BULLETIN OF THE
INSTITUTE OF MATHEMATICS
ACADEMIA SINICA
Volume 21, Number 3, September 1993

BIFURCATION OF POSITIVE SOLUTIONS OF GENERALIZED NONLINEAR UNDAMPED PENDULUM PROBLEMS

BY

SHIN-HWA WANG (王信華)

Abstract. We study the bifurcation of positive solutions of generalized nonlinear undamped pendulum problems $u'' + f(u) = 0$, $-L < x < L$, $u(-L) = u(L) = 0$ by refining the "time map" techniques of J. Smoller and A. Wasserman (1981). We are able to count the exact number of the time maps and hence are able to count the exact number of positive solutions for these sublinear nonlinearities f satisfying (i) $f(0) = f(1) = 0$, (ii) $f(x) > 0$ in $(0,1)$, and (iii) f'' changing sign at most twice in $(0,1)$. We study the monotonicity as well as the convexity if possible of the time maps in $(0,1)$.

1. Introduction. In this paper we consider the local bifurcation of positive solutions of the generalized nonlinear undamped pendulum problems

$$(1.1)\qquad \begin{aligned} &u'' + f(u) = 0, \qquad -L < x < L \\ &u(-L) = u(L) = 0, \end{aligned}$$

where $2L > 0$, the interval length, is a real bifurcation parameter. Throughout this paper, we assume that $f \in C^2[0,1]$ or $C^3[0,1]$ and satisfies

Received by the editors May 19, 1992 and in revised form December 9, 1992.

AMS 1991 Subject Classification:58F14, 34B15.

Key words: positive solution, bifurcation, time map, generalized nonlinear undamped pendulum problem.

This work was partially supported by the National Science Council of Republic of China.

$$f(0) = f(1) = 0,\ f(x) > 0 \text{ on } (0,1), \text{ and there exists a number } \delta > 0 \text{ such that } f'(x) \leq 0 \text{ on } (1-\delta, 1). \tag{1.2}$$

For different f's with f'' changing sign at most twice, we obtain local bifurcation diagrams of positive solutions of problem (1.1) satisfying

$$0 < \|u\|_\infty < 1; \tag{1.3}$$

i.e., we count the exact number of positive solutions in the order interval $(0,1)$. Note that when $f(x) = \sin(\pi x)$, (1.1) becomes the undamped pendulum problem. In particular, the functions $f(x) = x^p(1-x)^q$ with $p, q \geq 1$ are some interesting examples in consideration.

Problem (1.1) with f satisfying (1.2) also models steady states of a predator-prey system with no predator in which f describes the growth of the prey (whose population density is u) [2]. The differential equation in (1.1) with f satisfying (1.2) is also related to the classic Kolmogorov equation $u_t = \frac{1}{2}u_{xx} + f(u)$ which arose in the context of a genetics model for the spread of an advantageous gene through a population [1]. In addition to (1.2), the function f in the classic Kolmogorov equation satisfies and $f'(0) > f'(u)$ for $0 < u \leq 1$ [1].

Problem (1.1) with f satisfying (1.2) was discussed by Smoller and Wasserman [6], Conway [3], Schaaf [4], and Wang and Kazarinoff [8] under different assumptions on f. We are able to improve and generalize some results in [3, 6, 8] for general nonlinearities f with f'' changing sign at most twice in $(0,1)$ by refining the "time map" techniques introduced by J. Smoller and A. Wasserman [6] in which they studied the bifurcation of solutions of (1.1) for restricted cubic polynomials of the form $f(x) = -(x - a)(x-b)(x-c)$ with $a < b < c$.

As in [6], we rewrite the differential equation in (1.1) as a first order system

$$u' = v, \quad v' = -f(u), \quad -L < x < L. \tag{1.4}$$

It is clear that positive solutions of (1.1) correspond to those orbits of (1.4) which begin on the v-axis (i.e., the line $u = 0$), pass through the positive

u-axis, and end on the v-axis, and take "time" (parameter length) $2L$ to make the journey [5]. Then, as in [6], we define the time map

$$T(\alpha) = 2^{-1/2} \int_0^\alpha (F(\alpha) - F(u))^{-1/2} du, \tag{1.5}$$

where $F(u) = \int_0^u f(s)ds$. Note the solutions of (1.2) correspond to curves for which $T(\alpha) = L$. This leads us to investigate the shape of the time map T ([5, p.186]).

We study the monotonicity as well as the convexity if possible of the time map T in (0,1). We note that if $T''(\alpha) > 0$ in a subinterval (α_1, α_2) of $(0,1)$ then T has no critical point in (α_1, α_2) if $T'(\alpha_1)T'(\alpha_2) \geq 0$ and T has exactly one critical point in (α_1, α_2) if $T'(\alpha_1)T'(\alpha_2) < 0$. Sometimes, the proof of convexity of T might be easier and be more feasible to study the shape of T; see e.g. [4, 9]. The study of convexity of T can sometimes be useful in study the bifurcation of solutions of $u'' + f(u) = 0$ in $(-L, L)$ associated with homogeneous Neumann boundary conditions $u'(-L) = u'(L) = 0$; see, e.g. [6, Section 3].

$T'(\alpha)$ and $T''(\alpha)$ and be computed from (1.5). We write as (1.6) and (1.8) listed below (see [6, p.273]).

$$T'(\alpha) = 2^{-3/2} \int_0^\alpha \frac{\theta(\alpha) - \theta(u)}{(\Delta F)^{3/2}} \frac{du}{\alpha} \tag{1.6}$$

where $\Delta F = F(\alpha) - F(u)$, and

$$\theta(x) = 2F(x) - xf(x). \tag{1.7}$$

$$T''(\alpha) = \frac{2^{-3/2}}{\alpha^2} \int_0^\alpha \frac{-\frac{3}{2}\Delta\theta(\Delta\tilde{f}) + \Delta F(\Delta\tilde{\theta}')}{(\Delta F)^{5/2}} du, \tag{1.8}$$

where $\Delta\theta = \theta(\alpha) - \theta(u)$,

$$\Delta\tilde{f} = \alpha f(\alpha) - uf(u), \quad \text{and} \tag{1.9}$$

$$\Delta\tilde{\theta}' = \alpha\theta'(\alpha) - u\theta'(u). \tag{1.10}$$

Define

$$\psi(x) = 3\theta(x) - x\theta'(x). \tag{1.11}$$

The numerator of the integrand of the integral in (1.8), $N(\alpha)$, can be written as

$$\begin{aligned}
N(\alpha) &= -\frac{3}{2}\Delta\theta(\Delta\tilde{f}) + \Delta F(\Delta\tilde{\theta}') \\
&= -\frac{3}{2}\Delta\theta(\Delta\tilde{f}) + 3\Delta\theta\Delta F - 3\Delta\theta\Delta F + \Delta F(\Delta\tilde{\theta}') \\
&= \frac{3}{2}\Delta\theta(2\Delta F - \Delta\tilde{f}) - (\Delta F)(3\Delta\theta - \Delta\tilde{\theta}') \\
&= \frac{3}{2}(\Delta\theta)^2 - (\Delta F)(3\theta(\alpha) - 3\theta(u) - \alpha\theta'(\alpha) + u\theta'(u)) \\
&= \frac{3}{2}(\Delta\theta)^2 - (\Delta F)[(3\theta(\alpha) - \alpha\theta'(\alpha)) - (3\theta(u) - u\theta'(u))] \\
&= \frac{3}{2}(\Delta\theta)^2 - (\Delta F)(\psi(\alpha) - \psi(u))
\end{aligned}$$

(see [6, p.285]). That is, formula (1.8) can be written as

$$T''(\alpha) = \frac{2^{-3/2}}{\alpha^2}\int_0^\alpha \frac{N(\alpha)}{(\Delta F)^{5/2}}du. \tag{1.12}$$

By (1.7), we compute and find that

$$\psi(x) = 6F(x) - 4xf(x) + x^2 f'(x), \tag{1.13}$$

$$\psi'(x) = 2f(x) - 2xf'(x) + x^2 f''(x), \quad \text{and} \tag{1.14}$$

$$\psi''(x) = x^2 f'''(x). \tag{1.15}$$

From (1.6) and (1.8), by an easy computation, we have

$$T''(\alpha) + \frac{3}{\alpha}T'(\alpha) = \int_0^\alpha \frac{(3/2)(\Delta\theta)^2 + (\Delta F)(\Delta\tilde{\theta}')}{2\alpha^2(\Delta F)^{5/2}}du, \tag{1.16}$$

where $\Delta\tilde{\theta}'$ is defined as in (1.10) (see [6, p.273]).

Formulas (1.6) and (1.12) and estimate (1.16) are useful in our analysis of the time map T; cf. [8, 9, 10].

2. Main results. First, through asymptotic expansion of the integrand and direct computation, we have the following well-known proposition.

Proposition. *Suppose f satisfies* (1.2) *and let T be defined by* (1.5). *Then*

$$T(0) = \begin{cases} \frac{\pi}{2} f'(0)^{-1/2} & \textit{if } f'(0) > 0, \\ \infty & \textit{if } f'(0) = 0, \end{cases} \tag{2.1}$$

and

$$T(1) = \infty. \tag{2.2}$$

Having known the values of the time map T at the boundary points 0 and 1, we then study the time map $T(\alpha)$ defined by (1.5) for $\alpha \in (0,1)$ for different classes of nonlinearities f's with f'' changing sign at most twice. We classify and study the problems by the sign of f'' in $(0,1)$. We begin with the simplest case.

CASE I. f'' does not change sign in $(0,1)$.

Theorem 1. *Suppose that, in addition to* (1.2), $f \in C^2$ *and satisfies* $f''(x) < 0$ *in* $(0,1)$, *then, in addition to* (2.1) *and* (2.2),

$$T'(\alpha) > 0 \quad \textit{for} \quad 0 < \alpha < 1. \tag{2.3}$$

Moreover, if $f \in C^3$ *and satisfies either*

(2.4) $\quad f'''(x) < 0$ *in* $(0,1)$ *or*

(2.5) $\quad f'''(x) < 0$ *in* $(0,d)$, $f'''(x) \geq 0$ *in* $(d,1)$ *for some* $d \in (0,1)$ *and* $\psi'(1) \leq 0$ (*where* $\psi(x)$ *is defined in* (1.13)),

then $T''(\alpha) > 0$ *for* $0 < \alpha < 1$.

Examples to Theorem 1. Choose $f(x) = f_1(x) = -(x+2)x(x-1)$ satisfying (2.4) and $f(x) = f_2(x) = x(-10x^5 + 38x^4 - 45x^3 + 17)$ satisfying (2.5).

Proof of Theorem 1. Suppose $f''(x) < 0$ in $(0,1)$ conclusion (2.3), the monotonicity of the time map T, is well-known. It can also be easily shown by observing that (1.7) gives

(2.6) $\quad \theta'(x) = f(x) - xf'(x)$ and

(2.7) $\quad \theta''(x) = -xf''(x)$.

So $\theta(0) = \theta'(0) = 0$ and $\theta''(x) > 0$ in $(0,1)$. Thus θ is a strictly increasing function in $(0,1)$. So $\theta(\alpha) - \theta(u) > 0$ for $0 < u < \alpha < 1$. Thus $T'(\alpha) > 0$

for $0 < \alpha < 1$. Moreover, suppose that f satisfies either (2.4) or (2.5). Since $f(0) = 0$, by (1.13) and (1.14), $\psi(0) = \psi'(0) = 0$. If f satisfies (2.4), by (1.15), we have $\psi''(x) < 0$ in $(0,1)$; and if f satisfies (2.5), by (1.15), we have $\psi''(x) < 0$ in $(0,d)$, $\psi''(d) = 0$, $\psi''(x) \geq 0$ in $(d,1)$, and $\psi'(1) \leq 0$. In either cases, $\psi(x)$ is strictly decreasing for x near 0^+ and is decreasing for $0 < x < 1$. So $\psi(\alpha) - \psi(u) < 0$ for $0 < u < \alpha$, α near 0^+ and $\psi(\alpha) - \psi(u) \leq 0$ for $0 < u < \alpha < 1$. Thus, in (1.12), $N(\alpha) > 0$ for $0 < u < \alpha < 1$, and hence $T''(\alpha) > 0$ for $0 < \alpha < 1$. This completes the proof of Theorem 1.

Remark 1. In Theorem 1, condition (2.4) can be weakened a little bit as

$$f'''(0^+) < 0 \text{ and } f'''(x) \leq 0 \text{ in } (0,1). \tag{2.8}$$

Remark 2. If the function f satisfies (1.2) and $f''(x) < 0$ in $(0,1)$ but neither (2.4) nor (2.5), then it is not necessary that $T''(\alpha) > 0$ in $(0,1)$. For example for $f(x) = (x^4 - x^2 + x)(1-x)$, numerical evaluation shows that $T''(0^+) < 0$.

CASE II. f'' changes sign exactly once in $(0,1)$.

Theorem 2. *Suppose that, in addition to* (1.2), $f \in C^2$ *and*

$$\text{there exists a number } b \in (0,1) \text{ such that } f''(x) < 0 \text{ in } (0,b) \text{ and } f''(x) \geq 0 \text{ in } (b,1). \tag{2.9}$$

Then, in addition to (2.1) *and* (2.2),

$$T'(\alpha) > 0 \text{ for } 0 < \alpha < 1. \tag{2.10}$$

An example to Theorem 2. Choose $f(x) = x(1-x)^2$.

Proof of Theorem 2. The proof of Theorem 2 is quite similar to that of Theorem 1. By (2.6) and (2.7), $\theta(0) = \theta'(0) = 0$. By (2.9), $\theta''(x) > 0$ in $(0,b)$ and $\theta''(x) \leq 0$ in $(b,1)$. Since $\theta'(1) = -f(1) \geq 0$, θ is strictly increasing in $(0,b)$ and is increasing in $(b,1)$. So $\theta(\alpha) - \theta(u) > 0$ for $0 < u < \alpha < 1$, $0 < u < b$, and $\theta(\alpha) - \theta(u) \geq 0$ for $0 < u < \alpha < 1$. Thus, in (1.6), $T'(\alpha) > 0$ for $0 < \alpha < 1$. This completes the proof of Theorem 2.

The next theorem follows from [8] and the second part of Theorem 1 which improves [8].

Theorem 3. *Suppose that, in addition to* (1.2), $f \in C^2$ *and*

(2.11) *there exists a number* $a \in (0,1)$, *which is the first zero of* f'' *in* $(0,1)$ *and suppose that* $f''(x) > 0$ *in* $(0,a)$ *and* $f''(x) \leq 0$ *in* $(a,1)$.

Then, in addition to (2.1) *and* (2.2),

$$T \text{ has exactly one critical point, a minimum, in } (0,1). \tag{2.12}$$

Moreover, if $f \in C^3$ *and satisfies either* (2.4) *or* (2.5), *then*

$$T''(\alpha) > 0 \text{ for } 0 < \alpha < 1. \tag{2.13}$$

Examples to Theorem 3. Choose $f(x) = f_1(x) = -(x+\frac{1}{2})x(x-1)$ satisfying $f_1'(0) > 0$ and $f(x) = f_2(x) = x^2(1-x)$ satisfying $f_2'(0) = 0$. Both $f_1(x)$ and $f_2(x)$ satisfy (2.4).

CASE III. f'' **changes sign exactly twice in** $(0,1)$.

Theorem 4. *Suppose that, in addition to* (1.2), $f \in C^2$ *and satisfies*

(2.14) *there exist numbers* $a, b \in (0,1)$, $a < b$, *such that* $f''(x) > 0$ *in* $(0,a)$, $f''(x) \leq 0$ *in* (a,b), *and* $f''(x) \geq 0$ *in* $(b,1)$, *and*

(2.15) $\theta(b) \geq 0$.

Then, in addition to (2.1) *and* (2.2),

$$T \text{ has exactly one critical point, a minimum, in } (0,1). \tag{2.16}$$

Examples to Theorem 4. Choose $f(x) = f_1(x) = x^2(1-x)^3$ satisfying $f_1'(0) = 0$ and $f(x) = f_2(x) = x^2(1-x)^3 + 0.001x(1-x)^2$ satisfying $f_2'(0) > 0$.

Note that the method used in the proof of Theorem 3 in [8] does not apply to Theorem 4; see [8] for details.

Proof of Theorem 4. We prove (2.16) as follows. In (1.6), we have

$$T'(\alpha) = 2^{-3/2} \int_0^\alpha \frac{\theta(\alpha) - \theta(u)}{(\Delta F)^{3/2}} \frac{du}{\alpha}. \tag{2.17}$$

Now, by (1.7) and (2.6), we get

$$\theta(0) = 0, \tag{2.18}$$

$$\theta'(0) = 0,\ \theta'(1) = -f'(1) \geq 0, \text{ and} \tag{2.19}$$

$$\theta(1) = 2F(1) - f(1) = 2F(1) > 0. \tag{2.20}$$

In addition, by (2.7) and (2.14), we have

$$\begin{aligned} &\theta''(x) < 0 \text{ in } (0,a),\ \theta''(a) = 0,\ \theta'(a) < 0, \\ &\theta''(x) \geq 0 \text{ in } (a,b),\ \theta''(b) = 0,\ \theta'(b) \geq 0, \text{ and} \\ &\theta''(x) \leq 0 \text{ in } (b,1). \end{aligned} \tag{2.21}$$

So, by (2.15), there exist numbers p and q, the first positive zero of θ' and θ in $(0,1)$ respectively, such that

$$a < p < q \leq b, \tag{2.22}$$

and

$$\begin{aligned} &\theta(x) < 0 \text{ and } \theta'(x) < 0 \text{ in } (0,p), \\ &\theta(p) < 0,\ \theta'(p) = 0, \\ &\theta(q) = 0, \text{ and } \theta'(x) \geq 0 \text{ in } (p,1). \end{aligned} \tag{2.23}$$

Thus, for $0 < \alpha \leq p$, by (2.23),

$$\theta(\alpha) - \theta(u) \leq 0 \text{ if } 0 < u < \alpha \leq p. \tag{2.24}$$

Hence, for (2.17),

$$T'(\alpha) < 0 \text{ if } 0 < \alpha \leq p. \tag{2.25}$$

Similarly, for $q \leq \alpha < 1$, by (2.23),

$$\begin{aligned} &\theta(\alpha) - \theta(u) > 0 \text{ if } 0 < u < p,\ q \leq \alpha < 1, \text{ and} \\ &\theta(\alpha) - \theta(u) \geq 0 \text{ if } 0 < u < \alpha,\ q \leq \alpha < 1. \end{aligned} \tag{2.26}$$

Hence, for (2.17),

$$T'(\alpha) > 0 \text{ if } q \le \alpha < 1. \tag{2.27}$$

By above, T is strictly decreasing in $(0,p)$ and is strictly increasing in $(q,1)$. Thus, to show T has exactly one critical point in $(0,1)$, it suffices to show T has exactly one critical point in (p,q). In (1.16), we have

$$T''(\alpha) + \frac{3}{\alpha}T'(\alpha) = \int_0^\alpha \frac{(3/2)(\Delta\theta)^2 + (\Delta F)(\Delta\tilde{\theta}')}{2\alpha^2(\Delta F)^{5/2}}du, \tag{2.28}$$

where $\Delta\tilde{\theta}' = \alpha\theta'(\alpha) - u\theta'(u) = \phi(\alpha) - \phi(u)$ in which we define $\phi(x) = x\theta'(x)$. By (2.15) and (2.23), we have $\phi(0) = 0$, $\phi'(0) = 0$, and

$$\begin{aligned} &\phi(x) < 0 \text{ in } (0,p), \\ &\phi(p) = p\theta'(p) = 0, \text{ and} \\ &\phi(x) \ge 0 \text{ in } (p,q). \end{aligned} \tag{2.29}$$

In addition, differentiating ϕ and by (2.21), (2.22) and (2.23), we get

$$\phi'(x) = \theta'(x) + x\theta''(x) \ge 0 \text{ in } (p,q). \tag{2.30}$$

By above, we have

$$\begin{aligned} &\phi(\alpha) - \phi(u) > 0 \text{ if } p < \alpha < q \text{ and } 0 < u < p, \text{ and} \\ &\phi(\alpha) - \phi(u) \ge 0 \text{ if } p < \alpha < q \text{ and } 0 < u < \alpha. \end{aligned} \tag{2.31}$$

Hence, for (1.16), the integrand is always positive if $p < \alpha < q$ and $0 < u < p$ and is always nonnegative if $0 < \alpha < p$ and $0 < u < \alpha$. Thus, for (2.28), $T''(\alpha) + \frac{3}{\alpha}T'(\alpha) > 0$ if $p < \alpha < q$. If $T'(\bar{\alpha}) = 0$ for some $\bar{\alpha}$, $p < \bar{\alpha} < q$, then $T''(\bar{\alpha}) > 0$. Hence T has exactly one critical point, a minimum in (p,q). This completes the proof of Theorem 4.

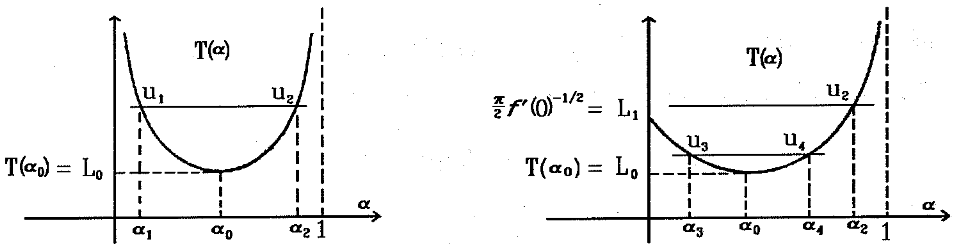
In the proof of Theorem 4, we require that $\theta(b) \ge 0$ to imply that $q < b$ and thus to imply

$$\phi'(x) = \theta'(x) + x\theta''(x) = f(x) - xf'(x) - x^2 f''(x) \ge 0 \text{ in } (p,q). \tag{2.32}$$

For f satisfying (1.2), (2.14), (2.32) except (2.15), we can assume that (2.33) holds and we have the next theorem. We omit the proof.

Theorem 5. *Suppose that $f \in C^2$ and satisfies* (1.2), (2.14), (2.32), *and* $\theta(b) < 0$. *Then, in addition to* (2.1) *and* (2.2), *T has exactly one critical point, a minimum, in* $(0,1)$.

An examples to Theorem 5. Choose $f(x) = x^{20}(1-x)^{20}$. So $b = (39+\sqrt{39})/78 \simeq 0.588$.



case (a). $f'(0) = 0$ case (b). $f'(0) > 0$

Figure 1. Time maps T.

Remark. It follows that the time maps T in Theorems 3, 4, and 5 have the form given in Figure 1. This means in case (a) ($f'(0) = 0$): for $0 < L < L_0$, there is no positive solution; when $L = L_0$, a local bifurcation occurs and we obtain a ("double") nonconstant positive solution; while for every $L > L_0$, there are precisely two nonconstant positive solutions. In case (b) ($f'(0) > 0$): for $0 < L < L_0$, there is no positive solution; when $L = L_0$, a local bifurcation occurs and we obtain a nonconstant positive solution; while for $L_1 > L > L_0$, there are precisely two nonconstant positive solutions; when $L = L_1$, a local bifurcation occurs again and we obtain only one nonconstant positive solution; when $L > L_1$, there is exactly one nonconstant positive solution.

Theorem 6. *Suppose that, in addition to* (1.2), $f \in C^2$,

(2.33) *there exist numbers $a, b \in (0,1)$, $a < b$, such that $f''(x) < 0$ in $(0,a)$, $f''(x) \geq 0$ in (a,b), $f''(x) \leq 0$ in $(b,1)$, and*

(2.34) $\theta'(b) \geq 0$.

Then, in addition to (2.1) *and* (2.2), $T'(\alpha) > 0$ *for* $0 < \alpha < 1$.

Condition (2.34) says that the tangent line to the curve $y = f(x)$ at the point $(b, f(b))$ on the curve intersects positive y-axis or the origin.

An example to Theorem 6. Choose $f(x) = x(1-x)(3-8x+8x^2)$ satisfying $f'(a) < 0$ and $f'(b) > 0$ in which $a = (6-\sqrt{3})/12 \simeq 0.356$ and $b = (6+\sqrt{3})/12 \simeq 0.644$. Numerical evaluation shows that, for $f(x) = x(1-x)(3-8x+8x^2)$, $T''(\alpha)$ changes sign in $(0,1)$.

The proof of Theorem 6 is quite similar to that of Theorem 2. We omit it.

In Theorem 6, condition (2.34) is used to implies that $\theta'(x) > 0$ in $(0,1)$ and hence $\theta(x) > 0$ in $(0,1)$. If condition (2.34) does not hold, the time map T may not be strictly increasing in $(0,1)$. We then in the following consider the time map T of problem (1.1) with f satisfying (1.2), (2.33) and

$$\text{there exists a number } x_0 \in (0,1) \text{ such that } \theta(x_0) < 0. \tag{2.35}$$

Considering $T'(\alpha)$ in $(0,1)$, by (1.7), (2.6), (2.7), and (2.33), we have

$$\begin{aligned}
&\theta(0) = 0,\\
&\theta'(0) = f(0) = 0,\\
&\theta''(x) > 0 \text{ in } (0,a),\ \theta''(x) \le 0 \text{ in } (a,b),\ \theta''(x) \ge 0 \text{ in } (b,1), \text{ and}\\
&\theta(1) = 2F(1) > 2F(x) - xf(x) = \theta(x) \text{ in } (0,1).
\end{aligned} \tag{2.36}$$

In addition, by (2.35), $\theta(x_0) < 0$ for some $x_0 \in (0,1)$. So θ has exactly two positive zeros, says at B, D with $0 < B < D < 1$, and has exactly one relative maximum at A, and one relative minimum, say at C, in $(0,1)$. The graph of function θ takes the form depicted in Figure 2, where E is the number in $(D,1)$ such that $\theta(E) = \theta(A)$. Thus, by (1.6), immediately, we have (i) T is strictly increasing in $(0,A)$, (ii) T has at least one critical point in (A,B), (iii) T is strictly decreasing in (B,C), (iv) T has at least one critical point in (C,E), and (v) T is strictly increasing in $(E,1)$.

Under additional suitable assumptions, we are able to show that T has at least one critical point, a local maximum, in (A,B), and has at least one

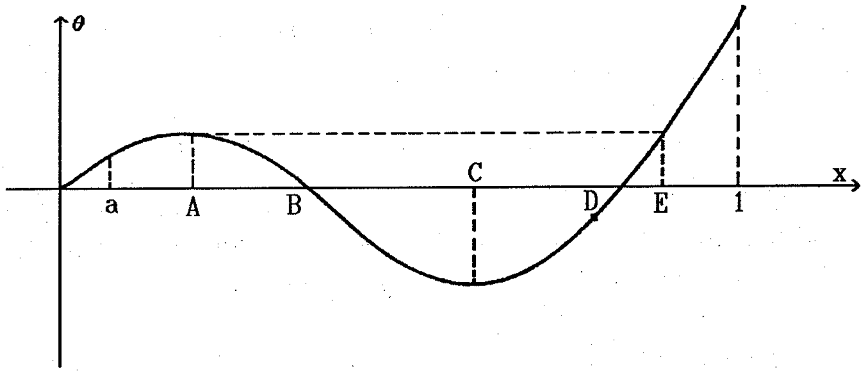


Figure 2. The graph of θ.

critical point, a local minimum, in (C, E); cf. [10].

Theorem 7. *Suppose that, in addition to* (1.2), (2.33) *and* (2.35), $f \in C^2$ *and satisfies*

$$\varphi(C) \geq \varphi(a), \text{ where } \varphi(x) = x\theta'(x) - \theta(x), \tag{2.37}$$

$$(x(\ln f(x))')' \geq 0 \text{ for } x \in (A, B), \text{ and} \tag{2.38}$$

$$\frac{xf'(x)}{f(x)} \geq \frac{-1}{3} \text{ in } (0, B). \tag{2.39}$$

Then, in addition to (2.1) *and* (2.2), *the time map* T *has exactly two critical points in* $(0,1)$. *More precisely,* T *is strictly increasing in* $(0, A)$, *has exactly one critical point, a local maximum, in* (A, B), *is strictly decreasing in* (B, C), *has exactly one critical point, a local minimum, in* (C, E), *and is strictly increasing in* $(E, 1)$.

An example to Theorem 7. Choose

$$f(x) = \begin{cases} x(100x^2 - 58x + 11), & 0 \leq x \leq 1/2, \\ (324x^2 - 254x + 53)(1 - x), & 1/2 < x \leq 1. \end{cases}$$

So $a = 29/150 \simeq 0.193$ and $b = 289/486 \simeq 0.594$. Note that $f'(a) > 0$ and $f'(b) < 0$.

Remark. In Theorem 7 let $L_1 = T(0) = \frac{\pi}{2} f'(0)^{-1/2}$, and L_2 and L_3 be the local minimum and maximum values of T in $(0,1)$. Then it follows from Theorem 7 that for $0 < L < \min(L_1, L_2)$, there is no positive solution; when $\max(L_1, L_2) < L < L_3$, there are precisely three nonconstant positive solutions; while for $L = L_3$, a local bifurcation occurs and we obtain precisely two nonconstant positive solutions; when $L > L_3$, there is exactly one nonconstant positive solution.

Proof of Theorem 7. By above, it suffices to show that the time map $T(\alpha)$ has exactly one critical point, a local maximum, in (B, C) and has exactly one critical point, a local minimum, in (C, E).

First, we consider the time map T in (C, E). By estimate (2.6) in [6], taking $c = -1/\alpha$, we have

$$T''(\alpha) + \frac{2}{\alpha}T'(\alpha) > \frac{2^{-3/2}}{\alpha^2}\int_0^\alpha \frac{\varphi(\alpha) - \varphi(u)}{(\Delta F)^{3/2}}du, \tag{2.40}$$

where

$$\varphi(x) = x\theta'(x) - \theta(x). \tag{2.41}$$

By (2.7) we have

$$\varphi'(x) = x\theta''(x) = -x^2 f''(x). \tag{2.42}$$

Hence, by (2.33), (2.37), (2.41) and (2.42), we have

$$\begin{aligned}
&\varphi(0) = 0,\ \varphi'(0) = 0,\\
&\varphi'(x) > 0 \text{ in } (0, a),\ \varphi'(x) \le 0 \text{ in } (a, b),\ \varphi'(x) \ge 0 \text{ in } (b, 1), \text{ and}\\
&\varphi(C) \ge \varphi(a) > 0.
\end{aligned} \tag{2.43}$$

Since $b < C$, (2.43) implies that $\varphi(\alpha) - \varphi(u) > 0$ for $0 < u < A$, $C < \alpha < E$, and $\varphi(\alpha) - \varphi(u) \ge 0$ for $0 < u < \alpha$, $C < \alpha < E$. Hence, in (2.40), $T''(\alpha) + \frac{2}{\alpha}T'(\alpha) > 0$ for $C < \alpha < E$. Thus T has exactly one critical point, a local minimum, in (C, E).

Secondly, we consider the time map T in (A, B). We show the following Lemma 1 which will be needed subsequently.

Lemma 1. *If* (2.38) *and* (2.39) *hold, then*

$$\max_{0 \le u \le \alpha} \frac{\Delta \tilde{f}}{\Delta F} - \min_{0 \le u \le \alpha} \frac{\Delta \tilde{\tilde{f}}'}{\Delta \tilde{f}} = 1 \textit{ for } \alpha \in (A, B),$$

where $\Delta\tilde{f} = \alpha f(\alpha) - uf(u)$ *and* $\Delta\tilde{\tilde{f'}} = \alpha^2 f'(\alpha) - u^2 f'(u)$.

Lemma 1 follows easily from the following Lemmas 2 and 3.

Lemma 2. *If* (2.38) *and* (2.39) *hold, then the maximum of* $\dfrac{\Delta\tilde{f}}{\Delta F}$ *in* $[0,\alpha]$ *occurs at* $u = \alpha$ *for* $\alpha \in (A,B)$, *and* $\max\limits_{0\le u\le\alpha} \dfrac{\Delta\tilde{f}}{\Delta F} = \dfrac{f(\alpha)+\alpha f'(\alpha)}{f(\alpha)}$ *for* $\alpha \in (A,B)$.

Proof of Lemma 2. For fixed $\alpha \in (A,B)$, $\theta(\alpha) = 2F(\alpha) - \alpha f(\alpha) > 0$, and $\theta'(\alpha) = f(\alpha) - \alpha f'(\alpha) < 0$. So

$$\left.\frac{\Delta\tilde{f}}{\Delta F}\right|_{u=0} = \left.\frac{\alpha f(\alpha) - uf(u)}{F(\alpha) - F(u)}\right|_{u=0} = \frac{\alpha f(\alpha)}{F(\alpha)} < 2, \text{ and}$$
$$\left.\frac{\Delta\tilde{f}}{\Delta F}\right|_{u=0} = \frac{f(\alpha)+\alpha f'(\alpha)}{f(\alpha)} > \frac{f(\alpha)+f(\alpha)}{f(\alpha)} = 2 \text{ (by L'Hospital's rule).}$$

So, for $\alpha \in (A,B)$, the maximum of $\dfrac{\Delta\tilde{f}}{\Delta F}$ occurs at α or at some internal point in $(0,\alpha)$.

Set $G(u) = \dfrac{f(u)+uf'(u)}{f(u)} = 1 + \dfrac{uf'(u)}{f(u)}$. Then, by (2.38) and L'Hospital's rule again, we have

$$G(0) = 2,$$
$$G(u) < 2 \text{ if } 0 < u < A \text{ (since } \theta'(u) = f(u) - uf'(u) > 0 \text{ in } (0,A)), \tag{2.44}$$
$$G(A) = 2, \text{ and } G'(u) = (u(\ln f(u))')' \ge 0 \text{ for } u \in (A,B).$$

Thus

$$G(\alpha) - G(u) \ge 0 \text{ for } 0 < u < \alpha \text{ and } \alpha \in (A,B). \tag{2.45}$$

For $\alpha \in (A,B)$, suppose the maximum of $\dfrac{\Delta\tilde{f}}{\Delta F}$ does not occur at $u = \alpha$, then the maximum of $\dfrac{\Delta\tilde{f}}{\Delta F}$ occurs at an internal point u_0 in $(0,\alpha)$, for some u_0. Hence $\left.\left(\dfrac{\Delta\tilde{f}}{\Delta F}\right)'\right|_{u=u_0} = 0$ which implies that $f(u_0)[\alpha f(\alpha) - u_0 f(u_0)] - [F(\alpha) - F(u_0)][u_0 f'(u_0) + f(u_0)] = 0$. So

$$\frac{\alpha f(\alpha) - u_0 f(u_0)}{F(\alpha) - F(u_0)} = \frac{f(u_0) + u_0 f'(u_0)}{f(u_0)}. \tag{2.46}$$

That is, $\left.\dfrac{\Delta\tilde{f}}{\Delta F}\right|_{u=u_0} = G(u_0)$. For $\alpha \in (A,B)$, by (2.45),

$$\left.\frac{\Delta\tilde{f}}{\Delta F}\right|_{u=u_0} = G(u_0) \leq G(\alpha) = \left.\frac{\Delta\tilde{f}}{\Delta F}\right|_{u=\alpha} \text{ (by L'Hospital's rule).}$$

This contradicts our assumption that the maximum of $\dfrac{\Delta\tilde{f}}{\Delta F}$ does not occur at $u = \alpha$ for $\alpha \in (A,B)$. Hence, for $\alpha \in (A,B)$, the maximum of $\dfrac{\Delta\tilde{f}}{\Delta F}$ occurs at $u = \alpha$, and $\max\limits_{0\leq u\leq\alpha} \dfrac{\Delta\tilde{f}}{\Delta F} = \dfrac{f(\alpha)+\alpha f'(\alpha)}{f(\alpha)}$ for $\alpha \in (A,B)$. This completes the proof of Lemma 2.

Lemma 3. *If* (2.38) *and* (2.39) *hold, then the minimum of* $\dfrac{\Delta\tilde{\tilde{f}}'}{\Delta\tilde{f}}$ *in* $[0,\alpha]$ *occurs at* $u = 0$ *for* $\alpha \in (A,B)$, *and* $\min\limits_{0\leq u\leq\alpha} \dfrac{\Delta\tilde{\tilde{f}}'}{\Delta\tilde{f}} = \dfrac{\alpha f'(\alpha)}{f(\alpha)}$ *for* $\alpha \in (A,B)$.

Proof of Lemma 3. By (2.39), $(xf(x))' = f(x)+xf'(x) > (1/3)(f(x)+3xf'(x)) \geq 0$ in $(0,B)$. So $\alpha f(\alpha) - uf(u) > 0$ for $0 < u < \alpha < B$; that is

$$\Delta\tilde{f} > 0 \text{ for } 0 < u < \alpha < B. \tag{2.47}$$

For $\alpha \in (A,B)$ and $0 < u < \alpha$, by (2.45) and (2.47),

$$\begin{aligned}\frac{\Delta\tilde{\tilde{f}}'}{\Delta\tilde{f}} - \left(\left.\frac{\Delta\tilde{\tilde{f}}'}{\Delta\tilde{f}}\right|_{u=0}\right) &= \frac{uf(u)\left[\frac{\alpha f'(\alpha)}{f(\alpha)} - \frac{uf'(u)}{f(u)}\right]}{\alpha f(\alpha) - uf(u)} \\ &= \frac{uf(u)(G(\alpha)-G(u))}{\alpha f(\alpha) - uf(u)} \geq 0.\end{aligned} \tag{2.48}$$

Hence $\min\limits_{0\leq u\leq\alpha} \dfrac{\Delta\tilde{\tilde{f}}'}{\Delta\tilde{f}} = \dfrac{\alpha f'(\alpha)}{f(\alpha)}$ for $\alpha \in (A,B)$. This completes the proof of Lemma 3.

Lemma 1 implies that suppose $M = \max\limits_{0\leq u\leq\alpha} \dfrac{\Delta\tilde{f}}{\Delta F}$ and $m = \min\limits_{0\leq u\leq\alpha} \dfrac{\Delta\tilde{\tilde{f}}'}{\Delta\tilde{f}}$ then $M - m = 1$. By [6, p.282, 1.10], we have

$$T''(\alpha)+\frac{M}{2\alpha}T'(\alpha)=$$
$$\int_0^\alpha \frac{(\frac{M}{2})[2(\Delta F)^2-(\Delta\tilde{f})(\Delta F)]+\frac{3}{2}(\Delta\tilde{f})^2-2(\Delta\tilde{f})(\Delta F)-(\Delta\tilde{\tilde{f}}')(\Delta F)}{2\alpha^2(\Delta F)^{5/2}}du.$$

Let the numerator of the integrand of the above integral be Q and $\lambda=\dfrac{\Delta\tilde{f}}{\Delta F}$. By (2.47), $\Delta\tilde{f}>0$. In addition, define $\Gamma(x)=xf(x)-(2/3)F(x)$. By (2.39), $\Gamma(0)=0$, $\Gamma'(x)=(1/3)(f(x)+3xf'(x))\geq 0$ in $(0,B)$. So, for $0<u<\alpha<B$, $(\alpha f(\alpha)-(2/3)F(\alpha))-(uf(u)-(2/3)F(u))\geq 0$. Hence $\dfrac{\alpha f(\alpha)-uf(u)}{F(\alpha)-F(u)}\geq\dfrac{2}{3}$; that is $\lambda=\dfrac{\Delta\tilde{f}}{\Delta F}\geq\dfrac{2}{3}$. Now by (2.47) and Lemma 3, for $\alpha\in(A,B)$, $0<u<\alpha$, we have

$$\begin{aligned}Q&\leq\frac{3}{2}(\Delta\tilde{f})^2-(2+m+\frac{M}{2})(\Delta\tilde{f})(\Delta F)+M(\Delta F)^2\\&=(\Delta F)^2(\frac{3}{2}\lambda^2-\lambda(2+m+\frac{M}{2})+M).\end{aligned}$$

Denoting the quadratic in λ by $p(\lambda)$, and noting that $m=M-1$ by Lemma 1, we have $p(\lambda)=\frac{3}{2}\lambda^2-\lambda(\frac{3}{2}M+1)+M=\frac{3}{2}(\lambda-M)(\lambda-\frac{2}{3})$. Since $\frac{2}{3}\leq\lambda\leq M$, $p(\lambda)\leq 0$. It follows that $Q\leq 0$. By more careful analysis, it can be shown that Q is not identically zero for fixed $\alpha\in(A,B)$, $0<u<\alpha$. Hence $T''(\alpha)+\dfrac{M}{2\alpha}T'(\alpha)<0$ for $\alpha\in(A,B)$. So T has exactly one critical point, a local maximum, in (A,B).

So T has exactly two critical points in $(0,1)$. This completes the proof of Theorem 7.

References

1. M. Bramson, *Convergence of Solutions of the Kolmogorov Equation to Travelling Waves*, Memoirs Amer. Math. Soc. **285**, 1983.

2. K. Brown, M. Ibrahim and R. Shivaji, *S-shaped bifurcation curves*, Nonlinear Analysis, Theory, Methods, and Applications **5**(1981), 475-486.

3. E. D. Conway, *Diffusion and predator-prey interaction: steady states with flux at the boundaries*, Contemporary Mathematics **17**(1983), 215-234.

4. R. Schaaf, *Golbal Solution Branches of Two Point Boundary Values Problems*, Lecture Notes in Math. **1458**, Springer-Verlag, Berlin-Heidelberg, 1990.

5. J. Smoller, *Shock Waves and Reaction-Diffusion Equations*, Springer-Verlag, New York, 1983.

6. J. Smoller and A. Wasserman, *Global bifurcation of steady-state solutions*, J. Differential Equations **39**(1981), 269-290.

7. S.–H. Wang, *A correction for a paper by J. Smoller and A. Wasserman*, J. Differential Equations **77**(1989), 199-202.

8. S.–H. Wang and N. D. Kazarinoff, *Bifurcation and stability of positive solutions of a two-point boundary value problem*, J. Aust. Math. Soc. (ser. A) **52**(1992), 334-342.

9. S.–H. Wang, *On the time map of a nonlinear two-point boundary value problem*, preprint.

10. S.–H. Wang and D.–M. Long, *Bifurcation of positive solutions of a generalized Gelfand problem*, Preprint.

Department of Mathematics, National Tsing Hua University, Hsinchu, Taiwan 300, R.O.C.