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ON THE ASYMPTOTIC BEHAVIOR OF POSITIVE RADIAL SOLUTIONS OF A CLASS OF SEMI-LINEAR ELLIPTIC EQUATIONS

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Abstract. Positive radial solutions of the semi-linear elliptic equation $\Delta u - u + Q(|x|)u^\gamma = 0$, $x \in R^n$, $Q(|x|) > 0$, $n \geq 3$, $\gamma > 1$, behaves like $r^{-(n-1)/2}e^{-r}$ or $r^{-(n-1)/2}e^{r}(Q(r)^{1/1-\gamma})$, $r = |x| \to \infty$, under suitable conditions of Q.

1. Introduction. For semi-linear elliptic equations of the form

$$(1) \quad \Delta u - u + Q(|x|)u^\gamma = 0, \quad x \in R^n, \quad Q(|x|) > 0, \quad n \geq 3, \text{ and } \gamma > 1,$$

there are already some results about the existence of positive entire solutions [6, 7]. As to uniqueness (of positive solutions tending to zero), it is confirmed for some special cases, eq. $Q \equiv 1$ [8]. In this article we will discuss the asymptotic behavior of positive radial solutions of (1).

In [2, 4], we studied the asymptotic behavior of positive radial solutions of

$$(2) \quad \Delta u + p(|x|)u^\gamma = 0, \quad x \in R^n, \quad p(|x|) > 0, \quad n \geq 3 \text{ and } \gamma > 1.$$

When $\int_0^\infty rp(r)dr < \infty$, there are only two kinds of positive solutions of (2), one tends to positive constants and the other behaves like $O(r^{2-n})$ as $r \to \infty$ [2]. When $\int_0^\infty rp(r)dr = \infty$, it is shown [4, 5] that the positive

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radial solutions of (2) behaves either like $O(r^{2-n})$ or $(\int_0^r \tau p(\tau)d\tau)^{1/1-\gamma}$, under suitable condition on p.

We will show below that similar situation holds for equation (1). The positive radial solutions of (1) behave like

$$r^{-(n-1)/2}e^{-r} \text{ or } r^{-(n-1)/2}e^{r}(Q(r)^{1/1-\gamma}),$$

under suitable conditions of Q.

When $Q \equiv$ constant, the latter in parenthesis is a constant, as desired [9].

We prove the results by transforming the corresponding ordinary differential equations of (1) to that of (2) and apply the results of [2, 4, 5].

2. The ordinary differential equations. For the radial solutions $u(r)$, $r = |x|$, of (1), it satisfies

$$u''(r) + \frac{n-1}{r}u'(r) - u(r) + Q(r)u^{\gamma}(r) = 0. \tag{3}$$

Let $V(r) = r^{(n-1)/2}u(r)$, then

$$V''(r) - \left\{1 + \frac{(n-1)(n-3)}{4r^2}\right\}V(r) + Q(r)r^{-(n-1)(\gamma-1)/2}V^{\gamma}(r) = 0. \tag{4}$$

We write this as

$$V''(r) - \alpha(r)V(r) + \beta(r)V^{\gamma}(r) = 0, \tag{5}$$

that is $0 < \alpha(r) \equiv 1 + ((n-1)(n-3))/(4r^2) \to 1$, $r \to \infty$, and $\beta \equiv Q(r)r^{-((n-1)(\gamma-1))/2} > 0$. If we look this equation as been transformed from

$$w''(r) + p(r)w'(r) + q(r)w^{\gamma}(r) = 0, \qquad \text{via} \tag{6}$$

$$w = VP, \qquad P(r) = \exp\left(-\frac{1}{2}\int^r p(\tau)d\tau\right), \tag{7}$$

then (5) is identical with

$$V''(r) + \left(-\frac{1}{2}p'(r) - \frac{p^2(r)}{4}\right)V(r) + \left(q(r)P^{\gamma-1}(r)\right)V^{\gamma}(r) = 0. \tag{8}$$

That is

$$\alpha(r) \equiv \frac{1}{2}p'(r) + \frac{p^2(r)}{4} \qquad \beta(r) \equiv q(r)p^{\gamma-1}(r). \tag{9}$$

Hence the equivalence of (5) and (6) via (7) is reduced to the solvability of (9), for given α and β, both are positive.

We assert that we can solve (9) for positive p and q. Notice that $p(r) > 0$ whenever $p(0) > 0$ as follows from comparison theorem because $\alpha(r) > 0$. Also, $q(r) > 0$, certainly.

For this, we let $X(r) \equiv p(2r)$, then the problem is reduced to solve the Ricatti equation

$$X'(r) + X^2(r) = 4\alpha(2r) \equiv \alpha_1(r), \qquad X(r) > 0. \tag{10}$$

This can be shown to be the case via [1, p. 127].

We can also consider the well known mapping related to Ricatti differential equation

$$X(r) = \frac{y'(r)}{y(r)}.$$

(10) is then reduced to

$$\begin{aligned} &y''(r) - \alpha_1(r)y(r) = 0 \qquad \text{or} \\ &y_1''(r) - \alpha(r)y_1(r) = 0, \qquad y_1(r) = y\Big(\frac{r}{2}\Big). \end{aligned} \tag{11}$$

Since $\alpha(r) > 0$ there exists [8, Corollary 6.4] a non-principal solution $y_1(r) > 0$ with $y_1'(r) > 0$. Therefore $X(r) = y_1'(r)/y(r)$, $p(r) = X(r/2)$ is what we wanted. We also have $\lim_{r\to\infty}(y_1'(r)/y_1(r)) = 1$ [8. Theorem 9.1], and $X(r) \to 2$, $p(r) \to 2$, $r \to \infty$, because $\int^\infty |\alpha(r) - 1| dr < \infty$.

In summary, we have

Proposition 1. *There exist positive $p(r)$, $q(r)$ satisfying (9). For these p and q, (5) and (6) are equivalent.*

Write equation (6) as

$$w''(r) + \frac{K'(r)}{K(r)}w'(r) + \frac{\ell(r)}{K(r)}w^\gamma(r) = 0, \tag{12}$$

where $p(r) = K'(r)/K(r)$, $q(r) = \ell(r)/K(r)$, or we may take $K(r) = e^{\int^r p(\tau)d\tau} \sim e^{2r}$, $r \to \infty$, and $\ell(r) = K(r)q(r)$. Since $\int_0^\infty \frac{d\tau}{K(\tau)} < \infty$, we may let [10]

$$(13) \qquad s = \left(\int_r^\infty \frac{d\tau}{K(\tau)}\right)^{-1}, \quad U(s) = sw(r), \quad s \geq \left(\int_0^\infty \frac{d\tau}{K(\tau)}\right)^{-1} \equiv c,$$

and (12) is equivalent to

$$(14) \qquad U''(s) + \frac{K(r)\ell(r)}{s^{\gamma+3}}U^\gamma(s) = 0. \qquad (r \text{ and } s \text{ are related by (13)}).$$

Theorem 2. *Equations* (3) *and* (14) *are equivalent.*

Proof. By transformations (4), (7), (12) and (13).

3. Asymptotic behavior of positive solutions. From Theorem 2, the asymptotic behavior of positive solution u of (1) can be studied by the behavior of positive solution U of (14). For the asymptotic behavior of U, we have [2, 4, and correction of 4]

Theorem 3. *Let U be a positive solution of* (14).

(a) *If*

$$(15) \qquad \int^\infty \frac{K(r)\ell(r)}{s^3} ds < \infty,$$

then U is increasing and tends to a positive constant or U is of order s ($w(r)$ tends to a positive constant) as $s \to \infty$ ($r \to \infty$).

(b) *If*

$$(16) \qquad \int^\infty \frac{K(r)\ell(r)}{s^3} ds = \infty, \ \textit{and} \ \frac{sq'(s)}{q(s)} \leq m < -\frac{\gamma+3}{2} \ \textit{for } s \textit{ large},$$

where m is a constant and $q(s) = (K(r)\ell(r)/s^{\gamma+3})$. Then U tends to a positive constant or $w(r)$ is of order

$$(17) \qquad \left[\int^\infty \frac{K(r)\ell(r)}{s^3} ds\right]^{1/1-\gamma}, \quad s \to \infty, \quad (r \to \infty).$$

Applying Theorem 2 and 3, the behavior of u can be determined as

Theorem 4. *Let u be a positive solution of* (3).

(a) *If*

$$(18)\qquad \int^{\infty} s^{(\gamma-3)/2} Q(r) r^{-(n-1)(\gamma-1)/2} ds < \infty, \qquad (r,\, s \text{ related by } (13)),$$

then u is either of order $r^{-(n-1)/2}e^{-r}$ or $r^{-(n-1)/2}e^{r}$, $r \to \infty$.

(b) *When*

$$(19)\qquad \int^{\infty} s^{(\gamma-3)/2} Q(r) r^{-(n-1)(\gamma-1)/2} ds = \infty, \quad \textit{and}$$

$$\frac{sq'(s)}{q(s)} \le m < -\frac{\gamma+3}{2} \quad \textit{for } s \textit{ large},$$

then u is either of order $r^{-(n-1)/2}e^{-r}$ or $O(Q(r)^{1/1-\gamma})$, $r \to \infty$.

Proof. Case (a).

$$\begin{aligned}\int^{\infty} \frac{K(r)\ell(r)}{s^3} ds &= \int^{\infty} \frac{K^2(r)Q(r)r^{-(n-1)(\gamma-1)/2}P^{1-\gamma}(r)}{s^3} ds\\ &\sim \int^{\infty} \frac{Q(r)r^{-(n-1)(\gamma-1)/2}}{s} \cdot s^{(\gamma-1)/2} ds\\ &= \int^{\infty} s^{(\gamma-3)/2} Q(r) r^{-(n-1)(\gamma-1)/2} ds.\end{aligned}$$

Hence, by Theorem 3, (a), U is of order s or constant. Therefore u is of order $r^{-(n-1)/2}e^{-r}$ or $r^{-(n-1)/2}e^{r}$, $r \to \infty$.

Case (b). (19) implies (16) and

$U(s)$ is of order constant or of order

$$s\Big[\int^{s} \frac{K(r)\ell(r)}{s^3}\Big]^{1/1-\gamma}.$$

In the first case, u is of order $r^{-(n-1)/2}e^{-r}$. For the second case,

$$w(r) \sim \Big[\int^{s} \frac{K(r)\ell(r)}{s^3} ds\Big]^{1/1-\gamma}, \qquad s \to \infty.$$

Or, $[u(r)r^{(n-1)/2}e^{-r}]^{\gamma-1}$ is of order $\Big[\int^{s} \frac{K(r)\ell(r)}{s^3} ds\Big]^{-1}$. That is $u^{\gamma-1}$ is of order

$$\frac{[r^{-(n-1)/2}e^r]^{\gamma-1}}{\int^\infty Q(r)r^{-(n-1)(\gamma-1)/2}e^{r(\gamma-1)}dr} \sim O\Big(\frac{1}{Q(r)}\Big), \quad r \to \infty, \tag{20}$$

by L'Hopital Rule. Therefore $u(r) \sim O(Q^{1/1-\gamma})$, $r \to \infty$. The proof is completed.

Example. When Q = constant, then u is of order $r^{-(n-1)/2}e^{-r}$ or constant, as desired [9].

Remark. Condition (18) is equivalent to

$$\int^\infty Q(r)\big(r^{-(n-1)/2}e^r\big)^{\gamma-1}dr < \infty. \tag{21}$$

4. Existence of $O(r^{-(n-1)/2}e^{-r})$ solutions. Using the result of [3] that there exists a positive bounded solution U of (14) when

$$\int_c^\infty \frac{K(r)\ell(r)}{s^{(\gamma+1)/2+2}}ds < \infty, \tag{22}$$

we have

Theorem 5. *If*

$$\int_0^\infty Q(r)r^{-(n-1)(\gamma-1)/2}dr < \infty \tag{23}$$

then (1) *has a positive solution u of order $r^{-(n-1)/2}e^{-r}$, $r \to \infty$.*

Proof. (23) is equivalent to (22).

Remarks. 1. Theorem 5 was proved in [6] by different method.

2. It is clear from the equivalence of (3) and (4) that the unique existence of a special kind of positive solution for one equation can be translated to that of the other. However, only quite a few uniqueness results were known.

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