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SYMMETRIC PERIODIC ORBITS OF CONTINUOUS ODD FUNCTIONS ON THE INTERVAL

BY

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Abstract. Let $I=[-a, a]$ for some positive real number a and let f be a continuous odd function in $C^0(I, I)$. A periodic orbit is called symmetric if it is symmetric with respect to the origin. A periodic orbit which is not symmetric is called asymmetric. Assume that, for some integer $n\geq 2$, f has a symmetric periodic orbit of least period $2n$ or an asymmetric periodic orbit of least period $2n-1$. In this paper, we compute the best possible lower bounds on the topological entropy and on the number of symmetric and asymmetric periodic orbits of other periods for this function f. We also study the C^0- and C^1-perturbation phenomena of f when f is in $C^1(I, I)$. In particular, we show shat, as far as continuous odd functions and the periods of symmetric periodic orbits are concerned, the C^1-perturbation phenomenon is quite different from that of C^0-perturbation.

1. **Introduction.** Let I denote a compact real interval of the form $[-a, a]$ for some positive real number a and let f be a continuous function from I into itself. For any positive integer n, we define the nth iterate of f by letting $f^1=f$ and $f^n=f\circ f^{n-1}$ for $n>1$. If $x_0\in I$, we call $\{f^i(x_0)\mid i\geq 0\}$ the orbit of x_0 under f. If $f^m(x_0)=x_0$ for some positive integer m, we call x_0 a periodic point of f and call the orbit of x_0 under f a periodic orbit of f. We also call the smallest such positive integer m the least period of x_0 and of the orbit of x_0 under f. Assume that f has a periodic orbit of least period m for some positive integer m, it is natural to ask the following question: Must f also have periodic points of other periods? In 1964, Sharkovskii [13] (see also [2], [6], [7], [9], [10], [12-15]) had given a complete answer to this question. For brevity we say property $P(n)$ holds for f if f has a periodic

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point of least period n. Sharkovskii's result can be stated as follows:

THEOREM A. *For any continuous function in* $C^0(I, I)$, $P(3) \Rightarrow P(5) \Rightarrow P(7) \Rightarrow \cdots \Rightarrow P(2 \cdot 3) \Rightarrow P(2 \cdot 5) \Rightarrow P(2 \cdot 7) \Rightarrow \cdots \Rightarrow P(2^k \cdot 3) \Rightarrow P(2^k \cdot 5) \Rightarrow P(2^k \cdot 7) \Rightarrow \cdots \Rightarrow P(2^3) \Rightarrow P(2^2) \Rightarrow P(2) \Rightarrow P(1)$.

Now if we also assume that f is odd, i.e., $f(-x) = -f(x)$ for all $x \in I$, then there are some special types of periodic orbits of f which are worth studying. These are the orbits which are symmetric with respect to the origin. If $0 \neq x_0 \in I$ satisfies $f^m(x_0) = -x_0$ for some positive integer m, then since f is odd, $f^{2m}(x_0) = x_0$. Hence, x_0 is a periodic point of f and the orbit of x_0 under f is symmetric with respect to the origin. In this case, we call the orbit of x_0 under f a symmetric periodic orbit of f and it is clear that the least period of x_0 under f is twice the smallest positive integer m such that $f^m(x_0) = -x_0$. Consequently, any symmetric periodic orbit has even period. A periodic orbit which is not symmetric is called asymmetric. Any asymmetric periodic orbit $\{x_1, x_2, \cdots, x_n\}$ has a twin orbit $\{-x_1, -x_2, \cdots, -x_n\}$ of the same period since f is odd. Note that under our terminology, the origin is an asymmetric fixed point of f. For simplicity, we say property $S(2n)$ holds for f if f has a symmetric periodic orbit of least period $2n$ and property $A(n)$ holds for f if f has an asymmetric periodic orbit of least period n. We also say property $ANP(n)$ holds for f if f has an asymmetric periodic orbit of least period n which contains both negative and positive elements. Using this terminology, it follows easily from the proof of Sharkovskii's theorem in e. g. [2] or in [9] that Theorem A can be rephrased as follows:

THEOREM A$'$. *For any continuous odd function in* $C^0(I, I)$, $A(3) \Rightarrow A(5) \Rightarrow A(7) \Rightarrow \cdots \Rightarrow A(2 \cdot 3) \Rightarrow A(2 \cdot 5) \Rightarrow A(2 \cdot 7) \Rightarrow \cdots \Rightarrow A(2^k \cdot 3) \Rightarrow A(2^k \cdot 5) \Rightarrow A(2^k \cdot 7) \Rightarrow \cdots \Rightarrow A(2^3) \Rightarrow A(2^2) \Rightarrow A(2) \Rightarrow A(1)$.

Now assume that f has a symmetric periodic orbit of least period $2m$ for some positive integer m. We can also ask the following question which is similar to the above one: Must f have symmetric or asymmetric periodic orbits of other periods?

In 1985, Branner [5] gave a complete answer to the above question for the class of all continuous odd functions in $C^0(I, I)$ which have exactly two extreme points. That is, for any continuous odd function g in $C^0(I, I)$ which has exactly two extreme points, Branner shows that if g is strictly increasing in a neighborhood of the origin, then

$$\begin{aligned}(*)\quad & S(4) \Rightarrow S(6) \Rightarrow \cdots \Rightarrow S(2(n+1)) \Rightarrow S(2(n+2)) \Rightarrow \cdots \\ & \Rightarrow A(3) \Rightarrow A(5) \Rightarrow \cdots \Rightarrow A(2n+1) \Rightarrow \cdots \\ & \Rightarrow A(2 \cdot 3) \Rightarrow A(2 \cdot 5) \Rightarrow \cdots \Rightarrow A(2(2n+1)) \Rightarrow \cdots \\ & \Rightarrow \cdots \\ & \Rightarrow A(2^k \cdot 3) \Rightarrow A(2^k \cdot 5) \Rightarrow \cdots \Rightarrow A(2^k(2n+1)) \Rightarrow \cdots \\ & \Rightarrow \cdots \\ & \Rightarrow \cdots \Rightarrow A(2^3) \Rightarrow A(2^2) \Rightarrow A(2) \Rightarrow A(1).\end{aligned}$$

On the other hand, if g is strictly decreasing on a neighborhood of the origin, then

$$\begin{aligned}(**)\quad & S(4) \Rightarrow ANP(3) \Rightarrow S(8) \Rightarrow \cdots \Rightarrow S(4n) \Rightarrow ANP(2n+1) \\ & \Rightarrow S(4n+4) \Rightarrow \cdots \\ & \Rightarrow S(2 \cdot 3) \Rightarrow S(2 \cdot 5) \Rightarrow \cdots \Rightarrow S(2(2n+1)) \\ & \Rightarrow S(2(2n+3)) \Rightarrow \cdots \\ & \Rightarrow ANP(2 \cdot 3) \Rightarrow ANP(2 \cdot 5) \Rightarrow \cdots \Rightarrow ANP(2(2n+1)) \Rightarrow \cdots \\ & \Rightarrow \cdots \\ & \Rightarrow ANP(2^k \cdot 3) \Rightarrow ANP(2^k \cdot 5) \Rightarrow \cdots \Rightarrow ANP(2^k(2n+1)) \\ & \Rightarrow \cdots \Rightarrow \cdots \\ & \Rightarrow \cdots \Rightarrow ANP(2^3) \Rightarrow ANP(2^2) \Rightarrow ANP(2).\end{aligned}$$

However, Branner's result does not hold in general. In this paper, we will generalize Branner's result to much larger class of continuous odd functions in $C^0(I, I)$ which may contain arbitrarily many extreme points. In fact, we have shown much more than this.

In Section 5, we show that, for any continuous odd function in $C^0(I, I)$ and any positive integer n, $S(2n+2) \Rightarrow S(2n+6)$ and $S(4n) \Rightarrow S(4n+2m)$ for all positive integers m. We also show that $S(4n) \Rightarrow A(2n+1)$ and $ANP(2n+1) \Rightarrow S(4n+4)$. Based on these results, we call any symmetric periodic orbit of f of least period $2n$ for some integer $n \geq 2$ minimal if f has no symmetric periodic orbit of least period $2n-4$ (and so any symmetric periodic orbit of least period 4 is minimal by our definition).

In Section 6, we determine the structures of all minimal symmetric periodic orbits. We find that some of these structures are fundamental. Let $P = \{\pm x_i \,|\, 1 \leq i \leq n\}$ with $0 < x_1 < x_2 < \cdots < x_n$ be a symmetric periodic orbit of f of least period $2n$ for some integer $n \geq 2$. We call P a simple symmetric periodic orbit of f of the first kind if $f(x_i) = x_{i+1}$ for all $1 \leq i \leq n-1$ and $f(x_n) = -x_1$ and call P a simple symmetric periodic orbit of f of the second kind if n is even and $f(x_i) = -x_{i+1}$ for all $1 \leq i \leq n-1$ and $f(x_n) = x_1$. We also call P a simple symmetric periodic orbit of f of the third kind with type "+" if $n = 2m+1$ is odd, $f(x_{m+2-i}) = -x_{m+1+i}$ and $f(x_{m+1+i}) = -x_{m+1-i}$ for all $1 \leq i \leq m$, and $f(x_1) = -x_{m+1}$. Finally we call P a simple symmetric periodic orbit of f of the third kind with type "−" if $n = 2m+1$ is odd, $f(x_{m+i}) = -x_{m+1-i}$ and $f(x_{m+1-i}) = -x_{m+1+i}$ for all $1 \leq i \leq m$, and $f(x_{2m+1}) = -x_{m+1}$. We then also show in this section that if f has a symmetric periodic orbit of least period $2n$ for some integer $n \geq 2$, then f must also have a simple symmetric periodic orbit of some kind.

In Section 7, we study those continuous odd functions in $C^0(I, I)$ which have a symmetric periodic orbit of even period ≥ 4. In particular, we compute the best possible lower bounds on the topological entropy and on the number of symmetric and asymmetric periodic orbits of periods guaranteed in Theorem 1 for these functions.

In Section 8, we study those continuous odd functions f in $C^0(I, I)$ with an asymmetric periodic orbit of least period $2n+1$ for some positive integer n which contains both negative and positive elements. We find that some special types of such periodic orbits are fundamental. Let Q be an asymmetric periodic orbit of f with least period $2n+1$ for some positive integer n which contains both negative and positive elements and let y be the unique element of Q which is closest to the origin. Then we call Q a simple

ANP periodic orbit of f if $0<|f^{i-1}(y)|<|f^{i}(y)|$ for all $1\leq i\leq 2n$, and exactly one of the following holds:

(a) $f^{2k-1}(y)<0<y<f^{2k}(y)$ for all $1\leq k\leq n$,

(b) $f^{2k}(y)<y<0<f^{2k-1}(y)$ for all $1\leq k\leq n$.

We show in this section that these functions f must contain a simple *ANP* periodic orbit of least period $2n+1$. We also compute the best possible lower bounds on the topological entropy and on the number of symmetric and asymmetric periodic orbits of periods guaranteed in Theorem 1 for these functions f.

Finally, we study in Section 9 C^0- and C^1-perturbations of those continuous odd functions which have a symmetric periodic orbit of even period ≥ 4. In particular, we find that, as far as continuous odd functions and the periods of symmetric periodic orbits are concerned, the C^1-perturbation phenomenon is quite diffierent from that of the C^0-perturbation.

For the proofs of our main results in Sections 5, 6, and 9, we use the method of directed graphs ([2], [6], [9], [10], [13–15]) which is briefly reviewed in Section 3. For the proofs of our main results in Sections 7 and 8, we use the method of symbolic representation ([7], [8]) which is briefly reviewed in Section 4. When we count the number of periodic orbits of various periods in Sections 7 and 8, we use a well-known result which is included in Section 2.

2. Counting the number of symmetric and asymmetric periodic orbits. Let $\phi(m)$ be an integer-valued function defined on the set of all positive integers. If $m=p_1^{k_1}p_2^{k_2}\cdots p_r^{k_r}$, where p_i's are distinct prime numbers, r and k_i's are positive integers, we define $\Phi_1(1,\phi)=\phi(1)$ and

$$\begin{aligned}\Phi_1(m,\phi)=\phi(m)-\sum_{i=1}^{r}\phi(m/p_i)+\sum_{i_1<i_2}\phi(m/(p_{i_1}p_{i_2}))\\ -\sum_{i_1<i_2<i_3}\phi(m/(p_{i_1}p_{i_2}p_{i_3}))+\cdots\\ +(-1)^r\phi(m/(p_1p_2\cdots p_r)),\end{aligned}$$

where the summation $\sum_{i_1<i_2<\cdots<i_j}$ is taken over all integers

$i_1, i_2, \cdots, i_j$ with $1 \le i_1 < i_2 < \cdots < i_j \le r$.

If $m = 2^{k_0} p_1^{k_1} p_2^{k_2} \cdots p_r^{k_r}$, where p_i's are distinct odd prime numbers, and $k_0 \ge 0$, r, k_i's ≥ 1 are integers, we define similarly

$$\begin{aligned}\Phi_2(m, \phi) = \phi(m) - \sum_{i=1}^{r} \phi(m/p_i) + \sum_{i_1<i_2} \phi(m/(p_{i_1} p_{i_2})) \\ - \sum_{i_1<i_2<i_3} \phi(m/(p_{i_1} p_{i_2} p_{i_3})) + \cdots \\ + (-1)^r \phi(m/(p_1 p_2 \cdots p_r)).\end{aligned}$$

If $m = 2^k$, where $k \ge 0$ is an integer, we define $\Phi_2(m, \phi) = \phi(m) - 1$.

On the other hand, if f is a continuous odd function from I into itself, then every periodic orbit (resp. symmetric periodic orbit) of f with least period m consists of exactly m distinct points. Since it is obvious that distinct periodic orbits (resp. symmetric periodic orbits) of f are pairwise disjoint, the number (if finite) of distinct periodic points (resp. symmetric periodic points) of f with least period m is divisible by m and the quotient equals the number of distinct periodic orbits (resp. symmetric periodic orbits) of f with least period m. This observation, together with a standard inclusion-exclusion argument, gives the following well-known result.

THEOREM B. *Let f be a continuous odd function from I into itself. Assume that, for every positive integer m, the equation $f^m(x) = x$ ($f^m(x) = -x$ resp.) has only finitely many distinct solutions. Let $\phi(m)$ ($\psi(m)$ resp.) denote the number of these solutions. Then, for every positive integer m, the following hold.*

(i) *The number of periodic points of f with least period m is $\Phi_1(m, \phi)$. Consequently, $\Phi_1(m, \phi) \equiv 0 \pmod m$.*

(ii) *The number of symmetric periodic points of f with least period $2m$ is $\Phi_2(m, \psi)$. Consequently, $\Phi_2(m, \psi) \equiv 0 \pmod{2m}$.*

(iii) *The number of asymmetric periodic points of f with least period $2m$ is $\Phi_1(2m, \phi) - \Phi_2(m, \psi)$.*

3. **The method of directed graphs.** In this section, we briefly review the method of directed graphs which will be used later in

the proofs of our main results. This method is based on the following three easy lemmas and is useful in showing the existence of symmetric and asymmetric periodic points of some periods.

From now on, for any closed subinterval $J = [b, c]$ of I, we let $-J$ denote the closed subinterval $[-c, -b]$ of I.

LEMMA 1. *Let f be a continuous odd function from I into itself. Then the following hold.*

(i) *If J is a closed subinterval of I with $f(J) \supset J$, then there is a point $y \in J$ such that $f(y) = y$.*

(ii) *If K is a closed subinterval of I with $f(K) \supset -K$, then there is a point $z \in K$ such that $f(z) = -z$ and $f^2(z) = z$.*

LEMMA 2. *Let f be a continuous odd function from I into itself and let J, L be closed subintervals of I with $f(J) \supset L$. Then there is a closed subinterval K of J such that $f(K) = L$.*

LEMMA 3. *Let f be a continuous odd function from I into itself and let $J_0, J_1, \cdots, J_n$, $n \geq 1$, be closed subintervals of I such that $f(J_i) \supset J_{i+1}$ for all $0 \leq i \leq n-1$. Then the following hold.*

(i) *If $J_n = J_0$, then there is a point $y \in J_0$ such that $f^i(y) \in J_i$ for all $0 \leq i \leq n-1$, $f^n(y) = y$, and the least period of y divides n.*

(ii) *If $J_n = -J_0$, then there is a point $z \in J_0$ such that $f^i(z) \in J_i$ for all $0 \leq i \leq n-1$, $f^n(z) = -z$, and the least period of z is $2m$ with $n/m \geq 1$ and odd.*

From now on, if $J_0, J_1, \cdots, J_n$, $n \geq 1$, are closed subintervals of I such that $f(J_i) \supset J_{i+1}$ for all $0 \leq i \leq n-1$, then we say that there is a path of length n from J_0 to J_n and denote it by $J_0 J_1 \cdots J_n$. A path of length n from J_0 to itself is also called a cycle of length n.

Let f be a continuous odd function from I into itself. Let x_0 be a symmetric periodic point of f with least period $2n$ for some positive integer n and let P denote the orbit of x_0 under f. Then the points in P divide the real line into $2n-1$ finite intervals. Label these finite intervals as $I_1, I_2, \cdots, I_{2n-1}$. A directed graph or digraph in short, can then be formed by using vertices corresponding to these intervals. The vertices are again labelled $I_1, I_2, \cdots, I_{2n-1}$. A directed arc is drawn from the vertex I_i to the vertex I_j if $f(I_i) \supset I_j$. This directed graph is called a $2n$-symmetric periodic digraph of f or the $2n$-symmetric periodic digraph of P. Sometimes, in order to obtain more detailed information on the location of each point in some periodic orbit, we also use the $2n$ finite intervals on the real line which are determined by the $2n+1$ points in the set $P \cup \{0\}$ as vertices and apply the same rule of drawing directed arcs from vertices to vertices as that just described above. The resulting digraph is also called a $2n$-symmetric periodic digraph of f or, more precisely, the $2n$-symmetric periodic digraph of $P \cup \{0\}$.

Based on Lemma 3 above, in order to find a symmetric periodic point of f in a $2n$-symmetric digraph of f, it suffices to find a vertex I_i and a path of some length m from I_i to $-I_i$. For example: If f is the continuous odd function from the interval $[-4, 4]$ onto itself such that $f(1)=-3$, $f(2)=-4$, $f(3)=-2$, $f(4)=1$, and f is linear on each component of the complement of the set $\{0, \pm 1, \pm 2, \pm 3\}$ in $[-4, 4]$. Then $P=\{1, -3, 2, -4, -1, 3, -2, 4\}$ is a symmetric periodic orbit of f of least period 8. If $I_i=[i-1, i]$ for all $i=1, 2, 3, 4$, then the 8-symmetric periodic digraph of $P \cup \{0\}$ looks as in Fig. 1 below:

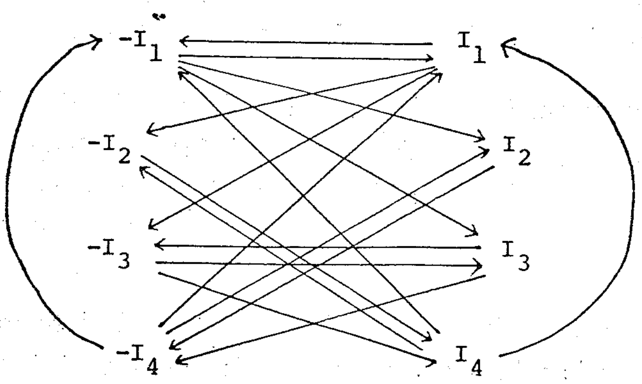

FIG. 1

Therefore, to find a symmetric periodic point of least period $2m$ for some odd integer $m>1$, we can apply Lemma 3 to the path $I_4(-I_1)I_2(-I_4)I_2\cdots(-I_4)I_2(-I_4)$ of length m. Similarly, to find a symmetric periodic point of least period $2n$ for some even integer $n\geq 4$, we can apply Lemma 3 to the path $I_4I_1(-I_1)I_1(-I_1)\cdots I_1(-I_1)I_3(-I_4)$ of length n. As for an asymmetric periodic point of least period $2k$ for some integer $k\geq 2$, we can apply Lemma 3 to the cycle $I_2(-I_4)I_1(-I_1)I_1(-I_1)\cdots I_1(-I_1)I_2$ of length $2k$. Note that this digraph of f also shows that f has no symmetric periodic orbit of least period 4.

4. **The method of symbolic representation.** In this section, we briefly review the method of symbolic representation which will be used later in the proofs of our main results. This method is useful in counting the number of symmetric and asymmetric periodic orbits of some periods (see also [7], [8]).

Let g be a continuous piecewise linear odd function from the interval $[-d, d]$ into itself. We call the set $\{(x_i, y_i)|i=1, 2,\cdots, k\}$ a set of nodes for (the graph of) $y=g(x)$ if the following three conditions hold:

(1) $k\geq 3$ and $x_j=0$ for some $1<j<k$,

(2) $x_1=-d$, $x_k=d$, $x_1<x_2<\cdots<x_k$, and

(3) g is linear on $[x_i, x_{i+1}]$ for all $1\leq i\leq k-1$ and $y_i=g(x_i)$ for all $1\leq i\leq k$.

For any such set, we will use its y-coordinates $y_1, y_2,\cdots, y_k$ to represent the graph of $y=g(x)$ and call $y_1y_2\cdots y_k$ (in that order) a (symbolic) representation for (the graph of) $y=g(x)$. For $1\leq i<j\leq k$, we will call $y_iy_{i+1}\cdots y_j$ the representation for $y=g(x)$ on $[x_i, x_j]$ obtained by restricting $y_1y_2\cdots y_k$ to $[x_i, x_j]$. For convenience, we will also call every y_i in $y_1y_2\cdots y_k$ a node. If $y_i=y_{i+1}$ for some i (i.e., g is constant on $[x_i, x_{i+1}]$), we will simply write $y_1\cdots y_iy_{i+2}\cdots y_k$ instead of $y_1\cdots y_iy_{i+1}y_{i+2}\cdots y_k$. Therefore, every two consecutive nodes in a (symbolic) representation are distinct.

Now assume that $\{(x_i, y_i)|i=1, 2,\cdots, k\}$ is a set of nodes for $y=g(x)$ and $a_1a_2\cdots a_r$ is a representation for $y=g(x)$ with

$\{a_1, a_2, \cdots, a_r\} \subset \{y_1, y_2, \cdots, y_k\}$ and $a_i \neq a_{i+1}$ for all $1 \leq i \leq r-1$. If $\{y_1, y_2, \cdots, y_k\} \subset \{x_1, x_2, \cdots, x_k\}$, then there is an easy way to obtain a representation for $y = g^2(x)$ from the one $a_1 a_2 \cdots a_r$ for $y = g(x)$. The procedure is as follows. First, for any two distinct real numbers u and v, let $[u : v]$ denote the closed interval with endpoints u and v. Then, let $b_{i,1} b_{i,2} \cdots b_{i,t_i}$ be the representation for $y = g(x)$ on $[a_i : a_{i+1}]$ which is obtained by restricting $a_1 a_2 \cdots a_r$ to $[a_i : a_{i+1}]$. We use the following notation to indicate this fact: $a_i a_{i+1} \rightarrow b_{i,1} b_{i,2} \cdots b_{i,t_i}$ (under g) if $a_i < a_{i+1}$, or $a_i a_{i+1} \rightarrow b_{i,t_i} \cdots b_{i,2} b_{i,1}$ (under g) if $a_i > a_{i+1}$. The above representation on $[a_i : a_{i+1}]$ exists since $\{a_1, a_2, \cdots, a_r\} \subset \{x_1, x_2, \cdots, x_k\}$. Finally, if $a_i < a_{i+1}$, let $z_{i,j} = b_{i,j}$ for all $1 \leq j \leq t_i$. If $a_i > a_{i+1}$, let $z_{i,j} = b_{i,t_i+1-j}$ for all $1 \leq j \leq t_i$. It is easy to see tht $z_{i,t_i} = z_{i+1,1}$ for all $1 \leq i \leq r-1$. So, if we define $Z = z_{1,1} \cdots z_{1,t_1} z_{2,2} \cdots z_{2,t_2} \cdots z_{r,2} \cdots z_{r,t_r}$, then it is obvious that Z is a representation for $y = g^2(x)$. It is also obvious that the above procedure can be applied to the representation Z for $y = g^2(x)$ to obtain one for $y = g^3(x)$, and so on.

Since g is odd, the graph of $y = g(x)$ is symmetric with respect to the origin. This special property of g can be used to simplify a symbolic representation for $y = g(x)$. For example: If $y_1 y_2 \cdots y_j y_{j+1} y_{j+2} y_{j+3} \cdots y_k$ is a symbolic representation for $y = g(x)$ with $y_{j+1} = 0$ and $x_{j+1} = 0$, then $y_1 y_2 \cdots y_j \circ (-y_j) \cdots (-y_2)(-y_1)$ is also a symbolic representation for $y = g(x)$.

5. **Ordering of the periods.** In this section, we discuss the ordering of the periods for general continuous odd functions. Because of the following easy lemma [5], this ordering cannot be as simple as the Sharkovskii's ordering stated in Theorem A in Section 1. However, we will soon see that some partial orderings do exist for arbitrary continuous odd functions.

LEMMA 4. *Let f be a continuous odd function in $C^0(I, I)$ and let $0 \neq x_0 \in I$. Then the following hold.*

(a) *The points x_0 and $-x_0$ determine a pair of asymmetric periodic twins of least (odd) period $2n+1$ for f if and only if x_0 determines a symmetric periodic orbit of least period $2(2n+1)$ for $-f$.*

(b) *The points* x_0 *and* $-x_0$ *determine a pair of asymmetric periodic twins of least (even) period* $2n$ *for* f *if and only if* x_0 *and* $-x_0$ *determine a pair of asymmetric periodic twins of the same least period* $2n$ *for* $-f$.

(c) *The point* x_0 *determines a symmetric periodic orbit of least period* $4n$ *for* f *if and only if* x_0 *determines a symmetric periodic orbit of the same least period* $4n$ *for* $-f$.

Recall that, for any continuous odd function f, we say property $S(2n)$ holds if f has a symmetric periodic orbit of least period $2n$ and property $A(n)$ holds if f has an asymmetric periodic orbit of least period n. We also say property $ANP(n)$ holds if f has an asymmetric periodic orbit of least period n which contains both negative and positive elements. It is clear that $ANP(n) \Rightarrow A(n)$. However, the reverse implication need not hold. The main result of this section is the following theorem.

THEOREM 1. *For any continuous odd function* f *in* $C^0(I, I)$ *and any positive integer* n, *the following hold.*

(i) $A(3) \Rightarrow A(5) \Rightarrow A(7) \Rightarrow \cdots \Rightarrow A(2\cdot 3) \Rightarrow A(2\cdot 5) \Rightarrow A(2\cdot 7) \Rightarrow \cdots \Rightarrow A(2^k\cdot 3) \Rightarrow A(2^k\cdot 5) \Rightarrow A(2^k\cdot 7) \Rightarrow \cdots \Rightarrow A(2^3) \Rightarrow A(2^2) \Rightarrow A(2) \Rightarrow A(1)$.

(ii) $S(4n) \Rightarrow S(4n+2m)$ *for all positive integers* m.

(iii) $S(4n) \Rightarrow A(2n+1)$.

(iv) $S(4n) \Rightarrow ANP(m)$ *for all integers* $m \geq 4n$.

(v) $ANP(2n+1) \Rightarrow S(4n+4)$.

(vi) $ANP(2n+1) \Rightarrow ANP(2n+3)$.

(vii) *Let* P *be a periodic orbit of* f *of least period* $2n+1$ *with* $y = \min\{|x| \mid x \in P\}$ *such that* $y \in P$ *and* $f^k(y) < 0$ *for some* $1 \leq k \leq 2n$. *If* $[\min P, -y] \cup [y, \max P]$ *contains a fixed point of* f, *then* $S(4n)$ *holds for* f.

(viii) $S(2(2n+1)) \Rightarrow S(2(2n+1)+4m)$ *for all positive integers* m.

(ix) $S(2(2n+1)) \Rightarrow A(6)$.

REMARK 1. In part (iii) of the above theorem, the number $2n+1$ is best possible in the sense that there exists (see Theorem 4 in Section 7 below) a continuous odd function in $C^0(I, I)$ with a symmetric periodic orbit of least period $4n$, but without periodic orbit of least period m for any odd integer m with $1 < m < 2n$.

REMARK 2. In part (iv) of the above theorem, the number $4n$ on the right-hand side is best possible in the sense that there exists (see Theorem 3 in Section 7 below) a continuous odd function in $C^0(I, I)$ with a symmetric periodic orbit of least period $4n$, but without any periodic orbit of least period m with $1 < m < 4n$ which contains both negative and positive elements.

REMARK 3. In part (v) of the above theorem, the number $4n+4$ is best possible in the sense that there exists (see Theorem 4 in Section 7 below) a continuous odd function in $C^0(I, I)$ with an asymmetric periodic orbit of least period $2n+1$ which contains both negative and positive elements, but without any symmetric periodic orbit of least period $4n$.

Proof of Theorem 1. The proof of part (i) is similar to that of Sharkovskii's theorem in [2] or in [9] and is omitted.

We now prove parts (ii), (iii) and (iv). Let $P = \{\pm x_1, \pm x_2, \cdots, \pm x_{2n}\}$ with $0 < x_1 < x_2 < \cdots < x_{2n}$ be a symmetric periodic orbit of f of least period $4n$. Then since $f^{2n}(y) = -y$ for every $y \in P$, there is a positive integer $j \leq 2n-1$ such that $f([x_j, x_{j+1}]) \supset [-x_1, x_1]$. Since $f^j([-x_1, x_1]) \supset [-x_{j+1}, x_{j+1}] \supset -[x_j, x_{j+1}]$, we have $f^{j+1}([x_j, x_{j+1}]) \supset -[x_j, x_{j+1}]$. Hence there is a path $J_0 J_1 J_2 \cdots J_r$ from $[x_j, x_{j+1}]$ to $-[x_j, x_{j+1}]$ via $[-x_1, x_1]$ such that $r = j+1$, $J_0 = [x_j, x_{j+1}]$, $J_1 = [-x_1, x_1]$, $J_r = -[x_j, x_{j+1}]$, and if $r > 2$, then $J_i = f^{i-1}([-x_1, x_1])$ for all $2 \leq i \leq r-1$.

For any positive integer m, we consider the path $K_0 K_1 \cdots K_{2n+m}$ from $[x_j, x_{j+1}]$ to $-[x_j, x_{j+1}]$ with $K_0 = [x_j, x_{j+1}]$, $K_i = [-x_1, x_1]$ for all $1 \leq i \leq 2n+m+1-r$, and $K_{2n+m+i-r} = J_i$ for all $2 \leq i \leq r$. It follows from Lemma 3 that this path $K_0 K_1 \cdots K_{2n+m}$ produces a symmetric periodic point of f of least period $2(2n+m)$ in $[x_j, x_{j+1}]$. This completes the proof of part (ii).

For the proof of part (iii), we note that $f([-x_1, x_1]) \supset [-x_1, x_1]$ and $f^k([-x_1, x_1]) \supset [-x_{k+1}, x_{k+1}]$ for all $1 \leq k \leq 2n-1$. Therefore, $f^m([x_j, x_{j+1}]) \supset [-x_{j+1}, x_{j+1}] \supset [x_j, x_{j+1}]$ for all integers $m \geq j+1$. Thus, f has a periodic point of least period $(2n+1)/s$ for some odd integer $1 \leq s < 2n+1$ and hence f has a periodic orbit of least period $2n+1$ by Theorem A (Sharkovskii's theorem) which is obviously asymmetric. This completes the proof of part (iii).

On the other hand, since $f^{2n}([x_j, x_{j+1}]) \supset -[x_j, x_{j+1}]$, there is a path of length $2n$ from $[x_j, x_{j+1}]$ to $-[x_j, x_{j+1}]$. Since $f^k([-x_1, x_1]) \supset [-x_{k+1}, x_{k+1}]$ for all $1 \leq k \leq 2n-1$ and since $f(\pm[x_j, x_{j+1}]) \supset [-x_1, x_1]$ and $f^i([-x_1, x_1]) \supset [-x_1, x_1]$ for all positive integers i, we obtain that $f^m([x_j, x_{j+1}]) \supset [x_j, x_{j+1}]$ for all integers $m \geq j+2n+1$. Therefore, for every integer $m \geq j+2n+1$, there is a cycle of length m from $[x_j, x_{j+1}]$ to itself via $[-x_1, x_1]$ and $-[x_j, x_{j+1}]$ which produces by Lemma 3 an asymmetric periodic orbit of f of least period m. It is clear that this orbit contains both negative and positive elements. This completes the proof of part (iv).

As for parts (v) and (vi), let P be a periodic orbit of f with least period $2n+1$ which contains both negative and positive elements. Let y be an element of P with $|y| = \min\{|x| \mid x \in P\}$. Without loss of generality, we may assume that $y > 0$. We now have two cases to consider.

If $y < f(y)$, then there is a smallest integer $2 \leq j \leq 2n$ such that $f^j(y) < 0 < f^{j-1}(y)$. By applying Lemma 3 to the path $J_0 J_1 J_2 \cdots J_{2n-1}(-J_0)$ with $J_i = [0, y]$ for all $0 \leq i \leq 2n-j$, $J_{2n+1+i-j} = f^i([y, f(y)])$ for all $0 \leq i \leq j-2$, we see that $S(4m)$ holds for f for some integer $1 \leq m \leq n+1$. By part (ii), $S(4n+4)$ holds for f. On the other hand, by applying Lemma 3 to the path $K_0 K_0 K_0 K_1 K_2 \cdots K_{2n} K_0$ with $K_0 = [0, y]$, $K_i = f^i([y, f(y)])$ for all $1 \leq i \leq j-1$, and $K_i = [f^i(y) : 0]$ for all $j \leq i \leq 2n$ (recall that, for any real numbers u and v, $[u : v]$ denotes the closed interval with u and v as endpoints), we see that $ANP(2n+3)$ holds for f.

If $f(y) < 0 < y$, then there is a smallest integer $2 \leq j \leq 2n+1$ such that $f^{j-1}(y) < 0 < f^j(y)$. If $j \geq 3$, we apply Lemma 3 to the path $J_0J_1J_2\cdots J_j$ or to the path $J_0J_1J_2\cdots J_j(-J_0)$ according as j is even or odd, where $J_0=[0, y]$, $J_1=[f(y), 0]$, $J_i=f^{i-2}([f(y): f^2(y)])$ for all $2\leq i\leq j$. Then $S(4m)$ holds for f for some integer $1\leq m\leq 2n+1$. By part (ii), $S(4n+4)$ holds for f. On the other hand, by applying Lemma 3 to the path $K_0K_1K_2\cdots K_{2n+2}K_0$ with $K_0=[0, y]$, $K_1=[f(y), 0]$, $K_i=f^{i-2}([f(y): f^2(y)])$ for all $2\leq i\leq j-1$, and $K_i=[-y, y]$ for all $j\leq i\leq 2n+2$, we see that $ANP(2n+3)$ holds for f. If $j=2$, then $f(y)<y<f^2(y)$ and we have two subcases to consider.

(a) There is a smallest integer $2\leq k\leq 2n$ such that $f^k(y)>0$ and $f^{k+1}(y)>0$. So, $f^{k-1}(y)<0$. By applying Lemma 3 to the path $J_0J_1J_2\cdots J_{k+1}$ or to the path $J_0J_1J_2\cdots J_{k+1}(-J_0)$ according as k is odd or even, where $J_0=[0, y]$, $J_1=[f(y), 0]$, $J_i=f^{i-2}([y, f^2(y)])$ for all $2\leq i\leq k$, and $J_{k+1}=[-y, y]$, and by part (ii), we see that $S(4n+4)$ holds for f. On the other hand, by applying Lemma 3 to the path $K_0K_1K_2\cdots K_{2n+2}K_0$ with $K_0=[0, y]$, $K_1=[f(y), 0]$, $K_i=f^{i-2}([y, f^2(y)])$ for all $2\leq i\leq k$, $K_i=[-y, y]$ for all $k+1\leq i\leq 2n+2$, we see that $ANP(2n+3)$ holds for f.

(b) There is a smallest integer $3\leq k\leq 2n-1$ such that $f^k(y)<0$ and $f^{k+1}(y)<0$. In this subcase, the proofs of parts (v) and (vi) are similar to (a) above and are omitted.

This completes the proofs of parts (v) and (vi).

To prove part (vii), we let y be the unique element in P which is nearest to the origin. Without loss of generality, we may assume that $y>0$.

If $f^{2n-1}([-y, y])\supset P$, then let z be a fixed point of f in $[y, \max P]\cup[y, -\min P]$ and let u be a point in P such that $y\leq u$ and $f(u)<0$. Let $J=[u:z]$ be the closed interval with u and z as endpoints. Then it is clear that $f(J)\supset[-y, y]$. Since $f^{2n-1}([-y, y])\supset P$, we see that $f^{2n-1}([-y, y])\supset[\min P, -\min P]\cup[-\max P, \max P]$. Consequently, $f^{2n}(J)\supset -J$. So, there is a path of length $2n$ from J to $-J$ via $[-y, y]$. By Lemma 3 and part (ii), $S(4n)$ holds for f.

If $f^{2n-1}([-y, y]) \not\supset P$, then it is clear that $0 < y < |f^i(y)| < |f^{i+1}(y)|$ for all $1 \le i \le 2n-1$. Let z be a fixed point of f in $(y, |f^{2n}(y)|)$. We now have four cases to consider.

(a) If $f^{2n}(y) < 0$, then $f^{2n}([y, z]) \supset [f^{2n}(y), z] \supset -[y, z]$.

(b) If $f^{2n}(y) > y$ and $f^{2k}(y) < 0$ for some $1 \le k \le n-1$, then $f^{2n-2k}([-f^{2k}(y) : z]) \supset [-f^{2n}(y), z] \supset -[-f^{2k}(y) : z]$.

(c) If $y < f^{2i}(y)$ for all $1 \le i \le n$ and if z is a fixed point of f in $(|f(y)|, f^{2n}(y))$, let k be an integer such that $1 \le k \le n$ and $f^{2k-1}(y) < 0$. Then $f^{2n}([f^{2k-2}(y) : z]) \supset f^{2n-1}([-|f(y)|, |f(y)|]) \supset [-f^{2n}(y), f^{2n}(y)] \supset -[f^{2k-2}(y) : z]$.

(d) If $y < f^{2i}(y)$ for all $1 \le i \le n$ and if z is a fixed point of f in $(y, |f(y)|)$, then we have two subcases to consider.

If $f(y) < 0$, then $f^2([y, z]) \supset f([f(y), z]) \supset f([y, z]) \supset [f(y), z] \supset -[y, z]$.

If $f(y) > 0$, then since $f^{2k+1}(y) < 0$ for some $1 \le k \le n-1$, we have $f^{2k}([z, f(y)]) \supset [f^{2k+1}(y), z] \supset -[z, f(y)]$.

In each of the four cases (a), (b), (c), and (d) above, we have shown that there exist a closed subinterval J of $[y, |f^{2n}(y)|]$ and an integer $1 \le k \le n$ such that $f^{2k}(J) \supset -J$. So, by Lemma 3 and part (ii), S(4n) holds for f. This proves part (vii).

For the proof of part (viii), we will apply Lemma 4 to part (i). Let x_0 be a symmetric periodic point of f of least period $2(2n+1)$. Then, by Lemma 4, x_0 is an asymmetric periodic point of $-f$ of least period $2n+1$. By Theorem A (Sharkovskii's theorem), $-f$ has, for every positive integer m, an asymmetric periodic point y_m of least period $2n+1+2m$. By Lemma 4, y_m is a symmetric periodic point of f of least period $2(2n+1)+4m$. This proves part (viii).

The proof of part (ix) is similar to that of part (viii) and is omitted.

The proof of Theorem 1 is now complete.

6. **The structure of minimal symmetric periodic orbits.** In this section, we will use the method of directed graphs as described

in Section 3 to determine the structure of all minimal symmetric periodic orbits of any continuous odd function in $C^0(I, I)$. These results are contained in the following theorem.

THEOREM 2. *Let f be a continuous odd function in $C^0(I, I)$ which has a minimal symmetric periodic orbit $P = \{\pm x_i \mid 1 \le i \le n\}$ with $0 < x_1 < x_2 < \cdots < x_n$ of least period $2n$ for some integer $n \ge 2$. Then the following hold.*

(1) *If $f(x_1) > 0$, then exactly one of the following holds.*

(i) $f(x_i) = x_{i+1}$ *for all* $1 \le i \le n-1$, *and* $f(x_n) = -x_1$.

(ii) $f(x_i) = x_{i+1}$ *for all* $1 \le i \le n-3$, $f(x_{n-2}) = x_n$, $f(x_n) = x_{n-1}$, *and* $f(x_{n-1}) = -x_1$.

(iii) $f(x_k) = x_{k+2}$, $f(x_{k+2}) = x_{k+1}$, $f(x_{k+1}) = x_{k+3}$ *for some* $1 \le k \le n-3$, $f(x_i) = x_{i+1}$ *for all* $1 \le i \le k-1$ *and all* $k+3 \le i \le n-1$, *and* $f(x_n) = -x_1$.

(iv) $f(x_2) = x_1$, $f(x_1) = x_3$, $f(x_i) = x_{i+1}$ *for all* $3 \le i \le n-1$, *and* $f(x_n) = -x_2$.

Furthermore, if (i) *holds, then for every integer $m \ge n$, f has a simple symmetric periodic orbit of the first kind with least period $2m$. If* (ii), (iii), *or* (iv) *holds, then for every integer $m \ge n-1$, f has a simple symmetric periodic orbit of the first kind with least period $2m$.*

(2) *If $f(x_1) < 0$ and n is even, then exactly one of the following holds.*

(i) $f(x_i) = -x_{i+1}$ *for all* $1 \le i \le n-1$, *and* $f(x_n) = x_1$.

(ii) $f(x_i) = -x_{i+1}$ *for all* $1 \le i \le n-3$, $f(x_{n-2}) = -x_n$, $f(x_n) = -x_{n-1}$, *and* $f(x_{n-1}) = x_1$.

(iii) $f(x_k) = -x_{k+2}$, $f(x_{k+2}) = -x_{k+1}$, $f(x_{k+1}) = -x_{k+3}$ *for some* $1 \le k \le n-3$, $f(x_i) = -x_{i+1}$ *for all* $1 \le i \le k-1$ *and all* $k+3 \le i \le n-1$, *and* $f(x_n) = x_1$.

(iv) $f(x_2) = -x_1$, $f(x_1) = -x_3$, $f(x_i) = -x_{i+1}$ *for all* $3 \le i \le n-1$, *and* $f(x_n) = x_2$.

Furthermore, for every even integer $m \ge n$, f has a simple symmetric periodic orbit of the second kind with least period $2m$ and if (ii), (iii), *or* (iv) *holds, then f also has a pair of asymmetric periodic orbits of least period $n-1$.*

(3) *If* $f(x_1)<0$ *and* n *is odd, then* P *is a simple symmetric periodic orbit of the third kind and, for every odd integer* $m\geq n$, f *also has a simple symmetric periodic orbit of the third kind with least period* $2m$ *and the same type as* P.

Proof. First we assume that $f(x_1)>0$. Then it is clear that there is a positive integer $k\leq n-1$ such that $f(x_k)f(x_{k+1})<0$ and, consequently, $f([x_k, x_{k+1}])\supset[-x_1, x_1]$. Let m denote the smallest such positive integer k. Note that, since m is the smallest positive integer such that $f(x_m)f(x_{m+1})<0$ and since $f(x_1)>0$, we have $f(x_i)>0$ for all $1\leq i\leq m$ and $f(x_{m+1})<0$.

If $m\leq n-3$, then since $f([x_m, x_{m+1}])\supset[-x_1, x_1]$ and $f^m([-x_1, x_1])\supset[-x_{m+1}, x_{m+1}]\supset-[x_m, x_{m+1}]$, there is a path of length $m+1\leq n-2$ from $[x_m, x_{m+1}]$ to $-[x_m, x_{m+1}]$ via $[-x_1, x_1]$. By Lemma 3 and Theorem 1, we see that $S(2n-4)$ holds for f. This is a contradiction. So, $m=n-2$ or $n-1$.

Assume that $m=n-2$. If $f^{n-3}([-x_1, x_1])\supset[-x_{n-1}, x_{n-1}]\supset-[x_{n-2}, x_{n-1}]$, then there is a path of length $n-2$ from $[x_{n-2}, x_{n-1}]$ to $-[x_{n-2}, x_{n-1}]$ via $[-x_1, x_1]$ and so $S(2n-4)$ holds for f. This is a contradiction. Thus $f^{n-3}([-x_1, x_1])\not\supset[-x_{n-1}, x_{n-1}]$. Consequently, since $f(x_1)>0$, we have $f(x_i)=x_{i+1}$ for all $1\leq i\leq n-3$ and, $f(x_{n-2})=x_{n-1}$ or x_n. If $f(x_{n-2})=x_{n-1}$, then since $f(x_{n-1})=f(x_{m+1})<0$, we have $f(x_{n-1})=-x_1$ or $-x_n$. If $f(x_{n-1})=-x_1$, then $-x_1=f^n(x_1)=f^2(x_{n-1})=f(-x_1)=-f(x_1)$. This is a contradiction. If $f(x_{n-1})=-x_n$, then $f([x_{n-2}, x_{n-1}])\supset[-x_n, x_{n-1}]$. So, if $J_0=J_1=[x_{n-2}, x_{n-1}]$, then, by applying Lemma 3 to the path $J_0J_1(-J_0)$ when $n>3$ or to the path $J_0(-J_0)$ when $n=3$, and by Theorem 1 when $n>3$, we obtain that $S(2n-4)$ holds for f. This is again a contradiction. Therefore, $f(x_{n-2})=x_n$. But then since $f(x_{n-2})f(x_{n-1})<0$, we must have $f(x_{n-1})<0$. Consequently, $f(x_n)=x_{n-1}$ and $f(x_{n-1})=-x_1$. This shows that (ii) of part (1) holds.

Assume that $m=n-1$. In this case, we have $f(x_i)>0$ for all $1\leq i\leq n-1$ and $f(x_n)<0$. If $f([x_{n-1}, x_n])\supset[-x_3, x_3]$, then since $f^{n-3}([-x_3, x_3])\supset[-x_n, x_n]\supset-[x_{n-1}, x_n]$, there is a path of length $n-2$ from $[x_{n-1}, x_n]$ to $-[x_{n-1}, x_n]$ via $[-x_3, x_3]$. So,

$S(2n-4)$ holds for f. This is a contradiction. Thus, $f([x_{n-1}, x_n]) \not\supset [-x_3, x_3]$. Now let $f([x_{n-1}, x_n]) \supset [-x_2, x_2]$ with $f(x_{n-1}) = x_2$ or $f(x_n) = -x_2$. If $f(x_1) \geq x_4$ or $f(x_i) \geq x_{i+2}$ for some $3 \leq i \leq n-2$, then we have $f^{n-2}([x_{n-1}, x_n]) \supset [-x_n, x_n] \supset -[x_{n-1}, x_n]$. Thus, $S(2n-4)$ holds for f. This is again a contradiction. So, $f(x_1) = x_3$ and $f(x_i) = x_{i+1}$ for all $3 \leq i \leq n-2$. Since $f(x_{n-1}) > 0$ and $f(x_2) > 0$, we must have $f(x_{n-1}) = x_n$ and $f(x_2) = x_1$. This shows that (iv) of part (1) holds. On the other hand, let $f([x_{n-1}, x_n]) \supset [-x_1, x_1]$ with $f(x_{n-1}) = x_1$ or $f(x_n) = -x_1$. But then, an easy argument will show that $f(x_{n-1}) \neq x_1$. So, let $f([x_{n-1}, x_n]) \supset [-x_1, x_1]$ and $f(x_n) = -x_1$. Since $f(x_1) > 0$, $f(x_j) = x_{j+i}$ for some integers i and j with $1 \leq j \leq n-1$ and $1 \leq i \leq n-j$. If $3 \leq i$, then we easily see that $S(2n-4)$ holds for f which is a contradiction. Thus, $i = 1$ or 2. If $f(x_j) = x_{j+1}$ for all $1 \leq j \leq n-1$, then (i) of part (1) holds. If $j = 1$ and $i = 2$, then since $0 < f(x_2) < x_4$, we must have $f(x_2) = x_1$. Thus, $f(x_i) = x_{i+1}$ for all $3 \leq i \leq n-1$ and, since $f(x_n) < 0$, $f(x_n) = -x_2$. This shows that (iv) of part (1) holds. If $j \geq 2$ and $i = 2$, then since $f(-x_n) = x_1$, we have $f(x_{j+1}) > x_{j+2}$. Consequently, we obtain that $f(x_{j+1}) = x_{j+3}$, $f(x_{j+2}) = x_{j+1}$, $f(x_i) = x_{i+1}$ for all $1 \leq i \leq n-1$, and $f(x_n) = -x_1$. This shows that (iii) of part (1) holds. As for the proofs of other statements of part (1), we only give one as an example since all others are similar. We assume that (iv) of part (1) holds for f. Let $J_0 = [0, x_1]$, $J_1 = [x_1, x_3]$, and $J_i = [x_{i+1}, x_{i+2}]$ for all $2 \leq i \leq n-2$. Then $J_0 J_1 J_2 \cdots J_{n-2}(-J_0)$ is a path of length $n-1$ which produces a simple symmetric periodic orbit of the first kind with least period $2(n-1)$ for f. On the other hand, for every integer $m \geq n$, since $f([0, x_1]) \supset [0, x_3]$, there are points $y_1, y_2, \cdots, y_{m-n+1}$ with $0 < y_1 < y_2 < \cdots < y_{m-n+1} < x_1$ such that $f(y_i) = y_{i+1}$ for all $1 \leq i \leq m-n$ if $m > n$, and $f(y_{m-n+1}) = x_1$. Let $J_0 = [0, y_1]$, $J_i = [y_i, y_{i+1}]$ for all $1 \leq i \leq m-n$ if $m > n$, $J_{m-n+1} = [y_{m-n+1}, x_1]$, $J_{m-n+2} = [x_1, x_3]$, and $J_{m-n+i} = [x_i, x_{i+1}]$ for all $3 \leq i \leq n-1$. Then $J_0 J_1 J_2 \cdots J_{m-1}(-J_0)$ is a path of length m which produces a simple symmetric periodic orbit of the first kind with least period $2m$ for f. This shows that if (iv) of part (1) holds for f, then for every integer $m \geq n-1$, f has a simple symmetric periodic orbit of the first kind with least period $2m$. Other cases can be proved similarly. This completes the proof of the case $f(x_1) > 0$.

Now assume that $f(x_1) < 0$. If n is even, then by applying part (1) of the theorem to the function $-f$, we obtain part (2). So, we assume that $n > 1$ is odd. In the following, we first show that, since $S(2n-4)$ does not hold for f, $f(x_i) < 0$ for all $1 \leq i \leq n$. For simplicity, let $n > 3$ and let m denote the smallest positive integer such that $f(x_m)f(x_{m+1}) < 0$. Consequently, $f([x_m, x_{m+1}]) \supset [-x_1, x_1]$. If $m \leq n-3$, then since

$$f^{n-3}([-x_1, x_1]) \supset [-x_{n-2}, x_{n-2}] \supset -[x_m, x_{m+1}],$$

there is a path of length $n-2$ from $[x_m, x_{m+1}]$ to $-[x_m, x_{m+1}]$ via $[-x_1, x_1]$. By Lemma 3 and Theorem 1, $S(2n-4)$ holds for f. This is a contradiction. So, $m = n-2$ or $n-1$.

If $m = n-2$, then it is clear that $f(x_i) < 0$ for all $1 \leq i \leq n-2$ and $f(x_{n-1}) > 0$. Since $m = n-2$ and $S(2n-4)$ does not hold for f, we have $f([-x_i, x_i]) \not\supset [-x_{i+2}, x_{i+2}]$ for all $1 \leq i \leq n-3$. Consequently, $f(x_i) = -x_{i+1}$ for all $1 \leq i \leq n-3$, and $f(x_{n-2}) = -x_{n-1}$ or $-x_n$. If $f(x_{n-2}) = -x_{n-1}$, then $-x_{n-1} = f(x_{n-2}) = f^{n-2}(x_1)$. Thus, $f(x_{n-1}) = x_n$ and $f(x_n) = x_1$. But then, $S(2)$ and $S(6)$ hold for f. This is a contradiction. If $f(x_{n-2}) = -x_n$, then $-x_n = f(x_{n-2}) = f^{n-2}(x_1)$. So, $f(x_n) = x_{n-1}$ and $f(x_{n-1}) = x_1$. But then, $S(2)$ and $S(6)$ hold for f. This is again a contradiction. So $m \neq n-2$.

If $m = n-1$, then since $f(x_1) < 0$, we have $f(x_i) < 0$ for all $1 \leq i \leq n-1$, and $f(x_n) > 0$. If $f(x_j) = -x_n$ for some $1 \leq j \leq n-3$, then $f^{n-3}([-x_1, x_1]) \supset [-x_n, x_n] \supset -[x_{n-1}, x_n]$. So, there is a path of length $n-2$ from $[x_{n-1}, x_n]$ to $-[x_{n-1}, x_n]$ via $[-x_1, x_1]$ which implies that $S(2n-4)$ holds for f. This is a contradiction. Thus, $f(x_j) = -x_n$ for $j = n-2$ or $n-1$. $f(x_{n-1}) = -x_n$, then let $J_0 = [x_{n-1}, x_n]$, $J_1 = [-x_{n-1}, -x_{n-2}]$. By applying Lemma 3 to the paths $J_0(-J_0)$ and $J_0 J_1 J_0(-J_0)$, we see that $S(2)$ and $S(6)$ hold for f. Consequently, $S(2n-4)$ holds for f. This is a contradiction. If $f(x_{n-2}) = -x_n$, then there is a positive integer $k \leq n-3$ such that $0 < f(-x_k) \leq x_{n-2}$. Let $J_0 = [x_{n-2}, x_n]$ and

$J_1=[-x_{n-2}, -x_k]$. By applying Lemma 3 to the paths $J_0(-J_0)$ and $J_0J_1J_0(-J_0)$, we see that $S(2)$ and $S(6)$ hold for f. So, $S(2n-4)$ holds for f. This is again a contradiction.

Therefore, there does not exist a positive integer m such that $f(x_m)f(x_{m+1})<0$. Consequently, $f(x_i)<0$ for all $1\le i\le n$. By Lemma 4, $\{x_i|1\le i\le n\}$ is a periodic orbit of $-f$ of least period n and, since $S(2n-4)$ does not hold for f, $-f$ has no periodic orbit of least period $n-2$. By a theorem of Stefan [14], if $n=2m+1$, then we have $(-f)^n(x_{m+1})=x_{m+1}$ and, (a) $x_{m+1-i}=(-f)^{2i-1}(x_{m+1})$ and $x_{m+1-i}=(-f)^{2i}(x_{m+1})$ for all $1\le i\le m$ or (b) $x_{m+1+i}=(-f)^{2i-1}(x_{m+1})$ and $x_{m+1-i}=(-f)^{2i}(x_{m+1})$ for all $1\le i\le m$. Since f is odd, we have $(-f)^{2i-1}=-f^{2i-1}$ and $(-f)^{2i}=f^{2i}$ for all positive integers i. Consequently, we have $f^{2i-1}(x_{m+1})=-x_{m+1-i}$ and $f^{2i}(x_{m+1})=x_{m+1+i}$ for all $1\le i\le m$ or, $f^{2i-1}(x_{m+1})=-x_{m+1+i}$ and $f^{2i}(x_{m+1})=x_{m+1-i}$ for all $1\le i\le m$. This shows that P is a simple symmetric periodic orbit of f of the third kind with type "$-$" or type "$+$". The proof of other statement of part (3) is easy and omitted. This completes the proof of the case $f(x_1)<0$ and hence the proof of the theorem.

The following result is an easy consequence of the above theorem.

COROLLARY 1. *Let f be a continuous odd function from the interval $[-1, 1]$ into itself. Assume that there is a point c with $0<c\le 1$ such that f has no symmetric periodic orbit of least period 2 in $(-c, c)$ and no symmetric periodic orbit of least period other than 2 in $[-1, -c)\cup(c, 1]$. Then*

$$\begin{aligned}
&S(4)\Rightarrow S(6)\Rightarrow\cdots\Rightarrow S(2(n+1))\Rightarrow S(2(n+2))\Rightarrow\cdots\\
&\quad\Rightarrow A(3)\Rightarrow A(5)\Rightarrow\cdots\Rightarrow A(2n+1)\Rightarrow\cdots\\
&\quad\Rightarrow A(2\cdot 3)\Rightarrow A(2\cdot 5)\Rightarrow\cdots\Rightarrow A(2(2n+1))\Rightarrow\cdots\\
&\quad\Rightarrow\cdots\\
&\quad\Rightarrow A(2^k\cdot 3)\Rightarrow A(2^k\cdot 5)\Rightarrow\cdots\Rightarrow A(2^k(2n+1))\Rightarrow\cdots\\
&\quad\Rightarrow\cdots\\
&\quad\Rightarrow\cdots\Rightarrow A(2^3)\Rightarrow A(2^2)\Rightarrow A(2)\Rightarrow A(1).
\end{aligned}$$

Note that the above corollary extends the result of Branner (Theorem 2.1) in [5]. (See also Section 1).

7. Continuous odd functions which have a symmetric periodic orbit. In this section, we use the method of symbolic representation as described in Section 4 to compute the best possible lower bounds on the number of symmetric and asymmetric periodic orbits of periods guaranteed in Theorem 1 in Section 5 for those continuous odd functions which have a symmetric periodic orbit of least period $2n$ for some integer $n \geq 2$. We have also found the best possible lower bound on the topological entropy of such continuous odd functions.

We first define some sequences of nonnegative integers.

Fix any integer $n \geq 2$. For all integers i, j, and k, with $1 \leq |i| \leq n$, $1 \leq |j| \leq n$, and $k \geq 1$, we define $b_{k,i,j,n}$ recursively as follows:

$$b_{1,i,j,n} = \begin{cases} 1 & \text{for all } i = j = \pm 1, \pm 2, \cdots, \pm n, \\ 0 & \text{otherwise,} \end{cases}$$

$$\begin{aligned} b_{k+1,i,1,n} &= b_{k,i,1,n} + b_{k,i,-n,n} + b_{k,i,n,n}, \\ b_{k+1,i,j,n} &= b_{k,i,j-1,n} + b_{k,i,n,n} \quad \text{for all } 2 \leq j \leq n, \\ b_{k+1,i,-1,n} &= b_{k,i,-1,n} + b_{k,i,-n,n} + b_{k,i,n,n}, \\ b_{k+1,i,-j,n} &= b_{k,i,-j+1,n} + b_{k,i,-n,n} \quad \text{for all } 2 \leq j \leq n. \end{aligned}$$

We also define $c_{k,n}$ and $d_{k,n}$ by letting

$$\begin{aligned} c_{k,n} = b_{k,1,1,n} &+ b_{k,1,-n,n} + b_{k,1,n,n} + \sum_{i=2}^{n} (b_{k,i,i-1,n} + b_{k,i,n,n}) \\ &+ b_{k,-1,-1,n} + b_{k,-1,-n,n} + b_{k,-1,n,n} \\ &+ \sum_{i=2}^{n} (b_{k,-i,-i+1,n} + b_{k,-i,-n,n}) - 1, \end{aligned}$$

$$\begin{aligned} d_{k,n} = b_{k,1,-1,n} &+ b_{k,1,-n,n} + b_{k,1,n,n} + \sum_{i=2}^{n} (b_{k,i,-i+1,n} + b_{k,i,-n,n}) \\ &+ b_{k,-1,1,n} + b_{k,-1,-n,n} + b_{k,-1,n,n} \\ &+ \sum_{i=2}^{n} (b_{k,-i,i-1,n} + b_{k,i,-n,n}) + 1. \end{aligned}$$

It is easy to see that these sequences $\langle b_{k,i,j,n} \rangle$, $\langle c_{k,n} \rangle$, and $\langle d_{k,n} \rangle$

have the following eight properties. Some of these will be used in the proof of Theorem 3 below.

(i) For all integers i, j, k with $1 \le i \le n$, $1 \le j \le n$, and $k \ge 1$, $b_{k,-i,j,n} = b_{k,i,-j,n}$ and $b_{k,-i,-j,n} = b_{k,i,j,n}$.

(ii) For all integers $1 \le |i| \le n$, $1 \le j \le n$, the sequence $\langle b_{k,i,j,n}\rangle$ is increasing. For all integers $1 \le i \le n$ and $k \ge 1$, $b_{k,i,1,n} \ge b_{k,i,2,n} \ge \cdots \ge b_{k,i,n,n}$ and $b_{k,i,-1,n} \ge b_{k,i,-2,n} \ge \cdots \ge b_{k,i,-n,n}$. Furthermore, for all positive integers k, $b_{k,n,1,n} = b_{k,n,-1,n}$ and $b_{k,1,1,n} - b_{k,1,-1,n} = (-1)^k$.

(iii) If $n > 2$, then for all integers i, j, and s with $2 \le i \le n-1$, $1 \le |j| \le n$, and $1 \le s \le n+1-i$,

$$b_{s,i,j,n} = \begin{cases} 1, & \text{if } j = i + s - 1, \\ 0, & \text{otherwise.} \end{cases}$$

Furthermore, for all integers i, j, and k with $2 \le i \le n-1$, $1 \le |j| \le n$, $k \ge 1$,

$$b_{n-i+k,i,j,n} = b_{k,n,j,n},$$

$$b_{n-i+k,i,-j,n} = b_{k,n,-j,n}.$$

(iv) For every integer $k \ge n-1$, $c_{k,n}$ can also be obtained by the following formulas:

$$\begin{aligned} c_{k,n} = 2 \sum_{i=1}^{n-1} (b_{k-i+1,n,n-i,n} + b_{k-i+1,n,n,n}) \\ + 2(b_{k,1,1,n} + b_{k,1,-n,n} + b_{k,1,n,n}) - 1 \\ = 2 \sum_{i=1}^{n-1} b_{k+2-i,n,n+1-i,n} + 2b_{k+1,1,1,n} - 1. \end{aligned}$$

These identities also hold for all integers k with $1 \le k \le n-2$ provided we define $b_{k,n,j,n} = 0$ for all $3-n \le k \le 0$ and $1 \le j \le n$.

(v) For every integer $k \ge n-1$, $d_{k,n}$ can also be obtained by the following formulas:

$$\begin{aligned} d_{k,n} = 2 \sum_{i=1}^{n-1} (b_{k-i+1,n,-n,n} + b_{k-i+1,n,i-n,n}) \\ + 2(b_{k,1,-1,n} + b_{k,1,-n,n} + b_{k,1,n,n}) + 1 \\ = 2 \sum_{i=1}^{n-1} b_{k+2-i,n,i-n-1,n} + 2b_{k+1,1,-1,n} + 1. \end{aligned}$$

These identities also hold for all integers k with $1 \leq k \leq n-2$ provided we define $b_{k,n,-j,n} = 0$ for all $3-n \leq k \leq 0$ and $1 \leq j \leq n$.

(vi) For all integers k with $1 \leq k \leq 2n-1$, $c_{k,n} = 2^{k+1} - 1$.

(vii) Since, for all positive integers k,

$$b_{k+2n-1,n,n,n} - 3b_{k+2n-2,n,n,n} + \sum_{i=0}^{2n-3} b_{k+i,n,n,n} = 0$$

$$\left(b_{k+2n-1,n,-n,n} - 3b_{k+2n-2,n,-n,n} + \sum_{i=0}^{2n-3} b_{k+i,n,-n,n} = 0 \quad \text{resp.}\right),$$

there exist $2n-1$ nonzero constants α_j's (α_{-j}'s resp.) such that

$$b_{k,n,n,n} = \sum_{j=1}^{2n-1} \alpha_j\, x_j^k \left(b_{k,n,-n,n} = \sum_{j=1}^{2n-1} \alpha_{-j}\, x_j^k \text{ resp.}\right)$$

for all positive integers k, where $\{x_j \mid 1 \leq j \leq 2n-1\}$ is the set of all zeros (including complex zeros) of the polynomial $(x^{2n} - 4x^{2n-1} + 4x^{2n-2} - 1)/(x-1)$.

(viii) Since, for all positive integers k,

$$1 + b_{k+2n-1,1,n,n} - 3b_{k+2n-2,1,n,n} + \sum_{i=1}^{2n-3} b_{k+i,1,n,n} = 0$$

$$\left(-1 + b_{k+2n-1,1,-n,n} - 3b_{k+2n-2,1,-n,n} + \sum_{i=1}^{2n-3} b_{k+i,1,-n,n} = 0 \quad \text{resp.}\right)$$

there exist $2n-1$ nonzero constants β_j's (β_{-j}'s resp.) such that

$$b_{k,1,n,n} = \sum_{j=1}^{2n-1} \beta_j\, x_j^k \left(b_{k,1,-n,n} = \sum_{j=1}^{2n-1} \beta_{-j}\, x_j^k \text{ resp.}\right)$$

for all positive integers k, where $\{x_j \mid 1 \leq j \leq 2n-1\}$ is defined as in (vii) above.

For all positive integers k, m, and n with $n \geq 2$, we let $\phi_{1,n}(k) = c_{k,n}$ and $\phi_{2,n}(k) = d_{k,n}$, and let $\Phi_{1,n}(m) = \Phi_1(m, \phi_{1,n})$ and $\Phi_{2,n}(m) = \Phi_2(m, \phi_{2,n})$, where Φ_1, Φ_2 are defined as in Section 2. Now we can state the following theorem.

THEOREM 3. *Let f be a continuous odd function in $C^0(I, I)$ which has a simple symmetric periodic orbit of the first kind with least period $2n$ for some integer $n \geq 2$. Then the following hold.*

(1) *For every integer $m \geq n$, f has a simple symmetric periodic orbit of the first kind with least period $2m$.*

(2) *For every positive integer k, $A(k)$ and $ANP(2n+k-1)$ hold for f.*

(3) *For every positive integer k, the equation $f^k(x) = x$ has at least $c_{k,n}$ (sharp) distinct solutions and the equation $f^k(x) = -x$ has at least $d_{k,n}$ (sharp) distinct solutions.*

(4) *For every positive integer m, f has at least $\Phi_{1,n}(m)/m$ (sharp) distinct periodic orbits of least period m, at least $\Phi_{2,n}(m)/(2m)$ (sharp) distinct symmetric periodic orbits of least period $2m$, and at least $\Phi_{1,n}(2m)/(2m) - \Phi_{2,n}(m)/(2m)$ (sharp) distinct asymmetric periodic orbits of least period $2m$.*

(5) $$\lim_{m\to\infty} (\log [\Phi_{1,n}(m)/m])/m = \log \lambda_n,$$

$$\lim_{m\to\infty} (\log [\Phi_{2,n}(m)/(2m)])/(2m) = (\log \lambda_n)/2,$$

and

$$\lim_{m\to\infty} (\log [\Phi_{1,n}(2m)/(2m) - \Phi_{2,n}(m)/(2m)])/(2m) = (\log (\lambda_n)/2,$$

where λ_n is the (unique) positive (and the largest in absolute value) zero of the polynomial $x^n - 2x^{n-1} - 1$.

(6) *The topological entropy of f is greater than or equal to* $\log \lambda_n$ *(sharp), where λ_n is defined as in* (5) *above.*

REMARK 4. For every integer $n \geq 2$, let $f_n : [-n, n] \to [-n, n]$ be the continuous odd function defined by (i) $f_n(k) = k+1$ for all integers $1 \leq k \leq n-1$; (ii) $f_n(n) = -1$; and (iii) f_n is linear on the interval $[k, k+1]$ for every integer k with $-n \leq k \leq n-1$. Then $\{\pm i \mid 1 \leq i \leq n\}$ is a symmetric periodic orbit of f_n with least period $2n$. However it is easy to see that no periodic orbit of f_n of least period m with $1 < m < 2n$ can contain both negative and positive elements.

REMARK 5. When $n=2$, the sequences $\langle c_{k,2}\rangle$ and $\langle d_{k,2}\rangle$ can also be obtained by the following formulas: Let $a_1=a_2=1$ and $a_{k+2}=2a_{k+1}+a_k-1$ for all positive integers. Then $c_{k,2}=4a_{k+1}-1$ and $d_{k,2}=c_{k,2}-2$ for all positive integers k. Consequently, $\Phi_{1,2}(m)/m=\Phi_{2,2}(m)/m$ for all odd integers $m>1$.

REMARK 6. Table 1 lists the first 25 values of $\Phi_{2,n}(m)/(2m)$ for $2\le n\le 6$. It seems that, for all positive integers $m\ge n\ge 2$, we have $\Phi_{2,n}(m)/(2m)=2^{m-n}$ for $n\le m\le 3n$, and $\Phi_{2,n}(m)/(2m)>2^{m-n}$ for $m>3n$. On the other hand, if, for all integers $m\ge 1$ and $n\ge 2$, we let $A_{m,n}=\Phi_{2,n}(m)/(2m)$ and define sequences $\langle B_{m,n,k}\rangle$ by letting $B_{m,n,1}=A_{m+3n,n}-2A_{m+3n-1,n}$ and $B_{m,n,k}=B_{m+2n,n,k-1}-B_{m+2n,n+1,k-1}$

TABLE 1

m	$\Phi_{2,2}(m)/(2m)$	$\Phi_{2,3}(m)/(2m)$	$\Phi_{2,4}(m)/(2m)$	$\Phi_{2,5}(m)/(2m)$	$\Phi_{2,6}(m)/(2m)$
1	0	0	0	0	0
2	1	0	0	0	0
3	2	1	0	0	0
4	4	2	1	0	0
5	8	4	2	1	0
6	16	8	4	2	1
7	34	16	8	4	2
8	72	32	16	8	4
9	154	64	32	16	8
10	336	130	64	32	16
11	738	264	128	64	32
12	1,632	538	256	128	64
13	3,640	1,104	514	256	128
14	8,160	2,272	1,032	512	256
15	18,384	4,692	2,074	1,024	512
16	41,616	9,730	4,176	2,050	1,024
17	94,560	20,236	8,416	4,104	2,048
18	215,600	42,208	16,980	8,218	4,096
19	493,122	88,288	34,304	16,464	8,194
20	1,130,976	185,126	69,376	32,992	16,392
21	2,600,388	389,072	140,458	66,132	32,794
22	5,992,560	819,458	284,684	132,608	65,616
23	13,838,306	1,729,296	577,592	265,984	131,296
24	32,016,576	3,655,936	1,173,040	533,672	262,740
25	74,203,112	7,742,124	2,384,678	1,071,104	525,824

for $k > 1$, then more extensive numerical computations seem to show that, for all positive integers k, we have (i) $B_{1,n,m} = 2$ for all $n \geq 2$, (ii) $B_{2,n,k} = 4k$ for all $n \geq 2$, (iii) $B_{3,n,k}$ and $B_{4,n,k}$ are constants depending only on k, and (iv) for all $1 \leq m \leq 2n$, $B_{m,n,k} = B_{m,j,k}$ for all $j \geq n$.

Proof of Theorem 3. First we prove part (1). Let $x_1 > 0$ be a simple symmetric periodic point of f of the first kind with least period $2n$ and let $x_{i+1} = f^i(x_1)$ for all $1 \leq i \leq n-1$. Then $0 < x_1 < x_2 < \cdots < x_n$. Since $f([0, x_1]) \supset [0, x_2] \supset [0, x_1]$, there is a point $y \in (0, x_1)$ such that $f(y) = x_1$. Let $J_0 = [0, y]$, $J_1 = [y, x_1]$, and $J_i = [x_{i-1}, x_i]$ for all $2 \leq i \leq n$. By applying Lemma 3 to $J_0 J_1 J_2 \cdots J_n(-J_0)$, we obtain a simple symmetric periodic point of f of the first kind with least period $2n+2$. The general case follows by inductive argument. This proves part (1).

The proof of part (2) is easy and omitted.

As for the proofs of parts (3), (4), and (5), we use the method of symbolic representation as described in Section 4. It suffices to consider, for every fixed integer $n \geq 2$, the continuous odd function $f_n : [-n, n] \to [-n, n]$ defined by (i) $f_n(k) = k+1$ for all integers $1 \leq k \leq n-1$; (ii) $f_n(n) = -1$; and (iii) f_n is linear on the interval $[k, k+1]$ for every integer k with $-n \leq k \leq n-1$.

The following lemma is easy.

LEMMA 5. *Under f_n, we have*

$$
\begin{aligned}
&02 \to 023, \qquad 20 \to 320\\
&n(-1) \to (-1)\,n(n-1)(n-2)\cdots 320(-2),\\
&(-1)\,n \to (-2)\,0234\cdots(n-1)\,n(-1),\\
&(n-1)n \to n(-1), \qquad n(n-1) \to (-1)\,n,\\
&0(-2) \to 0(-2)(-3), \qquad (-2)\,0 \to (-3)(-2)\,0,\\
&(-n)\,1 \to 1(-n)(-n+1)(-n+2)\cdots(-3)(-2)\,02,\\
&1(-n) \to 20(-2)(-3)(-4)\cdots(-n+1)(-n)1.\\
&(-n+1)(-n) \to (-n)1, \qquad (-n)(-n+1) \to 1(-n).
\end{aligned}
$$

If $n \geq 4$, we also have, for all integers $2 \leq i \leq n-2$,

$$i(i+1) \to (i+1)(i+2), \qquad (i+1)\,i \to (i+2)(i+1),$$

$$(-i)(-i-1)\to(-i-1)(-i-2),$$
$$(-i-1)(-i)\to(-i-2)(-i-1).$$

In the following, when we say the representation for $y=f_n^k(x)$, we mean the representation obtained, following the procedure as described in Section 4, by applying Lemma 5 to the representation

$$1(-n)(-n+1)(-n+2)\cdots$$
$$(-4)(-3)(-2)\,0234\cdots(n-2)(n-1)\,n(-1)$$

for $y=f_n(x)$ successively until we get to the one for $y=f_n^k(x)$.

For every positive integer k and all integers i, j with $1\le |i|\le n$, $1\le |j|\le n$, let $b_{k,i,j,n}$ denote the number of uv's and vu's in the representation for $y=f_n^k(x)$ whose corresponding x-coordinates are in $[i-1, i]$ if $i>0$ or in $[i, i+1]$ if $i<0$, where $uv=02$ if $j=1$, $uv=0(-2)$ if $j=-1$, $uv=j(j+1)$ if $2\le j\le n-1$, $uv=(j)(j-1)$ if $-n+1\le j\le -2$, $uv=n(-1)$ if $j=n$, and $uv=(-n)1$ if $j=-n$. It is obvious that $b_{1,i,i,n}=1$ for all integers i with $1\le |i|\le n$ and $b_{1,i,j,n}=0$ elsewhere. From Lemma 5, we easily see that the sequences $\langle b_{k,i,j,n}\rangle$ are exactly the same as those defined above.

Since

$$\begin{aligned}c_{k,n}=b_{k,1,1,n}+b_{k,1,-n,n}+b_{k,1,n,n}+\sum_{i=2}^{n}(b_{k,i,i-1,n}+b_{k,i,n,n})\\+b_{k,-1,-1,n}+b_{k,-1,-n,n}+b_{k,-1,n,n}\\+\sum_{i=2}^{n}(b_{k,-i,-i+1,n}+b_{k,-i,-n,n})-1,\end{aligned}$$

it is clear that $c_{k,n}$ is the number of intersection points of the graph of $y=f_n^k(x)$ with the diagonal $y=x$. On the other hand, since

$$\begin{aligned}d_{k,n}=b_{k,1,-1,n}+b_{k,1,-n,n}+b_{k,1,n,n}\\+\sum_{i=2}^{n}(b_{k,i,-i+1,n}+b_{k,i,-n,n})\\+b_{k,-1,1,n}+b_{k,-1,-n,n}+b_{k,-1,n,n}\\+\sum_{i=2}^{n}(b_{k,-i,i-1,n}+b_{k,i,-n,n})+1,\end{aligned}$$

it is clear that $d_{k,n}$ is the number of intersection points of the graph of $y=f_n^k(x)$ with the diagonal $y=-x$. This proves part

(3). Part (4) follows from the standard inclusion—exclusion argument. As for part (5), we note that there exist $2n-1$ nonzero constants α_j's (α_{-j}'s resp.) such that

$$b_{k,n,n,n}=\sum_{j=1}^{2n-1}\alpha_j\,x_j^k\quad\left(b_{k,n,-n,n}=\sum_{j=1}^{2n-1}\alpha_{-j}\,x_j^k\ \text{resp.}\right)$$

for all positive integers k, where $\{x_j \mid 1\le j\le 2n-1\}$ is the set of all zeros (including complex zeros) of the polynomial $(x^{2n}-4x^{2n-1}+4x^{2n-2}-1)/(x-1)$. Since $c_{k,n}$ and $d_{k,n}$ can also be expressed as

$$c_{k,n}=2\sum_{i=1}^{n-1}b_{k+2-i,n,n+1-i,n}+2b_{k+1,1,1,n}-1$$

$$d_{k,n}=2\sum_{i=1}^{n-1}b_{k+2-i,n,i-n-1,n}+2b_{k+1,1,-1,n}+1$$

in which $b_{k+1,n,n,n}$ and $b_{k+1,n,-n,n}$ are the dominant terms and since the largest (in absolute value) zero of the polynomial $x^{2n}-4x^{2n-1}+4x^{2n-2}-1=(x^n-2x^{n-1}-1)\cdot(x^n-2x^{n-1}+1)$ is the same as that of the polynomial $x^n-2x^{n-1}-1$, part (5) follows. This completes the proofs of parts (3), (4), and (5).

Part (6) can also be proved by the method of symbolic representation as described in Section 4. We omit the (easy but very technical) details. (See [9] for example).

The proof of Theorem 3 is now complete.

We now consider those continuous odd functions f in $C^0(I, I)$ which have a simple symmetric periodic orbit of the second kind with least period $2n$. We will find the best possible lower bounds on the topological entropy and on the number of symmetric and asymmetric periodic orbits of periods guaranteed in Theorem 1 in Section 5 for such continuous odd functions f. We note that since f has a simple symmetric periodic orbit of the second kind with least period $2n$, $n\ge 2$ is even and $-f$ has a simple symmetric periodic orbit of the first kind with least period $2n$. Hence, most of the statements in the following theorem (Theorem 4) can also be obtained by applying Lemma 4 to Theorem 3 above. However, since the method of symbolic representation is simple and can give more direct enumeration (which is also interesting from the number theoretic point of view) of these best possible lower bounds, we will give a proof Theorem 4 which is based on this method.

As in Theorem 3, we need to define some sequences of nonnegative integers first.

Fix any integer $n \geq 2$. For all integers i, j, and k, with $1 \leq |i| \leq n$, $1 \leq |j| \leq n$, and $k \geq 1$, we define $p_{k,i,j,n}$ recursively as follows:

$$p_{1,i,j,n} = \begin{cases} 1, & \text{if } i = -j = 1, 2, \cdots, n, \\ 0, & \text{otherwise.} \end{cases}$$

For $1 \leq |i| \leq n$, $k \geq 1$, and $j = \pm 1$,

$$p_{k+1,i,j,n} = p_{k,i,-n,n} + p_{k,i,n,n} + p_{k,i,-j,n}.$$

For $1 \leq |i| \leq n$, $k \geq 1$, and $1 \leq j \leq n-1$,

$$p_{k+1,i,-j-1,n} = p_{k,i,j,n} + p_{k,i,n,n},$$

$$p_{k+1,i,j+1,n} = p_{k,i,-j,n} + p_{k,i,-n,n}.$$

We also define $q_{k,n}$ and $r_{k,n}$ by letting

$$\begin{aligned} q_{k,n} = {} & p_{k,1,1,n} + p_{k,1,n,n} + p_{k,1,-n,n} \\ & + \sum_{i=2}^{n} (p_{k,i,i-1,n} + p_{k,i,n,n}) \\ & + p_{k,-1,-1,n} + p_{k,-1,-n,n} + p_{k,-1,n,n} \\ & + \sum_{i=2}^{n} (p_{k,-i,-i+1,n} + p_{k,-i,-n,n}) \end{aligned}$$

and

$$\begin{aligned} r_{k,n} = {} & p_{k,1,-1,n} + p_{k,1,-n,n} + p_{k,1,n,n} \\ & + \sum_{i=2}^{n} (p_{k,i,-i+1,n} + p_{k,i,-n,n}) \\ & + p_{k,-1,1,n} + p_{k,-1,n,n} + p_{k,-1,-n,n} \\ & + \sum_{i=2}^{n} (p_{k,-i,i-1,n} + p_{k,-i,n,n}). \end{aligned}$$

It is easy to see that these sequences $\langle p_{k,i,j,n} \rangle$, $\langle q_{k,n} \rangle$, and $\langle r_{k,n} \rangle$ have the following seven properties. Some of these will be used in the proof of Theorem 4 below.

(i) For all integers i, j, k with $1 \leq i \leq n$, $1 \leq j \leq n$, and $k \geq 1$, $p_{k,-i,j,n} = p_{k,i,-j,n}$ and $p_{k,-i,-j,n} = p_{k,i,j,n}$.

(ii) For all integers $1 \leq |i| \leq n$, $1 \leq j \leq n$, the sequence $\langle p_{k,i,j,n} \rangle$ is increasing. For all integers $1 \leq i \leq n$ and $k \geq 1$, $p_{k,i,1,n} \geq p_{k,i,2,n} \geq \cdots \geq p_{k,i,n,n}$ and $p_{k,i,-1,n} \geq p_{k,i,-2,n} \geq \cdots \geq p_{k,i,-n,n}$. Furthermore, for all positive integers k, $p_{k,n,1,n} = p_{k,n,-1,n}$ and $p_{k,1,1,n} - p_{k,1,-1,n} = (-1)^k$.

(iii) If $n > 2$, then for all integers i, j, and s with $2 \leq i \leq n-1$, $1 \leq |j| \leq n$, and $1 \leq s \leq n+1-i$,

$$p_{s,i,j,n} = \begin{cases} 1, & \text{if } j = (-1)^s(i+s-1), \\ 0, & \text{otherwise.} \end{cases}$$

Furthermore, for all integers i, j, and k with $2 \leq i \leq n-1$, $1 \leq |j| \leq n$, and $k \geq 1$,

$$p_{n-i+k,i,j,n} = p_{k,n,(-1)^{n-i}j,n},$$

$$p_{n-i+k,i,-j,n} = p_{k,n,(-1)^{n-i+1}j,n}.$$

(iv) For every integer $k \geq n-1$, $q_{k,n}$ can also be obtained by the following formula:

$$\begin{aligned} q_{k,n} = 2\Big\{ & p_{k+1,1,-1,n} \\ & + \sum_{j=1}^{[n/2]-1} (2p_{k-2j+2,n,n+1-2j,n} + p_{k-2j+2,n,n,n}) \\ & + ([1+(-1)^n]/2)(p_{k-n+2,n,1,n} + p_{k-n+2,n,n,n})\Big\} \\ & + (-1)^{k+1}, \end{aligned}$$

where $[n/2]$ is the largest integer $\leq n/2$. This identity also holds for all integers k with $1 \leq k \leq n-2$ provided we define $p_{k,n,j,n} = 0$ for all $3-n \leq k \leq 0$ and $1 \leq j \leq n$.

(v) For every integer $k \geq n-1$, $r_{k,n}$ can also be obtained by the following formula:

$$\begin{aligned} r_{k,n} = 2\Big\{ & p_{k+1,1,1,n} \\ & + \sum_{j=1}^{[n/2]-1} (2p_{k-2j+2,n,2j-n-1,n} + p_{k-2j+2,n,-n,n}) \\ & + ([1+(-1)^n]/2)(p_{k-n+2,n,-1,n} + p_{k-n+2,n,-n,n})\Big\} \\ & + (-1)^k, \end{aligned}$$

where $[n/2]$ is the largest integer $\leq n/2$. The identity also holds for all integers k with $1 \leq k \leq n-2$ provided we define $p_{k,n,-j,n} = 0$ for all $3-n \leq k \leq 0$ and $1 \leq j \leq n$.

(vi) For all integers k with $1 \leq k \leq n-1$, $p_{2k,n} = 2^{2k+1} - 1$.

(vii) Since, for all integers $k > n$, we have

$$p_{k,n,1,n} - 2p_{k-1,n,1,n} - p_{k-n,n,1,n} = 0$$

and

$$p_{k,1,1,n} - 2p_{k-1,1,1,n} - p_{k-n,1,1,n} = (-1)^k,$$

it is easy to see that

$$\lim_{k\to\infty} (\log (p_{k,n,1,n})/k = \lim_{k\to\infty} (\log p_{k,1,1,n})/k = \log \lambda_n,$$

where λ_n is the largest (in absolute value) zero of the polynomial $x^n - 2x^{n-1} - 1$. Note that $(x^n - 2x^{n-1} - 1) \cdot (x^n - 2x^{n-1} + 1) = x^{2n} - 4x^{2n-1} + 4x^{2n-2} - 1$ and λ_n is also the largest (in absolute value) zero of the polynomial $x^{2n} - 4x^{2n-1} + 4x^{2n-2} - 1$ which is the same as that defined in Theorem 3.

For all positive integers k, m, and n with $n \geq 2$, we let $\phi_{3,n}(k) = q_{k,n}$ and $\phi_{4,n}(k) = r_{k,n}$, and let $\Phi_{3,n}(m) = \Phi_1(m, \phi_{3,n})$ and $\Phi_{4,n}(m) = \Phi_2(m, \phi_{4,n})$, where Φ_1, Φ_2 are defined as in Section 2. Now we can state the following theorem.

THEOREM 4. *Let f be a continuous odd function in $C^0(I, I)$ such that $-f$ has a simple symmetric periodic orbit of the first kind with least period $2n$ for some integer $n \geq 2$. Then the following hold.*

(1) *For every integer $m \geq n$, $-f$ has a simple symmetric periodic orbit of the first kind with least period $2m$.*

(2) *For every positive integer k, the equation $f^k(x) = x$ has at least $q_{k,n}$ (sharp) distinct solutions and the equation $f^k(x) = -x$ has at least $r_{k,n}$ (sharp) distinct solutions.*

(3) *For every positive integer m, f has at least $\Phi_{3,n}(m)/m$ (sharp) distinct periodic orbits of least period m, at least $\Phi_{4,n}(m)/(2m)$ (sharp) distinct symmetric periodic orbits of least period $2m$, and at least $\Phi_{3,n}(2m)/(2m) - \Phi_{4,n}(m)/(2m)$ (sharp) distinct asymmetric periodic orbits of least period $2m$.*

(4) $$\lim_{m\to\infty} (\log [\Phi_{3,n}(m)/m])/m = \log \lambda_n,$$

$$\lim_{m\to\infty} (\log[\Phi_{4,n}(m)/(2m)])/(2m) = (\log \lambda_n)/2,$$

and

$$\lim_{m\to\infty} (\log [\Phi_{3,n}(2m)/(2m) - \Phi_{4,n}(m)/(2m)])/(2m)$$
$$= (\log \lambda_n)/2,$$

where λ_n is the (unique) positive (and the largest in absolute value) zero of the polynomial $x^n - 2x^{n-1} - 1$ which is the same as that defined in Theorem 3.

(5) *The topological entropy of f is greater than or equal to* $\log \lambda_n$ *(sharp), where λ_n is defined as in* (4) *above.*

REMARK 7. Because of Lemma 4, the collection $\{\langle b_{k,1,j,n}\rangle, \langle b_{k,n,j,n}\rangle | 1 \le |j| \le n\}$ of sequences and the collection $\{\langle p_{k,1,j,n}\rangle, \langle p_{k,n,j,n}\rangle | 1 \le |j| \le n\}$ of sequences can be said to be conjugate to each other. By counting the number of symmetric and asymmetric orbits of appropriate periods for f_n and g_n defined respectively in the proofs of Theorem 3 above and Theorem 4 below, we have, for every positive integer m, the following identities:

(i) $\Phi_{1,n}(2m+1) = \Phi_{4,n}(2m+1)$,

(ii) $\Phi_{3,n}(2m+1) = \Phi_{2,n}(2m+1)$,

(iii) $\Phi_{1,n}(4m) = \Phi_{3,n}(4m)$,

(iv) $\Phi_{2,n}(2m) = \Phi_{4,n}(2m)$,

(v) $\Phi_{1,n}(2(2m+1)) - \Phi_{2,n}(2m+1)$
$= \Phi_{3,n}(2(2m+1)) - \Phi_{4,n}(2m+1)$.

In the above theorem, we assume that $-f$ has a simple symmetric periodic orbit of the first kind with least period $2n$. This condition is equivalent to the following:

(i) If n is even, then f has a simple symmetric periodic orbit of the second kind with least period $2n$;

(ii) If n is odd, then f has a simple *ANP* periodic orbit (see Section 1 for definition) of least period n.

Consequently, from part (1) of the above theorem, we easily obtain the following result.

COROLLARY 2. *Let f be a continuous odd function in $C^0(I, I)$. Then the following hold.*

(a) *If f has a simple symmetric periodic orbit of the second kind with least period $2n$ for some even integer $n \geq 2$, then for every even integer $m \geq n$, f has a simple symmetric periodic orbit of the second kind with least period $2m$ and a simple ANP periodic orbit of least period $m+1$.*

(b) *If f has a simple ANP periodic orbit of least period n for some odd integer $n \geq 3$, then for every odd integer $m \geq n$, f has a simple symmetric periodic orbit of the second kind with least period $2m+2$ and a simple ANP periodic orbit of least period m.*

Proof of Theorem 4. Part (1) follows from Theorem 3. As for the proofs (2), (3), and (4), we use the method of symbolic representation as described in Section 4. It suffices to consider, for every fixed integer $n \geq 2$, the continuous odd funciton $g_n : [-n, n] \to [-n, n]$ defined by (i) $g_n(k) = -(k+1)$ for all integers $1 \leq k \leq n-1$; (ii) $g_n(n) = 1$; and (iii) g_n is linear on the interval $[k, k+1]$ for every integer k with $-n \leq k \leq n-1$.

The following lemma is easy.

LEMMA 6. *Under g_n, we have*

$$
\begin{aligned}
&02 \to 0(-2)(-3), \qquad 20 \to (-3)(-2)\,0,\\
&n(-1) \to 1(-n)(-n+1)\cdots(-3)(-2)\,02,\\
&(-1)\,n \to 20(-2)(-3)\cdots(-n+1)(-n)\,1,\\
&(n-1)\,n \to (-n)\,1, \qquad n(n-1) \to 1(-n),\\
&0(-2) \to 023, \qquad (-2)\,0 \to 320,\\
&(-n)\,1 \to (-1)\,n(n-1)\cdots 320(-2),\\
&1\,(-n) \to (-2)\,023\cdots(n-1)\,n(-1),\\
&(-n+1)(-n) \to n(-1), \qquad (-n)(-n+1) \to (-1)\,n.
\end{aligned}
$$

If $n \geq 4$, we also have, for all integers $2 \leq i \leq n-2$,

$$
\begin{aligned}
i(i+1) &\to (-i-1)(-i-2),\\
(i+1)\,i &\to (-i-2)(-i-1),\\
(-i)(-i-1) &\to (i+1)(i+2),\\
(-i-1)(-i) &\to (i+2)(i+1).
\end{aligned}
$$

In the following, when we say the representation for $y=g_n^k(x)$, we mean the representation obtained, following the procedure as described in Section 4, by applying Lemma 6 to the representation $(-1)n(n-1)(n-2)\cdots 320(-2)(-3)\cdots(-n+2)(-n+1)(-n)1$ for $y=g_n(x)$ successively until we get to the one for $y=g_n^k(x)$.

For every positive integer k and all integers i, j with $1\le|i|\le n$, $1\le|j|\le n$, let $p_{k,i,j,n}$ denote the number of uv's and vu's in the representation for $y=g_n^k(x)$ whose corresponding x-coordinates are in $[i-1, i]$ if $i>0$ or in $[i, i+1]$ if $i<0$, where $uv=02$ if $j=1$, $uv=0(-2)$ if $j=-1$, $uv=j(j+1)$ if $2\le j\le n-1$, $uv=j(j-1)$ if $-n+1\le j\le -2$, $uv=n(-1)$ if $j=n$, and $uv=(-n)1$ if $j=-n$. It is obvious that $p_{1,i,-i,n}=1$ for all integers i with $1\le|i|\le n$ and $p_{1,i,j,n}=0$ otherwise. From Lemma 6, we easily see that the sequences $\langle p_{k,i,j,n}\rangle$ are exactly the same as those defined above.

Since

$$\begin{aligned} q_{k,n} = p_{k,1,1,n} &+ p_{k,1,n,n} + p_{k,1,-n,n} \\ &+ \sum_{i=2}^{n} (p_{k,i,i-1,n} + p_{k,i,n,n}) \\ &+ p_{k,-1,-1,n} + p_{k,-1,-n,n} + p_{k,-1,n,n} \\ &+ \sum_{i=2}^{n} (p_{k,-i,-i+1,n} + p_{k,-i,-n,n}), \end{aligned}$$

it is clear that $q_{k,n}$ is the number of intersection points of the graph of $y=g_n^k(x)$ with the diagonal $y=x$. On the other hand, since

$$\begin{aligned} r_{k,n} = p_{k,1,-1,n} &+ p_{k,1,-n,n} + p_{k,1,n,n} \\ &+ \sum_{i=2}^{n} (p_{k,i,-i+1,n} + p_{k,i,-n,n}) \\ &+ p_{k,-1,1,n} + p_{k,-1,n,n} + p_{k,-1,-n,n} \\ &+ \sum_{i=2}^{n} (p_{k,-i,i-1,n} + p_{k,-i,n,n}), \end{aligned}$$

it is clear that $r_{k,n}$ is the number of intersection points of the graph of $y=g_n^k$ with the diagonal $y=-x$. This proves part (2). Part (3) follows from the standard inclusion-exclusion argument. Part (4) follows easily from the properties of the sequences $\langle p_{k,i,j,n}\rangle$ listed above. This completes the proofs of parts (2), (3), and (4).

Part (5) can also be proved by the method of symbolic representation. We omit the (easy but very technical) details. (see [9] for example).

The proof of Theorem 4 is now complete.

We now consider those continuous odd functions f in $C^0(I, I)$ which have a simple symmetric periodic orbit of the third kind with least period $2n$. We will find the best possible lower bounds on the topological entropy and on the number of symmetric periodic orbits of periods guaranteed in Theorem 1 for such continuous odd functions f. To find these best possible lower bounds, we use the method of symbolic representation. Since the derivation of these results is similar to that of Theorem 4, we will omit it.

Let n be a fixed positive integer. For all integers i and j with $1 \leq |i| \leq 2n$ and $1 \leq |j| \leq 2n$, we define seqnences $\langle u_{k,i,j,n} \rangle$ as follows:

$$
\begin{aligned}
&u_{1,1,-n-1,n} = 1, \\
&u_{1,i,-2n-2+i,n} = 1 \qquad \text{if} \quad 2 \leq i \leq n, \\
&u_{1,i,-2n-1+i,n} = 1 \qquad \text{if} \quad n+1 \leq i \leq 2n, \\
&u_{1,i,j,n} = 0 \qquad \text{elsewhere.}
\end{aligned}
$$

For $1 \leq i \leq 2n$ and all positive integers k,

$$
\begin{aligned}
&u_{k+1,i,j,n} = u_{k,i,-(n+1),n} + u_{k,i,-2n+j-1,n}, && 1 \leq j \leq n-1, \\
&u_{k+1,i,n,n} = u_{k,i,-(n+1),n} + u_{k,i,-n,n}, && \\
&u_{k+1,i,n+1,n} = u_{k,i,-1,n}, && \\
&u_{k+1,i,n+j,n} = u_{k,i,-n+j-2,n}, && 2 \leq j \leq n, \\
&u_{k+1,i,-j,n} = u_{k,i,n+1,n} + u_{k,i,2n+1-j,n}, && 1 \leq j \leq n-1, \\
&u_{k+1,i,-n,n} = u_{k,i,n+1,n} + u_{k,i,n,n}, && \\
&u_{k+1,i,-(n+1),n} = u_{k,i,1,n}, && \\
&u_{k+1,i,-(n+j),n} = u_{k,i,n+2-j,n}, && 2 \leq j \leq n.
\end{aligned}
$$

We also define sequences $\langle v_{k,n} \rangle$ and $\langle w_{k,n} \rangle$ by letting $u_{m,i,j,n} = 0$ if $m \leq 0$, and

$$v_{k,n} = 2\left(2 \sum_{j=1}^{n} u_{k+2-2j,1,j,n} + u_{k,n+1,n,n}\right) + 1,$$

$$w_{k,n} = 2\left(2 \sum_{j=1}^{n} u_{k+2-2j,1,-j,n} + u_{k,n+1,-n,n}\right) + 1.$$

It is easy to see that the sequences $\langle u_{k,1,j,n}\rangle$ and $\langle u_{k,n+1,j,n}\rangle$ have the following three properties which will be used later in the proof of Theorem 5 below.

(i) $u_{2k-1,i,j,n} = u_{2k,i,-j,n} = 0$ for all positive integers i, j, k with $i = 1$ or $n+1$, $1 \le j \le 2n$, and $k \ge 1$.

(ii) $u_{2k,n} = 2^{k+2} - 1$ for all integers $1 \le k \le 2n$.

(iii) Since, for every integer $k > 2n+1$, $i = 1$ or $n+1$, and $1 \le j \le 2n$,

$$u_{2k,i,j,n} - 4u_{2k-2,i,j,n} + 4u_{2k-4,i,j,n} - u_{2k-(4n+2),i,j,n} = 0$$

and

$$u_{2k+1,i,-j,n} - 4u_{2k-1,i,-j,n} + 4u_{2k-3,i,-j,n} - u_{2k+1-(4n+2),i,-j,n} = 0,$$

we have

$$\lim_{k\to\infty} (\log u_{2k,i,j,n})/k = \lim_{k\to\infty} (\log u_{2k+1,i,-j,n})/k = (\log \theta_n)/2,$$

where θ_n is the largest (in absolute value) zero of the polynomial $x^{4n+2} - 4x^{4n} + 4x^{4n-2} - 1$. Note that θ_n is also the (unique) positive zero of the polynomial $x^{2n+1} - 2x^{2n-1} - 1$.

For all positive integers k, m, n, we let $\phi_{5,n}(k) = v_{k,n}$ and $\phi_{6,n}(k) = w_{k,n}$. We also let $\Phi_{5,n}(m) = \Phi_1(m, \phi_{5,n})$ and $\Phi_{6,n}(m) = \Phi_2(m, \phi_{6,n})$, where Φ_1, Φ_2 are defined as in Section 2. Now we can state the following theorem whose proof is similar to that of Theorem 4 and is omitted.

THEOREM 5. *Let f be a continuous odd function in $C^0(I, I)$. Assume that, for some positive integer n, f has a simple symmetric periodic orbit of least period $2(2n+1)$ of the third kind with type "+" ("−" resp.). Then the following hold.*

(1) *For every integer $m \ge n$, f has a simple symmetric periodic orbit of least period $2(2m+1)$ of the third kind with type "+" ("−" resp.).*

(2) *For every positive integer k, the equation $f^k(x)=x$ has at least $v_{k,n}$ (sharp) distinct solutions and the equation $f^k(x)=-x$ has at least $w_{k,n}$ (sharp) distinct solutions.*

(3) *For every positive integer m, f has at least $\Phi_{5,n}(m)/m$ (sharp) distinct periodic orbits of least period m, at least $\Phi_{6,n}(m)/(2m)$ (sharp) distinct symmetric periodic orbits of least period $2m$, and at least $\Phi_{5,n}(2m)/(2m)-\Phi_{6,n}(m)/(2m)$ (sharp) distinct asymmetric periodic orbits of least period $2m$.*

(4) $\lim_{m\to\infty}(\log[\Phi_{5,n}(m)/m])/m=\log\theta_n$,

$$\lim_{m\to\infty}(\log([\Phi_{6,n}(m)/(2m)])/(2m)=(\log\theta_n)/2,$$

and

$$\lim_{m\to\infty}(\log[\Phi_{5,n}(2m)/(2m)-\Phi_{6,n}(m)/(2m)])/(2m)$$
$$=(\log\theta_n)/2,$$

where θ_n is the (unique) positive (and the largest in absolute value) zero of the polynomial $x^{2n+1}-2x^{2n-1}-1$.

(5) *The topological entropy of f is greater than or equal to $\log\theta_n$ (sharp), where θ_n is defined as in* (4) *above.*

REMARK 8. For the same reason as in Remark 7, the collection $\{\langle u_{k,1,j,n}\rangle, \langle u_{k,n+1,j,n}\rangle \mid 1\le j\le 2n\}$ of sequences and the collection $\{\langle b_{k,1,j,n}\rangle, \langle b_{k,n+1,j,n}\rangle \mid 1\le j\le 2n\}$ of sequences defined in Theorem 2 of [7] can be said to be conjugate to each other. Similar identities as those stated in Remark 7 also hold here for these two collections.

We can now use the the results obtained so far to find the best possible lower bounds on the topological entropy and on the number of symmetric and asymmetric periodic orbits of periods guaranteed in Theorem 1 for those continuous odd functions in $C^0(I, I)$ which have a symmetric periodic orbit of even period ≥ 4.

THEOREM 6. *Let f be a continuous odd function in $C^0(I, I)$ which has a symmetric periodic orbit of least period $4n$ for some positive integer n. Then the topological entropy of f is greater than or equal to $\log\lambda_{2n}$ (sharp), where λ_{2n} is defined as in Theorem* 3. *Furthermore, at least one of the following holds.*

(A) *The following four statements hold.*

(1) *For every integer* $m \geq 2n$, f *has a simple symmetric periodic orbit of the first kind with least period* $2m$.

(2) *For every positive integer* k, $A(k)$ *and* $ANP(4n+k-1)$ *hold for* f.

(3) *For every positive integer* k, *the equation* $f^k(x)=x$ *has at least* $c_{k,2n}$ *(sharp) distinct solutions and the equation* $f^k(x)=-x$ *has at least* $d_{k,2n}$ *(sharp) distinct solutions, where* $c_{k,2n}$ *and* $d_{k,2n}$ *are defined as in Theorem* 3.

(4) *For every positive integer* m, f *has at least* $\Phi_{1,2n}(m)/m$ *(sharp) distinct periodic orbits of least period* m, *at least* $\Phi_{2,2n}(m)/(2m)$ *(sharp) distinct symmetric periodic orbits of least period* $2m$, *and at least* $\Phi_{1,2n}(2m)/(2m)-\Phi_{2,2n}(m)/(2m)$ *(sharp) distinct asymmetric periodic orbits of least period* $2m$, *where* $\Phi_{1,2n}$ *and* $\Phi_{2,2n}$ *are defined as in Theorem* 3.

(B) *The following four statements hold.*

(1) *For every even integer* $m \geq n$, f *has a simple symmetric periodic orbit of the second kind with least period* $2m$ *and a simple ANP periodic orbit of least period* $m+1$.

(2) *For every positive integer* k, $ANP(2k)$ *holds for* f.

(3) *For every positive integer* k, *the equation* $f^k(x)=x$ *has at least* $p_{k,2n}$ *(sharp) distinct solutions and the equation* $f^k(x)=-x$ *has at least* $q_{k,2n}$ *(sharp) distinct solutions.*

(4) *For every positive integer* m, f *has at least* $\Phi_{3,2n}(m)/m$ *(sharp) distinct periodic orbits of least period* m, *at least* $\Phi_{4,2n}(m)/(2m)$ *(sharp) distinct symmetric periodic orbits of least period* $2m$, *and at least* $\Phi_{3,2n}(2m)/(2m)-\Phi_{4,2n}(m)/(2m)$ *(sharp) distinct asymmetric periodic orbits of least period* $2m$, *where* $\Phi_{3,2n}$, $\Phi_{4,2n}$ *are defined as in Theorem* 4.

Proof. Without loss of generality, we may assume that f has no symmeteric periodic orbit of least period $4(n-1)\geq 4$. By Theorem 2, we have two cases to consider. If part (1) of Theorem 2 holds, then, by Theorem 3, part (A) follows. If part (2) of Theorem 2 holds, then, by Theorem 4 and Corollary 2, part (B) follows. This completes the proof.

THEOREM 7. *Let* f *be a continuous odd function in* $C^0(I, I)$

which has a symmetric periodic orbit of least period $2(2n+1)$ *for some positive integer* n. *Then at least one of the following holds.*

(A) *The following five statements hold.*

(1) *For every integer* $m \geq 2n+1$, f *has a simple symmetric periodic orbit of the first kind with least period* $2m$.

(2) *For every positive integer* k, $A(k)$ *and* $ANP\,(2(2n+1)+k-1)$ *hold for* f.

(3) *For every positive integer* k, *the equation* $f^k(x) = x$ *has at least* $c_{k,2n+1}$ *(sharp) distinct solutions and the equation* $f^k(x) = -x$ *has at least* $d_{k,2n+1}$ *(sharp) distinct solutions, where* $c_{k,2n+1}$ *and* $d_{k,2n+1}$ *are defined as in Theorem* 3.

(4) *For every positive integer* m, f *has at least* $\Phi_{1,2n+1}(m)/m$ *(sharp) distinct periodic orbits of least period* m, *at least* $\Phi_{2,2n+1}(m)/(2m)$ *(sharp) distinct symmetric periodic orbits of least period* $2m$, *and at least* $\Phi_{1,2n+1}(2m)/(2m) - \Phi_{2,2n+1}(m)/(2m)$ *(sharp) distinct asymmetric periodic orbits of least period* $2m$, *where* $\Phi_{1,2n+1}$ *and* $\Phi_{2,2n+1}$ *are defined as in Theorem* 3.

(5) *The topological entropy of* f *is greater than or equal to* $\log \lambda_{2n+1}$, *where* λ_{2n+1} *is defined as in Theorem* 3.

(B) *The following five statements hold.*

(1) *At least one of the following holds.*

(a) *For every integer* $m \geq n$, f *has a simple symmetric periodic orbit of least period* $2(2m+1)$ *of the third kind with type* " + ".

(b) *For every integer* $m \geq n$, f *has a simple symmetric periodic orbit of least period* $2(2m+1)$ *of the third kind with type* " − ".

(2) *For every positive integer* k, $ANP(2k)$ *holds for* f.

(3) *For every positive integer* k, *the equation* $f^k(x) = x$ *has at least* $v_{k,n}$ *(sharp) distinct solutions and the equation* $f^k(x) = -x$ *has at least* $w_{k,n}$ *(sharp) distinct solutions, where* $v_{k,n}$ *and* $w_{k,n}$ *are defined as in Theorem* 5.

(4) *For every positive integer* m, f *has at least* $\Phi_{5,n}(m)/m$ *(sharp) distinct periodic orbits of least period* m, *at least* $\Phi_{6,n}(m)/(2m)$ *(sharp) distinct symmetric periodic orbits of*

least period $2m$, and at least $\Phi_{5,n}(2m)/(2m) - \Phi_{6,n}(m)/(2m)$ (sharp) distinct asymmetric periodic orbits of least period $2m$, where $\Phi_{5,n}$, $\Phi_{6,n}$ are defined as in Theorem 5.

(5) *The topological entropy of f is greater than or equal to* $\log \theta_n$, *where θ_n is defined as in Theorem* 5.

Proof. Without loss of generality, we may assume that f has no symmetric periodic orbit of least period $2(2n-1) \geq 6$. By Theorem 2, we have two cases to consider. If part (1) of Theorem 2 holds, then, by Theorem 3, part (A) follows. If part (3) of Theorem 2 holds, then, by Theorem 5, part (B) follows. This completes the proof.

8. **Continuous odd functions for which $ANP(2n+1)$ holds.** Let f be a continuous odd function in $C^0(I, I)$. Assume that f has a periodic orbit of least period $2n+1$ for some positive integer n which contains both negative and positive elements. From part (v) of Theorem 1, we see that f has a symmetric periodic orbit of least period $4n+4$. Consequently, we obtain, by Theorem 6, lower bounds on the topological entropy and on the number of symmetric and asymmetric periodic orbits of periods guaranteed in Theorem 1 for such function f. However, these lower bounds are not best possible. In this section, we will use the results obtained in Sections 5, 6, and 7 to obtain the beast possible lower bounds.

THEOREM 8. *Let f be a continuous odd function in $C^0(I, I)$. Assume that, for some positive integer n, f has a periodic orbit of least period $2n+1$ which contains both negative and positive elements. Then at least one of the following holds.*

(A) *The following five statements hold.*

(1) *For every integer $m \geq n$, f has a simple symmetric periodic orbit of the first kind with least period $2m$.*

(2) *For every positive integer k, $A(k)$ and $ANP(2n+k-1)$ hold for f.*

(3) *For every positive integer k, the equation $f^k(x) = x$ has at least $c_{k,2n}$ (sharp) distinct solutions and the equation*

$f^k(x) = -x$ has at least $d_{k,2n}$ (sharp) distinct solutions, where $c_{k,2n}$ and $d_{k,2n}$ are defined as in Theorem 3.

(4) *For every positive integer m, f has at least $\Phi_{1,2n}(m)/m$ (sharp) distinct periodic orbits of least period m, at least $\Phi_{2,2n}(m)/(2m)$ (sharp) distinct symmetric periodic orbits of least period $2m$, and at least $\Phi_{1,2n}(2m)/(2m) - \Phi_{2,2n}(m)/(2m)$ (sharp) distinct asymmetric periodic orbits of least period $2m$, where $\Phi_{1,2n}$ and $\Phi_{2,2n}$ are defined as in Theorem* 3.

(5) *The topological entropy of f is greater than or equal to* $\log \lambda_{2n}$ *(sharp), where λ_{2n} is defined as in Theorem* 3.

(B) *The following five statements hold.*

(1) *For every integer $m \geq n$, f has a simple symmetric periodic orbit of the second kind with least period $4m + 4$ and a simple ANP periodic orbit of least period $2m + 1$.*

(2) *For every positive integer k, ANP$(2k)$ holds for f.*

(3) *For every positive integer k, the equation $f^k(x) = x$ has at least $q_{k,2n+1}$ (sharp) distinct solutions and the equation $f^k(x) = -x$ has at least $r_{k,2n+1}$ (sharp) distinct solutions, where $q_{k,2n+1}$ and $r_{k,2n+1}$ are defined as in Theorem* 4.

(4) *For every positive integer m, f has at least $\Phi_{3,2n+1}(m)/m$ (sharp) distinct periodic orbits of least period m, at least $\Phi_{4,2n+1}(m)/(2m)$ (sharp) distinct symmetric periodic orbits of least period $2m$, and at least $(\Phi_{3,2n+1}(2m)/(2m)) - (\Phi_{4,2n+1}(m)/(2m))$ (sharp) distinct asymmetric periodic orbits of least period $2m$, where $\Phi_{3,2n+1}$ and $\Phi_{4,2n+1}$ are defined as in Theorem* 4.

(5) *The topological entropy of f is greater than or equal to* $\log \lambda_{2n+1}$, *where λ_{2n+1} is defined as in Theorem* 3.

Proof. Let P be a periodic orbit of f of least period $2n + 1$ which contains both negative and positive elements. Let $b = \max\{|x| \mid x \in P\}$ and let y be the unique element in P which is closest to the origin. Without loss of generality, we may assume that $y > 0$, $I = [-b, b]$, and I has no asymmetric periodic orbit of f of least period m with m odd and $1 < m < 2n + 1$ which contains both negative and positive elements. We have two cases to consider.

If $[y, b]$ contains a fixed point of f, then by Theorem 1 (vi), f has a symmetric periodic orbit of least period $4n$. By Theorem 6, f has a simple symmetric periodic orbit Q of least period $4n$ of the first kind or the second kind. If Q is of the first kind, then part (A) follows. If Q is of the second kind, then part (B) follows.

If $[y, b]$ contains no fixed point of f, then we define a continuous odd function h in $C^0(I, I)$ as follows. If the interval $(0, y)$ contains no fixed point of f, then let $h = f$ on I. If the interval $(0, y)$ contains a fixed point of f, let z be the largest fixed point of f in $(0, y)$ and let $h(x) = f(x)$ on $[-b, -z] \cup [z, b]$ and $h(x) = x$ on $[-z, z]$. By Theorem A (Sharkovskii's theorem), h has no periodic orbit of least period $2n - 1 \geq 3$ whose elements are all negative or all positive. Consequently, by Lemma 4, $-h$ has no symmetric periodic orbit of least period $2(2n-1) \geq 6$. So, by Lemma 4, $P \cup \{-x \mid x \in P\}$ is a minimal symmetric periodic orbit of $-h$ with least period $2(2n+1)$. From the definition of h, it is clear that $P \cup \{-x \mid x \in P\}$ is also a minimal symmetric periodic orbit of $-f$. By Theorem 2, $-f$ must have a simple symmetric periodic orbit of the first kind with least period $2(2n+1)$. By Theorem 3, part (B) follows.

9. **Perturbations of continuous odd functions with symmetric periodic orbits.** In this section, we study perturbations of those continuous odd functions f which have some symmetric periodic orbits of least period $2n \geq 4$. If f is in $C^0(I, I)$ and n is even (odd resp.), we show (Theorem 9) that there is a neighborhood V of f in $C^0(I, I)$ such that every continuous odd function in V has a symmetric periodic orbit of least period $2n + 2k$ ($2k + 4k$ resp.) for every positive integer k. On the other hand, it is easily seen that every neighborhood of f in $C^0(I, I)$ (in particular, V above) contains continuous odd functions with symmetric periodic orbits of all even periods. This implies that although those C^0-perturbations of f which are also odd preserve the periods following $2n$ in the ordering defined in (*) or in (**) in Section 1

as the periods of some symmetric periodic orbits, certain perturbations of f produce (possibly infinitely) many additional periods.

However, if f is in $C^1(I, I)$ with the C^1 metric d defined by $d(g, h) = \max\{|g(x) - h(x)| + |g'(x) - h'(x)| \,|\, x \in I\}$ and if f has no symmetric periodic orbit of least period $2m$ for some positive integer m, then we show (Theorem 10) that there is a neighborhood W of f in $C^1(I, I)$ such that every continuous odd function in W has no symmetric periodic orbit of least period $2m$. Therefore, as far as continuous odd functions and the periods of symmetric periodic orbits are concerned, the perturbation phenomenon in $C^1(I, I)$ is quite different from that in $C^0(I, I)$.

We first study C^0-perturbations.

THEOREM 9. *Let f be a continuous odd function in $C^0(I, I)$ which has a symmetric periodic orbit P of least period $2n$ for some integer $n \geq 2$. Then there is a neighborhood V of f in $C^0(I, I)$ such that the following hold.*

(i) *If P is a simple symmetric periodic orbit of the first kind, then for every integer $m > n$ and every continuous odd function g in V, g has a simple symmetric periodic orbit of the first kind with least period $2m$.*

(ii) *If P is a simple symmetric periodic orbit of the second kind, then n is even and, for every even integer $m > n$ and every continuous odd function g in V, g has a simple symmetric periodic orbit of the second kind with least period $2m$ and a simple ANP periodic orbit of least period $m - 1$.*

(iii) *If P is a simple symmetric periodic orbit of the third kind with type "+" (type "−" resp.), then n is odd and, for every odd integer $m > n$ and every continuous odd function g in V, g has a simple symmetric periodic orbit of the third kind with least period $2m$ and the same type as P.*

Proof. Let $P = \{\pm x_i \,|\, 1 \leq i \leq n\}$ with $0 < x_1 < x_2 < \cdots < x_n$ be a simple symmetric periodic orbit of f of the first kind with least period $2n$. Then $f^i(x_1) = x_{i+1}$ for all $1 \leq i \leq n-1$, and $f^n(x_1) = -x_1$. Since $f([0, x_1]) \supset [0, x_2]$, there is a point $y \in (0, x_1)$ such that $f(y) = x_1$. So, $f^{n+1}(y) = -x_1 < -y < 0 < y < f(y) < f^2(y) < \cdots < f^n(y)$. Consequently, there is a neighborhood V of f in $C^0(I, I)$ such that, for every continuous odd function g in V, $g^{n+1}(y) < -y < y < g(y) < g^2(y) < \cdots < g^n(y)$. Let $J_0 = [0, y]$, $J_i = [g^{i-1}(y), g^i(y)]$ for all $1 \leq i \leq n$. By applying Lemma 3 to the path $J_0 J_1 J_2 \cdots J_n(-J_0)$ of length $n+1$, we obtain that g has a simple symmetric periodic orbit of the first kind with least period $2n+2$. The general case follows from inductive argument. This proves part (i).

By applying Lemma 4 to part (i), we obtain part (ii). As for part (iii), we can apply Lemma 4 to Theorem A in [4]. We omit the details. This completes the proof of the theorem.

The following result is now an easy consequence of the above theorem and Theorem 2.

COROLLARY 3. *Let f be a continuous odd function in $C^0(I, I)$ with a symmetric periodic orbit of least period $2n$ for some integer $n \geq 2$. Then there is a neighborhood V of f in $C^0(I, I)$ such that if n is even (odd resp.), then every continuous odd function in V has a symmetric periodic orbit of least period $2n+2k$ ($2n+4k$ resp.) for every positive integer k.*

We now study C^1-perturbations. The following easy lemma [3] is needed.

LEMMA 7. *Let f be a continuous odd function in $C^1(I, I)$ and let $\{\pm p_i \mid 1 \leq i \leq k\}$ be a symmetric periodic orbit of f of least period $2k$ with $k \geq 3$ and $0 < p_1 < p_2 < \cdots < p_k$. Then there are points y and z in the interval $[-p_k, p_k]$ with $f'(y) > 0$ and $f'(z) \leq -1$. Furthermore, if $k \geq 3$ is odd and $f(p_i) < 0$ for all $1 \leq i \leq k$, then there are points u and v in the interval $[p_1, p_k]$ such that $f'(u) < 0$ and $f'(v) \geq 1$.*

THEOREM 10. *Let f be a continuous odd function in $C^1(I, I)$ which has no symmetric periodic orbit of least period $2m$ for some* *positive integer m. Then there is a neighborhood W of f in $C^1(I, I)$ such that every continuous odd function in W has no symmetric periodic orbit of least period 2m.*

REMARK 9. The above result shows that, as far as continuous odd functions and the periods of symmetric periodic orbits are concerned, the C^1-perturbation is better than the C^0-perturbation in the sense that we can more or less recover from the C^1-perturbation the periods of the symmetric periodic orbits of the original unperturbed function while in the C^0-perturbation we can not.

Proof of Theorem 10. Let ε_1 and ε_2 be two positive numbers with $\varepsilon_2 < \varepsilon_1$ such that (i) $|f'(x) - f'(y)| < \frac{1}{4}$ whenever $x, y \in I$ and $|x - y| < \varepsilon_1$, and (ii) $|f^i(x) - f^i(y)| < \varepsilon_1/4$ for all $1 \leq i \leq 2m$, whenever $x, y \in I$ and $|x - y| < \varepsilon_2$. It is clear that there is an open subset S of I containing the closed set $E = \{x \in I \mid f(x) = -x\}$ such that the Lebesgue measure of the set $S - E$ is strictly less than $\varepsilon_2/2$ and each open component of S contains at least one point of E. Choose a positive number ε_3 with $\varepsilon_3 < \frac{1}{8}$ such that every continuous odd function g in $C^1(I, I)$ with $d(f, g) < \varepsilon_3$ satisfies the following two conditions:

(a) $|f^i(x) - g^i(x)| < \varepsilon_1/4$ for all $1 \leq i \leq 2m$ and all $x \in I$.

(b) The graph of $y = g^m(x)$ has no intersection point in $I - S$ with the line $y = -x$.

Let g be a continuous odd function in $C^1(I, I)$ with $d(f, g) < \varepsilon_3$. Suppose that g has a symmetric periodic orbit Q of least period $2m$. From the choice of ε_3, this orbit Q must be contained entirely in S. If $y \in Q$, then there is a point z in E such that $|y - z| < \varepsilon_2/2$. So, $|f^i(z) - g^i(y)| \leq |f^i(z) - f^i(y)| + |f^i(y) - g^i(y)| < \varepsilon_1/4 + \varepsilon_1/4 = \varepsilon_1/2$ for all $1 \leq i \leq 2m$. If $z = 0$, then the orbit Q is contained entirely in a subinterval J_1 of I with length strictly less than ε_1. If $z \neq 0$, then since $f(z) = -z$, each half (i.e., $\{x \in Q \mid x > 0\}$ and $\{x \in Q \mid x < 0\}$) of the orbit Q is contained entirely in a subinterval J_2 of I with length strictly less than ε_1. In any case, each half of the orbit Q is contained entirely in a subinterval J of I with length strictly less than ε_1. But then, from the choice of ε_1, we have, for all $x, y \in J$,

$$
\begin{aligned}
|g'(x) - g'(y)| \le |g'(x) - f'(x)| & \\
+ |f'(x) - f'(y)| + |f'(y) - g'(y)| & \\
< \varepsilon_3 + \frac{1}{4} + \varepsilon_3 < \frac{1}{2}. &
\end{aligned}
$$

This contradicts Lemma 7. Therefore, every continuous odd function g in $C^1(I, I)$ with $d(f, g) < \varepsilon_3$ cannot have a symmetric periodic orbit of least period $2m$. This completes the proof of the theorem.

The following result is in contrast to the above theorem.

THEOREM 11. *There is a continuous odd function f in $C^1(I, I)$ which has the following two properties:*

(a) *f satisfies $S(6)$, but not $S(4m)$ or $A(2m+1)$ for any positive integer m.*

(b) *For every neighborhood W of f in $C^1(I, I)$, there is a continuous odd function in W which has symmetric periodic orbits of least period $4m$ and asymmetric periodic orbits of least period $2m+1$ for every sufficiently large positive integer m.*

Proof. For simplicity, let $I = [-1, 1]$. Let δ be a fixed number with $0 < \delta < \frac{1}{8}$ and let $f \in C^1(I, I)$ satisfy the following six conditions:

(i) $f(-x) = -f(x)$ for all $-1 \le x \le 1$.

(ii) $f([0, 1]) = [-1, 0]$.

(iii) $f(x) > -x$ for all $-\delta \le x < 0$.

(iv) f is strictly decreasing on $[-\delta, 0]$.

(v) $f(-\delta) = 1$.

(vi) $f(x) = 1 + x$ for all $-1 \le x \le -1 + 2\delta$.

Then it is clear that f has a symmetric periodic orbit of least period 6, but no symmetric periodic orbit of least period $4m$ and no asymmetric periodic orbit of odd period $2m+1$ for any positive integer m.

For any positive real number $\varepsilon < \delta$, let g_ε be a continuous odd function in $C^1(I, I)$ defined by $g_\varepsilon(x) = f(x)$ for $-1 + \delta \le x \le 0$ $g_\varepsilon(x) = f(x) - \varepsilon(x + 1 - \delta)^2$ for $-1 \le x \le -1 + \delta$. Then g_ε is in the ε-neighborhood of f in $C^1(I, I)$. However, for every sufficiently large odd integer k, there is a point x_0 (depending on k) which is close to the origin such that $-\varepsilon\delta^2 < x_0 < 0 < f(x_0)$, $f^i(x_0) = (-1)^{i+1}|f^i(x_0)|$ and $|f^{i-1}(x_0)| < |f^i(x_0)| \leq \delta$ for all $1 \leq i \leq k-3$, and $f^{k-2}(x_0) = \delta$. Since $g_\varepsilon = f$ on $[-1+\delta, 1]$, we have

$$\begin{aligned} g_\varepsilon^k(x_0) &= g_\varepsilon^2(g_\varepsilon^{k-2}(x_0)) = g_\varepsilon^2(f^{k-2}(x_0)) \\ &= g_\varepsilon^2(\delta) = g_\varepsilon(-1) = -\varepsilon\delta^2 < x_0 < 0 < f(x_0) = g_\varepsilon(x_0). \end{aligned}$$

Let $J_0 = [-1, -\delta]$, $J_1 = [x_0, 0]$, $J_{2i} = [-f^{2i-2}(x_0), f^{2i-1}(x_0)]$ and $J_{2i+1} = [f^{2i}(x_0), -f^{2i-1}(x_0)]$ for all $1 \leq i \leq (k-1)/2$. By applying Lemma 3 to $J_0 J_1 J_2 \cdots J_k(-J_0)$, we obtain a simple symmetric periodic orbit of f of the second kind with least period $2k+2$. It then follows from Theorem 4 that f has symmetric periodic orbits of least period $2k+2+4j$ and asymmetric periodic orbits of least period $k+2+2j$ for every positive integer j. This completes the proof of the theorem.

For continuous odd functions in $C^0(I, I)$ for which $ANP(2n+1)$ holds for some positive integer n, we also have a similar result on C^0-perturbation which is an easy consequence of Theorems 1, 8, and 9.

COROLLARY 4. *Let f be a continuous odd function in $C^0(I, I)$. Assume that f satisfies $ANP(2n+1)$ for some positive integer n. Then there is a neighborhood S of f in $C^0(I, I)$ such that, for every continuous odd function g in S, g satisfies $S((4n+4)+4m)$, $ANP(2n+1+2m)$, and $A(2m)$ for all positive integers m.*

For perturbations of continuous odd functions without any symmetric periodic orbit, see [1], [3], and [11].

REFERENCES

1. Louis Block, *Stability of periodic orbits in the theorem of Šarkovskii*, Proc. Amer. Math. Soc., 81 (1981), 333-336.

2. L. Block, J. Guckenheimer, M. Misiurewicz and L.-S. Young, *Periodic points and topological entropy of one dimensional maps.* 18-34 (Lecture Notes in Mathematics, 819, Springer-Verlag, Berlin, Heidelberg, New York, 1980).

3. Louis Block and David Hart, *The bifurcation of periodic orbits of one-dimensional maps*, Ergod. Th. & Dynam. Sys., 2 (1982), 125-129.

4. Louis Block and David Hart, *Stratification of the space of unimodal interval maps*, Ergod. Th. & Dynam Sys., 3 (1983), 533-539.

5. Bodil Branner, *Iterations by odd functions with two extrema*, J. Math. Anal. Appl., 105 (1985), 276-297.

6. U. Burkart, *Interval mapping graphs and periodic points of continuous functions*, J. Combin. Theory Ser. B, 32 (1982), 57-68.

7. Bau-Sen Du, *The minimal number of periodic orbits of periods guaranteed in Sharkovskii's theorem*, Bull. Austral. Math. Soc., 31 (1985), 89-103.

8. Bau-Sen Du, *Topological entropy and chaos of inerval maps*, Nonlinear Analysis 11 (1987), 105-114.

9. C.-W. Ho and C. Morris, *A graph theoretic proof of Sharkovsky's theorem on the periodic points of continuous functions*, Pacific J. Math., 96 (1981), 361-370.

10. Tien-Yien Li and James A. Yorke, *Period three implies chaos*, Amer. Math. Monthly, 82 (1975), 985-992.

11. M. Misiurewicz, *Structure of mappings of an interval with zero entropy*, Inst. Hautes Etudes Sci. Publ. Math., 53 (1981), 5-16.

12. Motosige Osikawa and Yoshitsugu Oono, *Chaos in c^0-endomorphism of interval*, Publ. RIMS, Kyoto Univ. 17., (1981), 165-177.

13. A. N. Sharkovskii, *Coexistence of cycles of a continuous map of line into itself*, Ukrain Mat. Ž., 16 (1964), 61-71.

14. P. Stefan, *A theorem of Sharkovsky on the existence of periodic orbits of continuous endomorphisms of the real line*, Comm. Math. Phys., 54 (1977), 237-248.

15. P. D. Straffin, Jr., *Periodic points of continuous functions*, Math. Mag., 51 (1978), 99-105.

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