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POINT BIFURCATIONS FOR SOME ONE-PARAMETER FAMILIES OF INTERVAL MAPS

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Abstract. For some one-parameter families of simple continuous interval maps f_c with c as the parameter, we show that f_c has a point bifurcation of period 6 orbit at some point $c = c_0$. That is, f_{c_0} has a periodic point p of least period 6, but there is a neighborhood $V(p)$ of p such that f_c has no periodic point of least period 6 in $V(p)$ for every c sufficiently close to and distinct from c_0.

1. Introduction. Let f_c be a one-parameter family of continuous maps from the compact interval I into itself. Assume that there exist two positive numbers δ and ε such that f_c has a periodic point p of some period n for $c = c_0$, but no periodic point (of same period) in $(p-\varepsilon, p+\varepsilon)$ for every c in $(c_0 - \delta, c_0) \cup (c_0, c_0 + \delta)$. Then we say that f_c has a point bifurcation of period n points (or orbits) at $c = c_0$. Trivial examples of point bifurcations of periodic orbits can be easily constructed. However, nontrivial examples seem lacking. As point bifurcations are very difficult to detect from the practical point of view, we may not notice them even when we encounter one such example. Therefore, it would be nice to have an explicit nontrivial example. In this note, we give some such examples.

For real numbers $1/2 > b \geq 0$ and $1 \geq c \geq 0$, let $F_{b,c}(x)$ denote the continuous map from [0,1] into itself defined by

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$$F_{b,c}(x) = \begin{cases} \frac{3}{4}, & 0 \le x \le b, \\ \frac{x}{(2-4b)} + \frac{(3-8b)}{(4-8b)}, & b \le x \le \frac{1}{2}, \\ 1 + (c-1)(2x-1), & \frac{1}{2} \le x \le 1. \end{cases}$$

Also, let $S(b)$ denote the set $\{0 < c < 1 | F_{b,c}(x)$ has periodic orbits of least period 6$\}$ and let $c_1(b) = \inf S(b)$ and $c_2(b) = \sup S(b)$. Then for suitable choices of b, the set $(c_1(b), c_2(b))$ (which may not be contained in the set $S(b)$) may not be empty. For example, it follows from [1, Theorem 3] that $c_2(b) = 1/2$ for $3/8 \le b < 1/2$. On the other hand, if $a \approx .228155$ is the unique positive zero of the polynomial $4x^3 - 8x^2 + 6x - 1$, then, for $a < b < 1/2$, we can easily show that $F^2_{b,b}(1/2) < 1/2 < F^6_{b,b}(1/2)$. It then follows from [2, Proposition 2.2] and [3] that $b \in S(b)$ for all $a < b < 1/2$. Consequently, $(c_1(b), c_2(b)) \supset [b, 1/2)$ for $3/8 \le b < 1/2$. By numerical computations, we find that (i) when $0 < b < a$, $S(b) = \phi$ and (ii) as b with $a < b < 1/2$ tends to a, the interval $(c_1(b), c_2(b))$ tends to the set $\{a\}$ consisting of the unique element a. That is, numerical results seem to indicate that $F_{a,c}(x)$ gives an example of point bifurcation of period 6 orbits at $c = a$. In this note, we prove this observation and some generalizations.

2. Statement of main results.

In this section, we state our main results. These results show the existence of point bifurcations for some one-parameter families of simple interval maps.

Theorem 1. *Let a denote the unique positive zero of the polynomial $4x^3 - 8x^2 + 6x - 1$ and let s be a fixed nonnegative number. For $0 \le c \le 1$, let $f_c(x)$ be the continuous map from $[0,1]$ into itself defined by*

$$f_c(x) = \begin{cases} \max\{\frac{1}{2}, sx - sa + \frac{3}{4}\}, & 0 \le x \le a, \\ \frac{x}{(2-4a)} + \frac{(3-8a)}{(4-8a)}, & a \le x \le \frac{1}{2}, \\ 1 + (c-1)(2x-1), & \frac{1}{2} \le x \le 1. \end{cases}$$

Then the following hold:

(A) *Assume that $0 \le s < (6a^2 - 8a + 3)/[8(1-a)^3]$. Then the following hold:*

(1) *If $a < c \le .25$, then f_c has no period 6 point in $[0, .25]$.*

(2) *If* $c = a$, *then* $\{1/2, 1, a, 3/4, (1+a)/2, a^2 - a + 1\}$ *is a period* 6 *orbit of* f_c.

(3) *There exist a number* α *(depending on* s*) with* $a > \alpha > 0$ *and a number* δ *(independent of* s*)* > 0 *such that, for every* $a - \alpha < c < a$, f_c *has no period* 6 *point in* $[0, a + \delta]$.

(B) *Assume that* $(6a^2 - 8a + 3)/[8(1-a)^3] \leq s$. *Then the following hold:*

(1) *If* $a < c \leq .25$, *then* f_c *has no period* 6 *point in* $[0, .25]$.

(2) *If* $c = a$, *then* $\{1/2, 1, a, 3/4, (1+a)/2, a^2 - a + 1\}$ *is a period* 6 *orbit of* f_c.

(3) *There exists a number* β *(depending on* s*) with* $a > \beta > 0$ *such that, for every* $a - \beta < c < a$, f_c *has at least one period* 6 *point in* (c, a).

Remarks. (1) Part (A) of Theorem 1 shows that, when $0 \leq s < (6a^2 - 8a + 3)/[8(1-a)^3]$, f_c has a point bifurcation of period 6 orbit at $c = a$.

(2) Part (B) of Theorem 1 shows that, when $(6a^2 - 8a + 3)/[8(1-a)^3] \leq s$, f_c has a bifurcation of period 6 orbit at $c = a$.

Theorem 2. *Let* a *denote the unique positive zero of the polynomial* $4x^3 - 8x^2 + 6x - 1$ *and let* z *be a fixed number with* $0 < z < a$. *Let* $g(x)$ *be any fixed continuous map from* $[0, a]$ *into* $[0, 1]$ *such that* $g(a) = 3/4$ *and* $g(x) \geq 3/4$ *on* $[z, a]$. *For* $0 \leq c \leq 1$, *let* $g_c(x)$ *be the continuous map from* $[0, 1]$ *into itself defined by*

$$g_c(x) = \begin{cases} g(x), & 0 \leq x \leq a, \\ \frac{x}{(2-4a)} + \frac{(3-8a)}{(4-8a)}, & a \leq x \leq \frac{1}{2}, \\ 1 + (c-1)(2x-1), & \frac{1}{2} \leq x \leq 1. \end{cases}$$

Then the following hold:

(1) *For* $a < c \leq .25$, g_c *has no period* 6 *point in* $[0, .25]$.

(2) *For* $c = a$, $\{1/2, 1, a, 3/4, (1+a)/2, a^2 - a + 1\}$ *is a period* 6 *orbit of* g_c.

(3) *There exists a number* δ *(depending on* g*) with* $a - z > \delta > 0$ *such*

that, for every $.2 \le c < a$, g_c *has no period* 6 *point in* $[a-\delta, a+\delta]$. *Consequently,* g_c *has a point bifurcation of period* 6 *orbit at* $c = a$.

Let $m > 6$ be a fixed even integer. Let $b, c, F_{b,c}(x)$ be defined as in Section 1. When $b = 3/8$, it is known [1] that there exists a number $0 < d < 1/2$ such that $F_{3/8,c}(x)$ has a periodic orbit of least period 6 for every c in $(d, 1/2)$. By Sharkovskii's theorem on the coexistence of periodic orbits [3], $F_{3/8,c}(x)$ has periodic orbits of least period m for every c in $(d, 1/2)$. When $b = 0$, it is clear that $F_{0,c}(x)$ can only have periodic orbits of periods ≤ 4 for every $0 \le c < 1/2$. Since $F_{b,c}$ is continuous in both parameters b and c, it is possible that there is a number $0 < a_m < 3/8$ such that $F_{a_m,c}(x)$ has a point bifurcation of least period m. Based on Theorem 1 above, a good candidate for such numbers will be numbers a_m such that the point $x = 1/2$ is a periodic point of $F_{a_m,a_m}(x)$ with least period m. So, we make the following

Conjecture. Let $m > 6$ be a fixed even integer. If a_m is a number such that the point $x = 1/2$ is a periodic point of $F_{a_m,a_m}(x)$ with least period m, then $F_{a_m,c}(x)$ has a point bifurcation of least period m at $c = a_m$.

3. Preliminary results. For the proofs of the main results, we need 3 lemmas. Note that the polynomial $4x^3-8x^2+6x-1 = 2(2x^3-4x^2+3x-1/2)$ is a strictly increasing map. Throughout this section, let $a \approx .228155$ denote the unique positive zero of the polynomial $4x^3-8x^2+6x-1$. Consequently, $2c^3-4c^2+3c < 1/2$ for $c < a$, $2c^3-4c^2+3c = 1/2$ for $c = a$, and $2c^3-4c^2+3c > 1/2$ for $c > a$. These facts will be used implicitly throughout this section.

Lemma 1. *For any fixed* $a < c \le .25$, *let* $h_c(x)$ *be the continuous map from* $[a, 1]$ *into itself defined by*

$$h_c(x) = \begin{cases} \frac{x}{(2-4a)} + \frac{(3-8a)}{(4-8a)}, & a \le x \le \frac{1}{2}, \\ 1 + (c-1)(2x-1), & \frac{1}{2} \le x \le 1. \end{cases}$$

Then $h_c^6(x) > x$ *for all* $a \le x \le .25$ *and all* $a < c \le .25$.

Proof. The idea of the proof is as follows: For any fixed $a < c \leq .25$, we show that, both $h_c(a)$ and $h_c(.25)$, both $h_c^2(a)$ and $h_c^2(.25)$, and both $h_c^3(a)$ and $h_c^3(.25)$ all lie on the same side of the point $1/2$. It then follows from the monotonicity of h_c^k, $1 \leq k \leq 3$, on $[a, .25]$ that the graphs of h_c, h_c^2 and h_c^3 are all straight lines on $[a, .25]$. Then we show that there is a unique point b (depending on c) in $(a, .25)$ such that $h_c^4(b) = 1/2$. Therefore, h_c^6 are straight lines on $[a, b]$ and on $[b, .25]$ with negative slope on $[a, b]$ and slope > 1 on $[b, .25]$ (see Fig.1). This suffices to obtain the desired result. We now proceed to the proof.

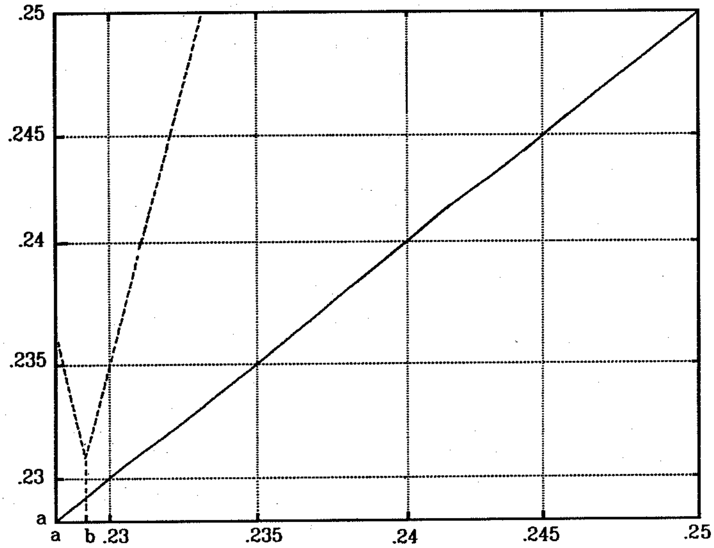

Figure 1. Typical graph of $y = f_c^6(x)$ on $[a, .25]$ for $a < c \leq .25$.

Fix any $a < c \leq .25$. Then $h_c(a) = 3/4$, $h_c^2(a) = (c+1)/2$, $h_c^3(a) = (c - 1/2)^2 + 3/4 > 1/2$, $h_c^4(a) = 2c^3 - 4c^2 + 3c > 1/2$. On the other hand, it is easy to check that, on $[a, .25]$, the following hold: (i) h_c is increasing and $> 1/2$, (ii) h_c^2 is decreasing and $> 1/2$, (iii) h_c^3 is increasing and $> 1/2$. Consequently, h_c^4 is linear and decreasing on $[a, .25]$. Let $b = b(c)$ denote the (unique) point (if it exists) in $[a, .25]$ such that $h_c^4(b) = 1/2$. Then since $2a^3 - 4a^2 + 3a = 1/2$, we see that $h_c^4(a) - 1/2 = 2c^3 - 4c^2 + 3c - (2a^3 - 4a^2 + 3a) = (c - a)[2(c^2 + ca + a^2) - 4(c + a) + 3] \leq (c - a)[6(.25)^2 - 8(.2) + 3] =$

$1.775(c-a)$. Since the graph of $h_c^4(x)$ on $[a,b]$ is a straight line with slope $-t$, where $t = 8(1-c)^3/(2-4a) \geq 8(.75)^3/(2-4a) > 1.775$, we easily obtain that $b-a < c-a$. That is, $a < b < c$. So, there exists a (unique) point $b = b(c)$ in $[a, .25]$ such that $h_c^4(b) = 1/2$. Consequently, $h_c^6(x)$ is linear on $[a,b]$ and on $[b, .25]$.

Now $h_c^6(x)$ is decreasing on $[a,b]$ and increasing on $[b, .25]$ with $h_c^6(b) = c > b$. Furthermore, $h_c^6(x)$ is a straight line on $[b, .25]$ with slope $32(1-c)^5/(2-4a) > 1$. Therefore, $h_c^6(x) > x$ on $[a, .25]$.

This completes the proof of Lemma 1.

Lemma 2. *Let $h(x)$ be any fixed continuous map from $[0,a]$ into $[0,1]$ with $h(a) = 3/4$. For $.2 \leq c \leq a$, let $h_c(x)$ be the continuous map from $[0,1]$ into itself defined by*

$$h_c(x) = \begin{cases} h(x), & 0 \leq x \leq a, \\ \frac{x}{(2-4a)} + \frac{(3-8a)}{(4-8a)}, & a \leq x \leq \frac{1}{2}, \\ 1 + (c-1)(2x-1), & \frac{1}{2} \leq x \leq 1. \end{cases}$$

Then there exists a positive number δ (independent of $h(x)$) such that $h_c^6(x) > x$ for all $.2 \leq c < a$ and all $a \leq x \leq a + \delta$.

Proof. The idea of the proof is as follows: For any fixed $.2 \leq c < a$, there is a positive number δ (independent of c and $h(x)$) such that $h_c^6(x)$ is linear on $[a, a+\delta]$. Then we show that $h_c^6(a) > a$. Since the slope of the graph of $h_c^6(x)$ on $[a, a+\delta]$ is > 1 with $h_c^6(a) > a$ (see Fig. 2), we easily obtain the desired result. We now proceed to the proof.

It is clear that, for $.2 \leq c < a$, we have $h_c(a) = 3/4$, $h_c^2(a) = (c+1)/2 > 1/2$, $.85 > h_c^3(a) = c^2 - c + 1 > 1/2$, $a < .3 < h_c^4(a) = 2c^3 - 4c^2 + 3c < 1/2$, $h_c^5(a) = (2c^3 - 4c^2 + 3c)/(2-4a) + (3-8a)/(4-8a) > 1/2$, and $h_c^6(a) = (4c^4 - 12c^3 + 14c^2 - 7c + 1)/(2-4a) + c$. Note that the values $h_c^k(a)$, $1 \leq k \leq 5$, are all $> a$. Therefore, there exists a positive number δ (independent of c and $h(x)$) such that, for each $.2 \leq c < a$ and on $[a, a+\delta]$, the following hold: (i) h_c is increasing and $> 1/2$; (ii) h_c^2 is decreasing and $> 1/2$; (iii) h_c^3 is increasing and $> 1/2$; (iv) h_c^4 is decreasing and $< 1/2$; and (v) h_c^5 is decreasing and $> 1/2$. Consequently, for each $.2 \leq c < a$, $h_c^6(x)$ is linear on $[a, a+\delta]$.

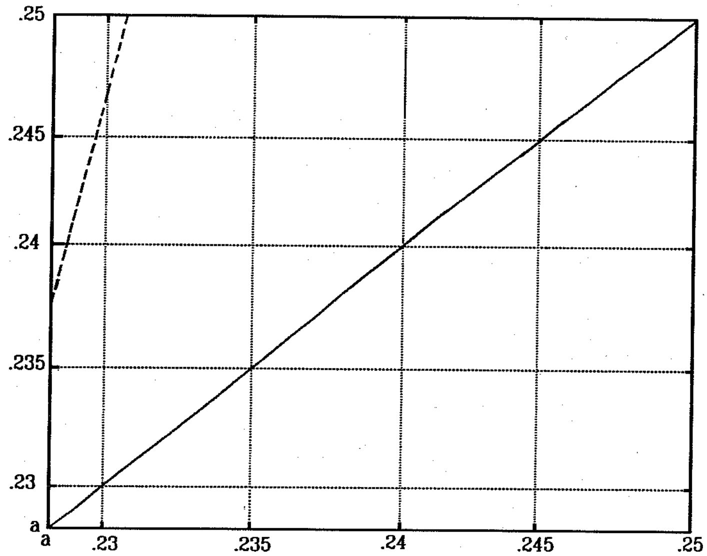

Figure 2. Typical graph of $y = f_c^6(x)$ on $[a, .25]$ for $.2 \leq c < a$.

For $.2 \leq c \leq a$, let $p(c) = h_c^6(a) = (4c^4 - 12c^3 + 14c^2 - 7c + 1)/(2-4a) + c$. Then $p'(c) = (16c^3 - 36c^2 + 28c - 7)/(2-4a) + 1 < -36c^2/(2-4a) + 1 < 0$ for $.2 \leq c \leq a$. So, $p(c)$ is strictly decreasing on $[.2, a]$. In particular, $p(c) > p(a)$ for $.2 \leq c < a$. Consequently, $h_c^6(a) = p(c) > p(a) = h_a^6(a) = a$ for all $.2 \leq c < a$.

On the other hand, for $.2 \leq c < a$, $h_c^6(x)$ is linear on $[a, a+\delta]$ with slope $4(1-c)^4/(1-2a)^2 > 1$. Since $h_c^6(a) > a$, we obtain that $h_c^6(x) > x$ on $[a, a+\delta]$ for all $.2 \leq c < a$.

This completes the proof of Lemma 2.

The following lemma whose proof is rather straightforward will be used in the proof of Theorem 1. For the sake of easy reference, we consider c as the variable.

Lemma 3. *Let s be a fixed nonnegative number. For $0 \leq c \leq a$, let $h(c) = c + 2(c-1)^2 + 2(c-1)^3 + 8s(c-a)(c-1)^3$. Then the following hold:*

(1) *If* $0 \le s < (6a^2-8a+3)/[8(1-a)^3]$, *then there is a number* α *(depending on* s*)* > 0 *such that* $h(c) < 1/2$ *for all* $a - \alpha < c < a$.

(2) *If* $(6a^2 - 8a + 3)/[8(1 - a)^3] \le s$, *then there is a number* β *(depending on* s*)* > 0 *such that* $h(c) > 1/2$ *for all* $a - \beta < c < a$.

Proof. It is clear that $h(a) = 1/2$, $h'(c) = 6c^2 - 8c + 3 + 8s(c-1)^2(4c - 3a - 1)$ and $h''(c) = 12c - 8 - 16s(c-1)(1 + 3a - 4c) + 32s(c-1)^2$.

If $0 \le s < (6a^2 - 8a + 3)/[8(1-a)^3]$, then $h'(a) = 6a^2 - 8a + 3 - 8s(1-a)^3 = 8(1-a)^3\{(6a^2 - 8a + 3)/[8(1-a)^3] - s\} > 0$. So, there is a number α (depending on s) > 0 such that $h'(c) > 0$ for all $a - \alpha < c \le a$. Thus, $h(c)$ is strictly increasing on $(a - \alpha, a]$. Consequently, $h(c) < h(a) = 1/2$ for all $a - \alpha < c < a$. This proves (1).

If $s = (6a^2 - 8a + 3)/[8(1-a)^3]$, then $h'(a) = 0$ and $h''(a) = 12a - 8 + 48s(1-a)^2 > 0$. So, there is a number β (depending on s) > 0 such that $h(c) > h(a) = 1/2$ for all $a - \beta < c < a$.

On the other hand, if $(6a^2 - 8a + 3)/[8(1-a)^3] < s$, then $h'(a) = 6a^2 - 8a + 3 - 8s(1-a)^3 = 8(1-a)^3\{(6a^2 - 8a + 3)/[8(1-a)^3] - s\} < 0$. So, there is a number β (depending on s) > 0 such that $h'(c) < 0$ for all $a - \beta < c \le a$. Thus, $h(c)$ is strictly decreasing on $(a - \beta, a]$. Consequently, $h(c) > h(a) = 1/2$ for all $a - \beta < c < a$. This proves (2).

The proof of Lemma 3 is now complete.

4. Proofs of main results.

We can now prove our main results.

Proof of Theorem 1. The idea of the proof is as follows: Depending on the values of s, we split the proof into two cases.

If $0 \le s < (6a^2 - 8a + 3)/[8(1-a)^3]$, the proof is more complicated. First, we apply part (1) of Lemma 3 to show that there exists a positive number α (depending on s) such that $f_c^4(c) < 1/2$ for each $a - \alpha < c < a$. Then we split the proof into two cases depending on whether the graph of f_c^4 on $[0, a]$ intersects with the diagonal line $y = x$. In any case, we show that $f_c^6(x) > x$ on $[0, a]$ for every $a - \alpha < c < a$. This, combined with Lemma 2, implies the desired result for these values of s.

If $(6a^2 - 8a + 3)/[8(1-a)^3] \le s$, then we apply part (2) of Lemma 3 to show that there exists a positive number β (depending on s) such that, for all $a - \beta < c < a$, $f_c^4(z) = 1/2$ for some $z \in (c, a)$. From this, we easily obtain that $f_c^6(y) = y$ for some $y \in (z, a)$ which is the desired result for these values of s. We now proceed to the proof of the theorem.

If $a < c \le .25$, then f_c maps $[a, 1]$ into itself. So, in this case, f_c is independent of s. By Lemma 1, $f_c^6(x) > x$ on $[a, .25]$. On $[0, a]$, $f_c^6(x) \ge \min f_c(x) = c > x$. So, $f_c^6(x) > x$ on $[0, .25]$ for every $a < c \le .25$. This proves (1) of part (A) and (1) of part (B).

The proofs of (2) of part (A) and (2) of part (B) are straightforward and omitted.

For the rest of this proof, we fix any $.2 \le c < a$. Then, by Lemma 2, we obtain that $f_c^6(a) > a$. Also note that the polynomial $2c^3 - 4c^2 + 3c = f_c^4(a)$ is a strictly increasing map of the variable c. So, $f_c^4(a) < f_a^4(a) = 2a^3 - 4a^2 + 3a = 1/2$ for $.2 \le c < a$.

If $0 \le s < (6a^2 - 8a + 3)/[8(1-a)^3] \approx .404256 < .41$, then it is easy to check that, on $[0, a]$, the following hold: (i) f_c is linear, increasing, and $> 1/2$; (ii) f_c^2 is linear, decreasing, and $> 1/2$; (iii) f_c^3 is linear, increasing, and $> 1/2$; and (iv) $f_c^4(x) = c + 2(c-1)^2 + 2(c-1)^3 + 8s(x-a)(c-1)^3 = 2c^3 - 4c^2 + 3c + 8s(x-a)(c-1)^3$ is linear and decreasing. Note that, since the map $f_c^4(x)$ is decreasing on $[0, a]$, there exists at most on point $b = b(c, s)$ in $[0, a]$ such that $f_c^4(b) = 1/2$. Since $0 \le s < (6a^2 - 8a + 3)/[8(1-a)^3]$, it follows from part (1) of Lemma 3 that there is a number α (depending on s) > 0 such that $f_c^4(c) < 1/2$ for all $a - \alpha < c < a$. We now have two cases to consider:

Case 1. If the set $\{0 \le x \le a | f_c^4(x) = 1/2\}$ is not empty, let $b = b(c, s)$ denote the unique point in the set $\{0 \le x \le a | f_c^4(x) = 1/2\}$. Then, since f_c^4 is decreasing on $[0, a]$ and both $f_c^4(c)$ and $f_c^4(a)$ are less than 1/2 , we have $b < c$. Consequently, $f_c^6(x)$ is linear on $[b, a]$ with $f_c^6(b) = c > b$ and $f_c^6(a) > a$. Thus, $f_c^6(x) > x$ on $[b, a]$ (see Fig. 3). On the other hand, $f_c^6(x) \ge \min f_c(x) = c > x$ on $[0, b]$. So, $f_c^6(x) > x$ on $[0, a]$ for every

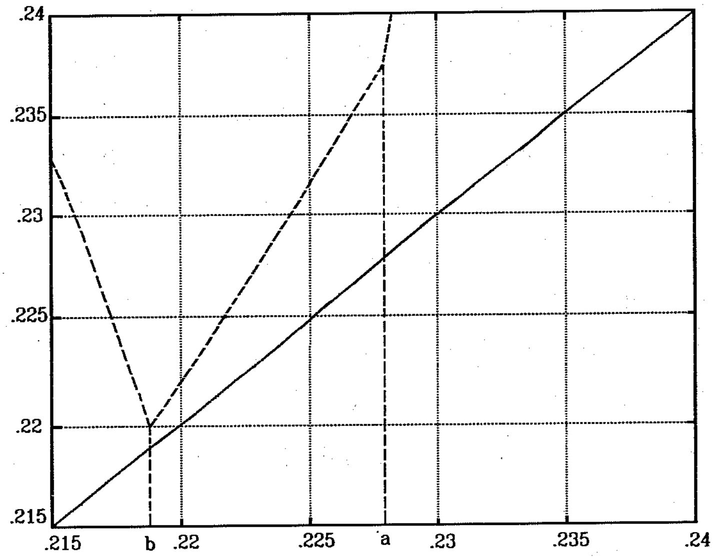

Figure 3. Typical Graph of $y = f_c^6(x)$ on $[0, a]$ for which Case 1 holds.

$a - \alpha < c < a$.

Case 2. If the set $\{0 \le x \le a | f_c^4(x) = 1/2\}$ is empty, then, since $f_c^4(c) < 1/2$, we obtain that $f_c^4(x) < 1/2$ on $[0, a]$. Since f_c^4 is linear and decreasing on $[0, a]$, we see that $f_c^4(x) \ge f_c^4(a) = 2c^3 - 4c^2 + 3c > 2(.2)^3 - 4(.23)^2 + 3(.2) = .4044 > a$. Consequently, f_c^5 is linear, decreasing, and $> 1/2$ on $[0, a]$. Therefore, f_c^6 is linear and increasing on $[0, a]$ with slope $= 8s^2(c-1)^4 < .55 < 1$. Since $f_c^6(a) > a$, this implies that $f_c^6(x) > x$ on $[0, a]$ for every $a - \alpha < c < a$ (see Fig. 4).

By Lemma 2, there exists a positive number δ such that, for every $a - \alpha < c < a$, $f_c^6(x) > x$ on $[a, a + \delta]$. This, combined with Cases 1 and 2 above, shows that, for every $a - \alpha < c < a$, $f_c^6(x) > x$ on $[0, a + \delta]$. This proves (3) of Part (A).

If $(6a^2 - 8a + 3)/[8(1-a)^3] \le s$, then by part (2) of Lemma 3, there exists a number β (depending on s)> 0 such that $f_c^4(c) > 1/2$ for all $a - \beta < c < a$. Since $f_c^4(a) < 1/2$, there exists $c < z < a$ such that $f_c^4(z) = 1/2$ and hence $f_c^6(z) = c < z$. Since $f_c^6(a) > a$, we obtain that there is a point y in $(z, a) \subset (c, a)$ such that $f_c^6(y) = y$. This proves (3) of part (B).

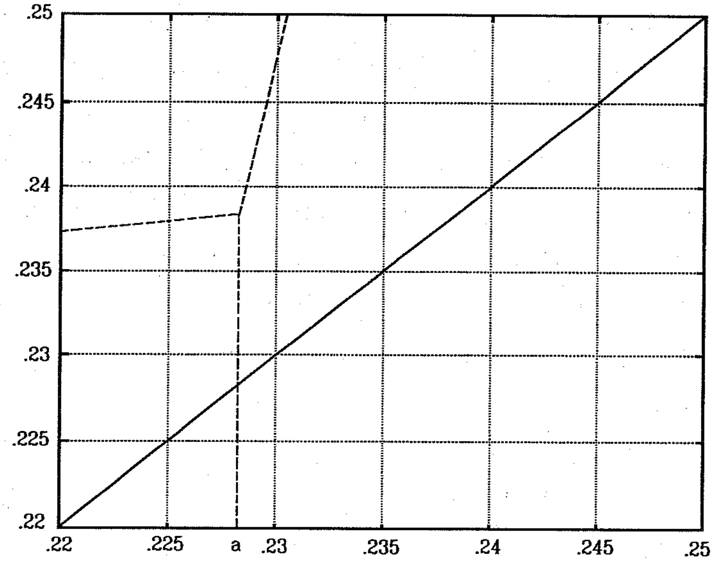

Figure 4. Typical Graph of $y = f_c^6(x)$ on $[0, a]$ for which Case 2 holds.

The Proof of Theorem 1 is now complete.

Proof of Theorem 2. The proof of Theorem 2 is rather straightforward. Part (1) follows from Lemma 1. Part (2) is trivial. We now prove part (3). By Lemma 2, there exists a positive number ε such that $g_c^6(x) > x$ for all $.2 \le c < a$ and all $a \le x \le a + \varepsilon$. On the other hand, fix any number δ with $a - z > \delta > 0$ and $\varepsilon > \delta$ such that $g_c([a - \delta, a]) \subset g_c([a, a + \varepsilon])$. Then it is clear that $g_c^6([a-\delta, a]) = g_c^5(g_c([a-\delta, a])) \subset g_c^5(g_c([a, a+\varepsilon])) = g_c^6([a, a+\varepsilon])$. So, for all $a - \delta \le x < a$, $g_c^6(x) = g_c^6(y)$ for some y in $[a, a + \varepsilon]$. But then, it follows from the above that $g_c^6(x) = g_c^6(y) > y \ge a > x$. This, together with the above, shows that $g_c^6(x) > x$ for all $a - \delta \le x \le a + \varepsilon$. Since $\delta < \varepsilon$, it follows that $g_c^6(x) > x$ for all $.2 \le c < a$ and all $a - \delta \le x \le a + \delta$. This completes the proof of part (3) and hence the proof of Theorem 2.

5. Concluding remarks. In this section, we let c and s be two fixed numbers with $0 \le c < a$ and $0 \le s < (6a^2 - 8a + 3)/[8(1 - a)^3]$ and let $f_c(x)$ be defined as in Theorem 1. Then the following hold: (i) $f_c([0, a])$ is contained in $[.65, .75]$ and f_c is linear and increasing on $[0, a]$; (ii) $f_c^2([0, a])$ is contained in $[(c + 1)/2, .3c + .7] \subset [1/2, 1]$ and hence f_c^2 is linear and decreasing on $[0,a]$; and (iii) $f_c^3([0,a])$ is contained in $[.6c^2-.2c+.6, c^2-c+1] \subset [1/2,1]$ and hence f_c^3 is linear and increasing on $[0,a]$. Consequently, $f_c^4(x)=2c^3-4c^2+3c+8s(x-a)(c-1)^3$ is linear and decreasing on $[0,a]$. So, the equation $f_c^4(x)=1/2$ has at most one solution in $[0,a]$. Note that $f_c^4(a)=2c^3-4c^2+3c$ is a strictly increasing map of the parameter c. Since $f_c^4(a) < f_a^4(a) = 2a^3-4a^2+3a = 1/2$, we see that if $f_c^4(c) > 1/2$ then there must exist a point $y \in (c,a)$ such that $f_c^4(y) = 1/2$. Consequently, $f_c^6(y) = c < y$. Since $0 \le c < a$, we have $f_c^4(a) = 2c^3-4c^2+3c < f_a^4(a) = 1/2$. If $a < f_c^4(a) < 1/2$, then as in Lemma 2, $f_c^6(a) > a$. If $0 \le f_c^4(a) \le a$, then $f_c^5(a) = s(2c^3-4c^2+3c)-sa+3/4 > 1/2$, and so, $f_c^6(a) = 1+(c-1)[2s(2c^3-4c^2+3c)-2sa+1/2] = (c+1)/2+2s(2c^3-4c^2+3c-a) > 1/2 > a$. Therefore, for $0 \le c < a$, we have $f_c^6(a) > a$. Consequently, there exists a point $z \in (y,a)$ such that $f_c^6(z) = z$. This point z must be a periodic point of f_c with least period 6. Therefore, the condition that $f_c^4(c) > 1/2$ is a sufficient condition for f_c to have a periodic point of least period 6 in (c,a). In the following, we explore further the equation $f_c^4(c) = 2c^3-4c^2+3c+8s(c-a)(c-1)^3$.

Since $2a^3-4a^2+3a = 1/2$, we obtain that $f_c^4(c)-1/2 = (c-a)[2(c^2+ca+a^2)-4(c+a)+3-8s(1-c)^3]$. Write $c=a-\varepsilon$ with $a \ge \varepsilon > 0$. Then

$$
\begin{aligned}
&f_c^4(c)-1/2\\
(*)\quad &= -\varepsilon[6a^2-8a+3+(4-6a+2\varepsilon)\varepsilon-8s(1-a+\varepsilon)^3]\\
&= -8(1-a+\varepsilon)^3\varepsilon\left[\frac{6a^2-8a+3}{8(1-a+\varepsilon)^3}+\frac{4-6a+2\varepsilon}{8(1-a+\varepsilon)^3}\varepsilon-s\right]\\
&= -8(1-a+\varepsilon)^3\varepsilon\left[\frac{6a^2-8a+3}{8(1-a)^3}\left(\frac{1-a}{1-a+\varepsilon}\right)^3\right.\\
&\quad\left.+\frac{4-6a}{8(1-a)^3}\left(\frac{1-a}{1-a+\varepsilon}\right)^3+\frac{1}{4(1-a+\varepsilon)^3}\varepsilon^2-s\right]\\
&= -8(1-a+\varepsilon)^3\varepsilon\left\{\frac{6a^2-8a+3}{8(1-a)^3}\left[1-\frac{3}{1-a}\varepsilon+\frac{6}{(1-a)^2}\varepsilon^2\right.\right.\\
&\quad\left.-\frac{10}{(1-a)^3}\varepsilon^3+\cdots\right]+\frac{4-6a}{8(1-a)^3}\left[1-\frac{3}{1-a}\varepsilon+\frac{6}{(1-a)^2}\varepsilon^2\right.
\end{aligned}
$$

$$- \frac{10}{(1-a)^3}\varepsilon^3 + \cdots \Big] + \frac{1}{4(1-a+\varepsilon)^3}\varepsilon^2 - s \Big\}$$

$$= -8(1-a+\varepsilon)^3\varepsilon\Big\{\Big[\frac{6a^2-8a+3}{8(1-a)^3} - s\Big] - \Big[\frac{18a^2-42a+21}{8(1-a)^4}\Big]\varepsilon$$

$$+ \Big[\frac{19a^2-44a+22}{4(1-a)^5}\Big]\varepsilon^2 - \Big[\frac{33a^2-76a+38}{4(1-a)^6}\Big]\varepsilon^3 + \cdots\Big\}$$

$$(**) \quad = -8(1-a+\varepsilon)^3\varepsilon\{[A_0 - s] - A_1\varepsilon + A_2\varepsilon^2 - A_3\varepsilon^3 + \cdots\}$$

where A_i, $0 \le i \le 3$, are defined in obvious way.

$$(***) \quad \approx -8(1-a+\varepsilon)^3\varepsilon[(.404256 - s) - 4.3512\varepsilon + 11.8186\varepsilon^2 - 26.4595\varepsilon^3 + \cdots].$$

Now since we have assumed that $0 \le s < (6a^2 - 8a + 3)/[8(1-a)^3]$, we easily see from above that $f_c^4(c) - 1/2 < 0$ for all $c < a$ and c sufficiently close to a. This is exactly what we have shown in the proof of Theorem 1. We now split the discussions into cases depending on the values of s.

When the fixed number s is strictly less than but sufficiently close to the value $A_0 = (6a^2 - 8a + 3)/[8(1-a)^3]$, we have already known that $f_c^4(c) < 1/2$ for all $c < a$ and c sufficiently close to a. But when c is not so close to a, that is, when $c = a - \varepsilon$ and ε is not small enough, then the second term 4.3512ε in $(***)$ above dominates. So, $f_c^4(c) > 1/2$. Consequently, it follows from our discussions above that f_c has a periodic point of least period 6 in (c, a). However, this periodic point does not bifurcate from the point a at $c = a$ because $f_c^4(c) < 1/2$ for all $c < a$ and c sufficiently close to a. Therefore, when we vary the values of c from larger number to smaller number, we shall see the following bifurcation phenomenon: f_c has a point bifurcation at $c = a$ and a bifurcation of period 6 points at $c = c_0$ for some $c_0 < a$. How the bifurcation value c_0 is close to a depends on how s is close to $A_0 = (6a^2 - 8a + 3)/[8(1-a)^3]$. Indeed, if the fixed number s is sufficiently close to A_0, then the map $G_s(\varepsilon) = (A_0 - s) - A_1\varepsilon + A_2\varepsilon^2$, where A_i, $i = 0, 1, 2$ are defined as in $(**)$ above, has exactly two positive zeros, say $\varepsilon_1, \varepsilon_2$ with $\varepsilon_1 < \varepsilon_2$. It is clear that $G_s(\varepsilon) < 0$ on $(\varepsilon_1, \varepsilon_2)$. That is, $f_c^4(c) > 1/2$ and so f_c has a period 6 point in (c, a) for each $c \in (a - \varepsilon_2, a - \varepsilon_1)$.

When s is sufficiently close to A_0, we see that the smaller zero ε_1 of $G_s(\varepsilon)$ is very close to zero. Consequently, the bifurcation value c_0 of period 6 points of $f_c(x)$ which belongs to $(a-\varepsilon, a)$ will be very close to a. See, for example, Figures 5 and 6 which show the graph of $y = f_c^6(x)$ with $s = .403 \lessapprox (6a^2 - 8a + 3)/[8(1-a)^3]$ and $c = .225 \lessapprox a$. So, for this family f_c with $s = .403$ and parameter c changing from larger number to smaller number, we shall see a point bifurcation of period 6 points at $c = a$ and a bifurcation of period 6 points somewhere in $(.225, a)$.

When the fixed number s satisfies $1/(16a) \approx .273936 < s < (6a^2 - 8a + 3)/[8(10a)^3 \approx .404256$, we see in $(*)$ above that $f_0^4(0) - 1/2 = -\varepsilon[2a^2 - 4a + 3 - 8s] = -\varepsilon[1/(2a) - 8s] > 0$. So, for all sufficiently small number $\delta > 0$, we have $f_\delta^4(\delta) > 1/2$. On the other hand, it is easy to see that $f_0^4(a) = 0 < 1/2$. So, for all sufficiently small number $\delta > 0$, we have $f_\delta^4(a) < 1/2$. Thus, for all sufficiently small number $\delta > 0$, we have $f_\delta^4(\delta) > 1/2$ and $f_\delta^4(a) < 1/2$. This shows that $f_\delta^4(y) = 1/2$ for some $y \in (\delta, a)$. Consequently, $f_\delta^6(y) = \delta < y$ for all sufficiently small number $\delta > 0$. Since it is easy to check that $f_0^6(a) > 1/2 > a$ (note that we have actually shown that, for $0 \le c < a$, $f_c^6(a) > a$ in the beginning of this section), we obtain that $f_\delta^6(a) > a$ for all sufficiently small number $\delta > 0$. This implies that, for all sufficiently small number $\delta > 0$, $f_\delta^6(z) = z$ for some $z \in (\delta, a)$. Therefore, for such numbers s, we shall see that the family $f_c(x)$ has a point bifurcation of period 6 points at $c = a$ and a bifurcation of period 6 points somewhere in $(0, a)$.

When the fixed number s equals the number $1/(16a) \approx .273936$, it is easy to check that the family $f_c(x)$ has a point bifurcation of period 6 points at $c = a$ and at $c = 0$.

When the fixed number s satisfies $0 \le s < 1/(16a)$, we see that, for $0 \le c < a$, $f_c^4(c) - 1/2 = 2c^3 - 4c^2 + 3c + 8s(c-a)(c-1)^3 - (2a^3 - 4a^2 + 3a) = (c-a)[2(c^2 + ca + a^2) - 4(c+a) + 3 - 8s(1-c)^3] < (c-a)[-8a + 3 - 8(1/16a)(1-a)^3] < .167(c-a)$. So, for all $0 \le c < a$, $f_c^4(c) < 1/2$. Since the map $f_c^4(x)$ is strictly decreasing on $[0, a]$, we see that the graph of $y = f_c^4(x)$ can have at most one intersection point with the diagonal line. Consequently, using

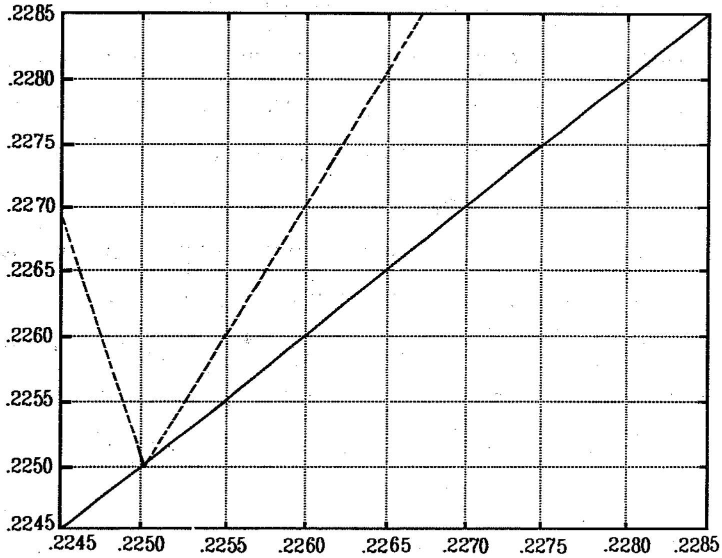

Figure 5. The graph of $y = f_c^6(x)$ with $s = .403$ and $c = .225$ on aneighborhood of a.

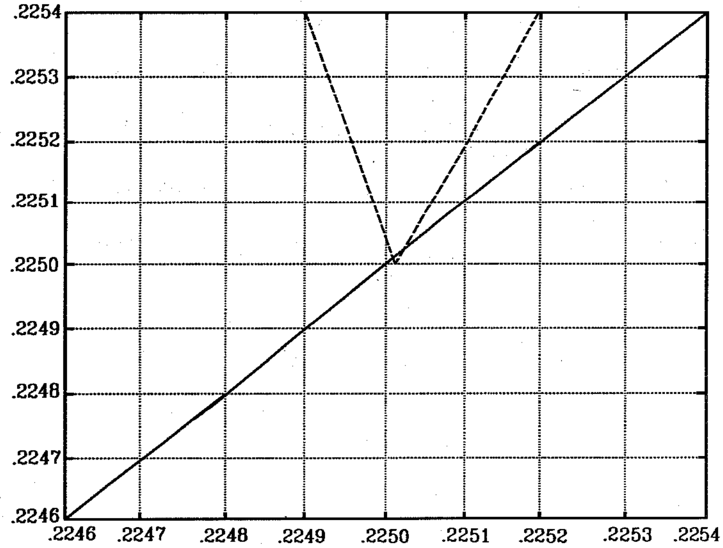

Figure 6. Enlargement of Figure 5 on the interval $[.2246, .2254]$.

similar arguments as in the proof of Theorem 1, we can obtain that $f_c^6(x) > x$ on $[0, a]$ for all $0 \leq c < a$. This shows that when the fixed number s satisfies $0 \leq s < 1/(16a)$, the family $f_c(x)$ has a point bifurcation of period 6 points at $c = a$ and no period 6 points bifurcation for $c \in [0, a)$.

When the fixed number s equals zero, we also note by computer experiments that the first period doubling bifurcation sequence of f_c has the property of supergeometric convergence as described in [4, p.4] and that the family f_c does not have flat top.

Our families are not families of smooth maps. To give a family of smooth maps with nontrivial point bifurcations, we can consider the family $G_{a,c}(x) = a - c(1+x^2)$ with c as the parameter. The computer experiments seem to imply that, for every positive integer $n \geq 3$, there is a number $a_n \in (1,3)$ such that $G_{a_n,c}(x)$ has a point bifurcation of least period n in $(0, a_n)$. However, we are unable to prove it.

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