such that $\Delta x_n > 0$ for $n \geq M$. Other concepts related to montonicity of solutions can be similarly defined.

Theorem 1. *Suppose the following equation*

(3) $\Delta(p_{n-1}(\Delta y_{n-1})^{\sigma-1}) + s_n y_n^{\sigma-1} = 0, \quad n = 1,2,3,\ldots, \quad \sigma \neq 0,\ \sigma \neq 1$

where $p_n > 0$ for $n \geq 0$, has an eventually positive nondecreasing solution $\{y_n\}$. Suppose further that $q_n \leq s_n$ for $n = 1,2,3,\ldots$. Then (2) has an eventually positive nondecreasing solution also.

Proof. Suppose for convenience that $y_n > 0$ and $\Delta y_n > 0$ for $n \geq 0$. Dividing equation (3) by $y_n^{\sigma-1}$, we obtain

(4) $p_n(y_{n+1}/y_n - 1)^{\sigma-1} - p_{n-1}(1 - y_{n-1}/y_n)^{\sigma-1} + s_n = 0, \ n = 1,2,3,\ldots$

Defining $c_k = y_{k+1}/y_k$ for $k \geq 0$, (4) can be rewritten as

(5) $p_n(c_n - 1)^{\sigma-1} - p_{n-1}(1 - 1/c_{n-1})^{\sigma-1} + s_n = 0, \ n = 1,2,3,\ldots$

Next, we let $u_k = (c_k - 1)^{\sigma-1}$ for $k \geq 0$ and write (5) as

(6) $$u_n = (p_{n-1}/p_n)F(u_{n-1}) - s_n/p_n, \quad n \geq 1$$

where $F(t) = \dfrac{t}{(t^{1/(\sigma-1)} + 1)^{\sigma-1}}$. Note that $u_k \geq 0$ for $k \geq 0$. Note further that by straightforward calculations, we may show that $F'(t) = (t^{1/(\sigma-1)} + 1)^{-\sigma}$ so that $F'(t) > 0$ for $t \geq 0$.

We now assert that

(7) $$v_n = (p_{n-1}/p_n)F(v_{n-1}) - q_n/p_n, \quad n \geq 1$$

has a solution $\{v_n\}$ such that $v_n \geq u_n$ for $n \geq 0$. Indeed, choose $v_0 \geq u_0$, then defining v_1 by (7), we see that

$$v_1 - u_1 = (p_0/p_1)(F(v_0) - F(u_0)) + \frac{s_0 - q_0}{p_1} \geq (p_0/p_1)F'(r)(v_0 - u_0) \geq 0$$

where $r \geq u_0 \geq 0$. An easy induction then shows that $v_n - u_n \geq 0$ for $n \geq 1$, which proves our assertion.

Finally, let $d_k = 1 + v_k^{1/(\sigma-1)}$ for $k \geq 0$; and $x_0 = 1$, $x_k = d_0 d_1 d_2 \ldots d_{k-1}$ for $k \geq 1$, we may then verify that $\{x_n\}$ is a positive increasing solution of (2).

We remark that when $\sigma = 2$, the above Theorem can be proved by means of discrete Wirtinger type inequalities (see for example Cheng [1]). Also, when $\sigma > 1$ and $p_n \equiv 1$, an argument similar to that given above has been described recently [5].

We say that a solution $\{x_n\}$ of (1) or (2) is nonoscillatory if it is eventually positive or eventually negative, and oscillatory otherwise. The following is an application of Theorem 1.

Theorem 2. *For each $\lambda > 0$, suppose every solution of*

$$\Delta(p_{n-1}(\Delta z_{n-1})^{\sigma-1}) + \lambda q_n z_n^{\sigma-1} = 0, \quad q_n \geq 0, \quad n = 1, 2, 3, \ldots \tag{8}$$

is oscillatory. Suppose further that $f(x)/x^{\sigma-1}$ is nondecreasing for $x > 0$. Then (2) *cannot have an eventually positive nondecreasing solution.*

Proof. Suppose (1) has an eventually positive nondecreasing solution $\{x_n\}$ such that $x_n > 0$ and $\Delta x_n \geq 0$ for $n \geq N$. Let $e_n = f(x_n)/x_n^{\sigma-1}$ for each $n \geq N$. Then $\{e_n\}$ is positive nondecreasing for $n \geq N$. Thus equation

$$\Delta(p_{n-1}(\Delta w_{n-1})^{\sigma-1}) + e_n q_n w_n^{\sigma-1} = 0, \quad n = 1, 2, 3, \ldots$$

has an eventually positive nondecreasing solution, namely, $\{x_n\}$. By Theorem 1,

$$\Delta(p_{n-1}(\Delta v_{n-1})^{\sigma-1}) + c_N q_n v_n^{\sigma-1} = 0, \quad n = 1, 2, 3, \ldots$$

would also have a positive solution which contradicts the assumption of our Theorem.

When $\sigma = 2$, the corresponding linear equation is said to strongly oscillatory if for each $\lambda > 0$, every solution is oscillatory. A result in [3] states when $\sigma = 2$, $p_n \equiv 1$ and $\{q_n\}$ is a nonnegative sequence with infinitely many positive terms, (8) is strongly oscillatory if and only if

$$\lim_{n\to\infty} \sup n \sum_{k=n+1}^{\infty} q_k = \infty. \tag{9}$$

As a consequence, under the above stated conditions on σ, p_n and q_n, if $f(x)/x$ is nondecreasing for $x > 0$ and if (9) holds, then equation (1) cannot have an eventually positive nondecreasing solution. Such a result, together with sufficient conditions for the existence of positive increasing solutions, will then yield oscillation theorems for equation (1) (see for example Li and Cheng [8]).

References

1. S. S. Cheng, *Sturmian comparison theorems for three-term recurrence equations*, J. Math. Anal. Appl., **111** (1985), 465-474.

2. S. S. Cheng and H. J. Li, *Bounded solutions of nonlinear difference equations*, Tamkang J. Math., **21** (1990), 137-142.

3. S. S. Cheng, T. C. Yan and H. J. Li, *Oscillation criteria for second order difference equation*, Funkcialaj Ekvacioj, **34** (1991), 223-239.

4. S. S. Cheng, H. j. Li and W. T. Patula, *Bounded and zero convergent solutions of second order difference equations*, J. Math. Anal. Appl., **141** (1989), 463-483.

5. S. S. Cheng and W. T. Patula, *An existence theorem for a nonlinear difference equation*, Nonlinear Anal., Theory Methods Appl., **20** (1993), 193-203.

6. J. W. Hooker and W. T. Patula, *Riccati type transformations for second-order linear difference equations*, J. Math. Anal. Appl., **82** (1981), 451-462.

7. J. W. Hooker and W. T. Patula, *A second order nonlinear difference equations: oscillation and asymptotic behavior*, J. Math. Anal. Appl., **91** (1983), 9-29.

8. H. J. Li and S. S. Cheng, *Asymptotically monotone solutions of a nonlinear difference equation*, Utilitas Math., to appear.

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