

AN A PRIORI ESTIMATE ON THE BOUNDARY OF SMALL ADMISSIBLE DOMAIN

BY

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1. Introduction. The purpose of this paper is to prove an a priori estimate on the boundary of small admissible domain along the line developed by M. Kuranishi [1] [2]. More precisely, let M be a manifold of real dimension $2n - 1$ with a given strongly pseudoconvex CR-structure. Let p_0 be a point of M . Suppose that there exists an admissible distance function t (Definition 2.23) on M with p_0 as the reference point. Set

$$b\Omega_r = \{p \in M \mid t(p) = r\}$$

and

$$b\Omega'_r = b\Omega_r - \{p \in M \mid t(p) = r \text{ and } b(p) = 0\},$$

where $b^2 = \sum_j |\sigma_j|^2$, $\sigma_j = Y_j t$ and Y_1, \dots, Y_{n-1} is an orthonormal base of the given CR-structure with respect to the Levi metric. Hence $b\Omega'_r$ can be viewed as a manifold with induced CR-structure which is of codimension two. We can thus define the Cauchy-Riemann operator D_b on $b\Omega'_r$. Let u be a q -form on $b\Omega'_r$ such that (i) u is C^1 on $b\Omega'_r$, and (ii) $D_b u$, $D_b^* u$, $b^{-1} u$ and $W_j u$ are in L^2 , then we prove

$$\begin{aligned} \|D_b u\|^2 + \|D_b^* u\|^2 &\geq \frac{n-2-q}{n-2} \sum_j \|W_j u\|^2 \\ &\quad + \frac{q}{n-2} \sum_j \|\tilde{W}_j u\|^2 \\ &\quad + \left(\frac{q(n-2)^2 - q^2}{n-2} - (q-1)q \right) \left\| \frac{ru}{b} \right\|^2, \end{aligned}$$

Received by the editors April 7, 1986.

* After the author had completed this paper, he was informed that T. Akahori had obtained a similar theorem and used that estimate to prove the local embedding of strongly pseudoconvex CR-structure.

where W_j 's are vector fields tangent to the level set of t and $\tilde{W}_j = \sum_k Q_{jk} W_k^*$, provided r is sufficiently small.

We note that this estimate is stronger than the L^2 -estimate and we hope that this gain could give us more information to overcome the difficulty caused by the existence of two characteristic points. Hence if $B_1 \cap B_2$ is the intersection of two balls B_1 and B_2 in C^n , we also wish that this estimate can be used to study the behavior of the canonical solution of the $\bar{\partial}$ -problem around the "corner", the intersection of bB_1 and bB_2 .

The author would like to thank Prof. J. J. Kohn for suggesting this problem and helpful conversations during the preparation of this paper.

2. Preliminaries. In this section we will review some definitions which had been set up by Kuranishi in [1] and [2].

Let M be a real smooth manifold of dimension $2n - 1$. Denote by CTM the bundle of complex tangent vectors of M . If E is a smooth integrable subbundle of CTM , namely, for any $X, Y \in C^\infty E$, $[X, Y] \in C^\infty E$, where $C^\infty E$ denotes the vector space of smooth sections of E , then we can define the complex of differential operators D associated with E

$$(2.1) \quad D : C^\infty \Lambda^q E \longrightarrow C^\infty \Lambda^{q+1} E, \quad \text{for } q = 0, 1, 2, \dots,$$

where $\Lambda^q E$ is the vector bundle of q -multilinear alternating functions on the fibre of CTM . D is defined as follows, for $u \in C^\infty \Lambda^q E$ and $X_0, \dots, X_q \in C^\infty E$,

$$(2.2) \quad \begin{aligned} Du(X_0, \dots, X_q) &= \sum_{i=0}^q (-1)^i X_i u(X_0, \dots, \hat{X}_i, \dots, X_q) \\ &+ \sum_{0 \leq i < j \leq q} (-1)^{i+j} u([X_i, X_j], X_0, \dots, \hat{X}_i, \dots, \hat{X}_j, \dots, X_q), \end{aligned}$$

where $\hat{\cdot}$ means that the variable is omitted. On the other hand, for $u \in \Lambda^p E$ and $v \in \Lambda^q E$, we have the exterior product $u \wedge v \in \Lambda^{p+q} E$, so that $\Lambda^* E = \sum_q C^\infty \Lambda^q E$ forms an algebra, and the following formulas hold

$$(2.3) \quad u \wedge v = (-1)^{pq} v \wedge u, \quad \text{for any } u \in \Lambda^p E \text{ and } v \in \Lambda^q E,$$

$$(2.4) \quad D(u \wedge v) = (Du) \wedge v + (-1)^p u \wedge Dv.$$

Consider a fixed C^∞ volume element on M , then we can define L^2 -norm and inner product which will be denoted by $\| \cdot \|$ and $\langle \cdot, \cdot \rangle$ respectively. Pick a C^∞ hermitian metric on the vector bundle E . Let Y_1, \dots, Y_m be a C^∞ orthonormal base of E on an open subset M_1 . For the multiindex $K = (k_1, \dots, k_q)$, where $k_a = 1, \dots, m$, we set

$$(2.5) \quad u_K = u_{k_1, \dots, k_q} = u(Y_{k_1}, \dots, Y_{k_q}), \quad \text{for } u \in C^\infty \Lambda^q E|_{M_1}.$$

Hence u_K is considered as a C^∞ function on M_1 , and set

$$(2.6) \quad \|u\|^2 = \sum_K \frac{1}{q!} \|u_K\|^2.$$

If $\|u\|$ and $\|v\|$ are finite, define $\langle u, v \rangle = \sum_K (1/q!) \langle u_K, v_K \rangle$. Since $\|u\|$ is defined independent of the choice of the orthonormal base, it is not hard to see that they are globally defined.

Next we define the interior product $u \lrcorner \beta \in C^\infty \Lambda^{q-p} E$, for $u \in C^\infty \Lambda^q E$ and $\beta \in C^\infty \Lambda^p E$, by duality:

$$(2.7) \quad \langle u \lrcorner \beta, v \rangle = \langle u, \beta \wedge v \rangle,$$

for all $v \in C^\infty \Lambda^{q-p} E$. Then it follows that for $v \in \Lambda^p E$ and $\beta \in \Lambda^1 E$, we have

$$(2.8) \quad (v \wedge u) \lrcorner \beta = (v \lrcorner \beta) \wedge u + (-1)^p v \wedge (u \lrcorner \beta).$$

Let e_1, \dots, e_m be the orthonormal base of $\Lambda^1 E = E^*$ dual to Y_1, \dots, Y_m , and $g \in \text{Hom}(\Lambda^1 E, \Lambda^1 E)$. We define $[g]_q \in \text{Hom}(\Lambda^q E, \Lambda^q E)$ by

$$(2.9) \quad [g]_q u = \sum_k (ge_k) \wedge (u \lrcorner e_k).$$

Clearly this definition is independent of the choice of the orthonormal base. Denote by $[g]^*([g]_q)^*$ the adjoint of $[g]_q$ respectively, then we have

$$(2.10) \quad [g]_q^* = [g^*]_q.$$

Hence for simplicity we write

$$gu = [g]_q u, \quad \text{for } u \in \Lambda^q E.$$

Then the following holds

$$(2.11) \quad g(v \wedge u) = (gv) \wedge u + v \wedge gu,$$

$$(2.12) \quad g(u \llcorner v) = (gu) \llcorner v - u \llcorner g^* v,$$

$$(2.13) \quad (\text{id}) \quad u = qu. \quad \text{for } u \in \Lambda^q E,$$

where id means the identity map. Define $r_{(k)} \in \text{Hom}(\Lambda^1 E, \Lambda^1 E)$, $k = 1, \dots, m$ by

$$(2.14) \quad r_{(k)} e_l = \sum_j r_{kjl} e_j, \quad \text{where } [Y_j, Y_k] = \sum_l r_{jkl} Y_l.$$

For a linear map G of the vector space of complex C^∞ functions on M_1 into itself, we let G operate on $C^\infty \Lambda^q E|_{M_1}$, by

$$(2.15) \quad (Gu)_K = Gu_K.$$

The operation depends on the choice of Y_1, \dots, Y_m . Then we have

PROPOSITION 2.16. *For $u \in C^\infty \Lambda^q E|_{M_1}$, we have*

$$(2.17) \quad Du = \sum_k e_k \wedge Y_k u - \frac{1}{2} e_k \wedge r_{(k)} u,$$

$$(2.18) \quad D^* u = \sum_k Y_k^*(u \llcorner e_k) - \frac{1}{2} r_{(k)}^*(u \llcorner e_k),$$

where D^* is the formal adjoint of D defined by duality.

Proof. See [1].

From now on we will assume that $E \cap \bar{E} = \{0\}$, and the CR-structure defined by E is strongly pseudoconvex. That is, pick a fixed real vector field S such that

$$(2.19) \quad CTM = \bar{E} \oplus E \oplus CS,$$

and the hermitian form on E given by the matrix C_s

$$(2.20) \quad [X, \bar{Y}] = iC_s(X, Y) S, \quad \text{mod}(E + \bar{E})$$

is positive definite if we choose the suitable S . The hermitian form is called the Levi form of E . Since the matrix of the Levi form is positive definite, it defines a metric on E . From now on we will use this particular metric.

Denote by π and $\tilde{\pi}$ the projection of CTM onto E and \bar{E} with respect to the decomposition of (2.19). Let f be a C^∞ function on

M . Define the E hessian of f by (with respect to $F = CS$)

$$(2.21) \quad H_F^f(X, Y) = X\bar{Y}f - (\bar{\pi}[X, \bar{Y}])f, \quad \text{for } X, Y \in C^\infty E.$$

It is easy to see that for any C^∞ function g on M , we have

$$(2.22) \quad H_F^f(gX, Y) = H_F^f(X, gY) = gH_F^f(X, Y).$$

Next we introduce the definition of admissible distance function.

DEFINITION 2.23. A C^∞ function t defined on M is called an admissible distance function to a point p_0 in M when the following conditions are satisfied

- (i) $t(p) \geq 0$, for all p in M , $t(p) = 0$ if and only if $p = p_0$,
- (ii) the gradient of t at p is zero if and only if $p = p_0$,
- (iii) $H_s^t(X, Y) = r C_s(X, Y)$ for a C^∞ function r with $r(p_0) \neq 0$,
- (iv) if $X, Y \in C^\infty E$, then $XY(t) = 0$ at p_0 .

If M is embedded in C^n , then the admissible distance function t can be constructed as follows. Let $p_0 = 0$ be the origin and ρ be the defining function of M . Then locally there exists a coordinate chart (z_1, \dots, z_n) , $z_k = x_k + iy_k$, such that $TM(0) = \{y_n = 0\}$ and ρ has the following form

$$\rho = y_n - h(z_1, \dots, z_{n-1}, x_n),$$

such that $h(0) = \nabla h(0) = 0$. If necessary, apply a coordinate change, we may also assume that

$$\frac{\partial^2 h}{\partial z_j \partial z_k}(0) = \frac{\partial^2 h}{\partial z_j \partial x_n}(0) = \frac{\partial^2 h}{\partial x_n^2}(0) = 0.$$

Under these assumptions, then for each $\lambda > 0$,

$$t = h + \lambda \operatorname{Re} z_n^2$$

is an admissible distance function on a sufficiently small neighborhood of $p_0 = 0$. For details see [1].

Hence from now on we assume the existence of the admissible distance function t on M with p_0 as the reference point, and consider the closed domain U_r defined by

$$(2.24) \quad U_r = \{p \in M \mid t(p) \leq r\}.$$

Define

$$(2.25) \quad Y_j t = \sigma_j, \quad b^2 = \sum |\sigma_j|^2,$$

$$(2.26) \quad \omega_j = b^{-1} \sigma_j,$$

$$(2.27) \quad Y^0 = \sum_k \bar{\omega}_k Y_k, \quad Y_j = \omega_j Y^0 + W_j, \quad \text{for } j = 1, \dots, n-1.$$

Hence the vector fields W_j 's are tangent to the level set of t , and Y^0 is transversal to the level set of t , and we have

$$(2.28) \quad \sum_{j=1}^{n-1} \bar{\omega}_j W_j = 0,$$

$$(2.29) \quad W_j = \sum_k Q_{kj} Y_k, \quad \text{where } Q_{kj} = \delta_{kj} - \bar{\omega}_k \omega_j,$$

$$(2.30) \quad \sum_{k=1}^{n-1} Q_{jk} Q_{kl} = Q_{jl}, \quad \bar{Q}_{jk} = Q_{kj},$$

$$(2.31) \quad \sum_{k=1}^{n-1} Q_{kj} \sigma_k = 0, \quad \sum_{k=1}^{n-1} Q_{jk} \bar{\sigma}_k = 0,$$

$$(2.32) \quad [Y_j, Y_k^*] = \frac{1}{i} C_s(Y_j, Y_k) S$$

$$+ \sum_l \bar{q}_{kjl} Y_l + \sum_l Y_l^* q_{jkl} + q_{jk}.$$

Next we introduce some definitions for the error terms, these terms eventually will be absorbed by the main term of our estimate.

Denote by Θ_{-1} any polynomial (with coefficients in C) of $r, \bar{r}, \omega_1, \dots, \omega_{n-1}, \bar{\omega}_1, \dots, \bar{\omega}_{n-1}$. We also denote by Θ_{-1} any matrix (considered as a linear transformation of $\Lambda^q(E)$) with coefficients Θ_{-1} in the above sense. By Θ_0 we denote any first degree polynomial with coefficients Θ_{-1} in $q_{jk}, q_{jkl}, r_{jkl}, [Y_j, r], [\bar{Y}_j, r], b^{-1}[Y_j, \sigma_k]$, and their bars. By Θ_1 we mean a linear combination in

$$[Y_j, \Theta_0], [\bar{Y}_j, \Theta_0], b^{-1}\Theta_0, \Theta_0\Theta_0,$$

where coefficients are $\Theta_{-1}, \Theta_{-1}, \Theta_0, \Theta_1$ may be different from term to term even when they appear in the same equation.

PROPOSITION 2.33. *For each Θ_0, Θ_1 and for any $\epsilon > 0$,*

$$|\Theta_0| \leq b^{-1} \epsilon, \quad |\Theta_1| \leq b^{-2} \epsilon,$$

on $U_r = \{p \mid t(p) \leq r\}$ for sufficiently small $r > 0$.

If we set $X^0 = ibS + \bar{\tau}Y^0 - r\bar{Y}^0$, then we see that

$$(2.34) \quad [W_j, \bar{W}_k] = b^{-1}Q_{kj}X^0 + \sum_l \theta_0 W_l + \sum_l \theta_0 \bar{W}_l,$$

and

$$(2.35) \quad X^0 t = 0,$$

namely, X^0 is tangent to the level set of t .

For the details of the above definitions and proofs, see Kuranishi [1] and [2].

3. An a priori estimate. We denote by $b\Omega_r = \{p \in M \mid t(p) = r\}$ the level set of t . It is a real submanifold of dimension $2n - 2$, and if we delete two characteristic points from $b\Omega_r$, namely,

$$b\Omega'_r = b\Omega_r - \{p \in M \mid b(p) = 0\},$$

then $b\Omega'_r$ is a strongly pseudoconvex CR-manifold of codimension two. Hence one can define the Cauchy-Riemann operator D_b on the induced CR-structure on $b\Omega'_r$. If u is a q -form defined on $b\Omega'_r$ and consider u is the restriction of some q -form \tilde{u} defined on a small neighborhood of $b\Omega'_r$, then we have

$$\begin{aligned} D\tilde{u}(W_0, \dots, W_q) &= \sum_{i=0}^q (-1)^i W_i \tilde{u}(W_0, \dots, \hat{W}_i, \dots, W_q) \\ &\quad + \sum_{i < j} (-1)^{i+j} \tilde{u}([W_i, W_j], \\ &\quad \quad \quad W_0, \dots, \hat{W}_i, \dots, \hat{W}_j, \dots, W_q), \end{aligned}$$

and

$$\begin{aligned} D_b u(W_0, \dots, W_q) &= \sum_{i=0}^q (-1)^i W_i u(W_0, \dots, \hat{W}_i, \dots, W_q) \\ &\quad + \sum_{i < j} (-1)^{i+j} u([W_i, W_j], \\ &\quad \quad \quad W_0, \dots, \hat{W}_i, \dots, \hat{W}_j, \dots, W_q). \end{aligned}$$

This shows that $D_b u$ is equal to the restriction of $D\tilde{u}$ to $b\Omega'_r$, and it is independent of the choice of the extension. Thus we have

$$(3.1) \quad D_b u = D\tilde{u}|_{b\Omega'_r} = \sum_k e_k \wedge Y_k \tilde{u} - \frac{1}{2} e_k \wedge r_{(k)} \tilde{u}|_{b\Omega'_r},$$

where \tilde{u} is an extension of u . The crucial thing is that each term in (3.1) is not well-defined for the extension \tilde{u} , so later we will pick a particular one and make use of that.

Now we reexamine the equation (3.1) carefully, and write a q -form u as the sum of a tangential part and transversal part. Hence we can remove the transversal part from (3.1). Define

$$(3.2) \quad \eta_0 = \sum_k \omega_k e_k, \quad \eta_k = \sum_j Q_{kj} e_j.$$

Then the following holds

$$(3.3) \quad \langle \eta_0, Y^0 \rangle = 1, \quad \langle \eta_0, W_j \rangle = \sum_k \omega_k Q_{kj} = 0,$$

$$(3.4) \quad \langle \eta_k, Y^0 \rangle = \sum_j \bar{\omega}_j Q_{kj} = 0, \quad \langle \eta_k, W_j \rangle = Q_{kj}.$$

Next we calculate $r_{(k)} \tilde{u}$ explicitly. By using the increasing multiindices denoted by ', we see that if

$$\tilde{u} = \sum'_{|I|=q} \tilde{u}_I e_I = \sum'_{|I|=q} \tilde{u}_I e_{i_1} \wedge \cdots \wedge e_{i_q},$$

then

$$\begin{aligned} r_{(k)} \tilde{u} &= \sum'_I \tilde{u}_I r_{(k)} (e_{i_1} \wedge \cdots \wedge e_{i_q}) \\ &= \sum'_I \tilde{u}_I \left(\sum_{j=1}^{n-1} r_{(k)} e_j \wedge (e_{i_1} \wedge \cdots \wedge e_{i_q} \llcorner e_j) \right) \\ &= \sum'_I \tilde{u}_I \left(\sum_{j \in I} r_{(k)} e_j \wedge (e_{i_1} \wedge \cdots \wedge e_{i_q} \llcorner e_j) \right) \\ &= \sum'_I \tilde{u}_I \left(\sum_{p=1}^q \sum_{m=1}^{n-1} r_{kmi_p} e_m \wedge (-1)^{p+1} e_{i_1} \wedge \cdots \wedge \hat{e}_{i_p} \wedge \cdots \wedge e_{i_q} \right) \\ &= \sum'_{|K|=q} \left(\sum_{m \in K} \sum_{i \in \hat{K}_m} \epsilon_{m \hat{K}_m}^K \epsilon_{i \hat{K}_m}^{\langle i \hat{K}_m \rangle} \tilde{u}_{\langle i \hat{K}_m \rangle} r_{kmi} \right) e_K, \end{aligned}$$

where \hat{K}_m means the increasing multiindex after deleting m from K , and $\langle i \hat{K}_m \rangle$ means the increasing multiindex after adding i to \hat{K}_m , and $\epsilon_{m \hat{K}_m}^K$, $\epsilon_{i \hat{K}_m}^{\langle i \hat{K}_m \rangle}$ mean the sign associated to the permutations.

On the other hand, we have

$$\begin{aligned} \sum_k e_k \wedge Y_k \tilde{u}|_{\delta\Omega'_r} &= \sum_k \eta_k \wedge (W_k + \omega_k Y^0) \tilde{u}|_{\delta\Omega'_r} \\ &= \sum_k \eta_k \wedge W_k \tilde{u}|_{\delta\Omega'_r} + \left(\sum_k \omega_k \eta_k \right) \wedge Y^0 \tilde{u}|_{\delta\Omega'_r} \\ &= \sum_k \eta_k \wedge W_k \tilde{u}|_{\delta\Omega'_r}. \end{aligned}$$

If $\tilde{u} = \sum_I' \tilde{u}_I e_I$, write $u = \sum_I u_I \eta_I$, where u_I is the restriction of \tilde{u}_I to $b\Omega'_r$. Then formally we define

FORMAL DEFINITION 3.5. By $W_k u$ we mean $W_k u = \sum_I' W_k(u_I) \eta_I$.

FORMAL DEFINITION 3.6. Define

$$\mathcal{R}_{(k)} u = \sum_{|I|=q}' \left(\sum_{m \in I} \sum_{i \in \hat{I}_m} \varepsilon_m^I \hat{\tau}_m^{i \hat{I}_m} u_{i \hat{I}_m} \mathcal{R}_{kmi} \right) \eta_I.$$

It is important to note that these two formal definitions depend on the choice of the extension \tilde{u} , so they are not well-defined. However by (3.1) we see that the following proposition is well-defined.

PROPOSITION 3.7. Let u be a q -form defined on $b\Omega'_r$ and \tilde{u} be an extension of u . Write $\tilde{u} = \sum_I' \tilde{u}_I e_I$ and $u = \sum_I u_I \eta_I$, where $u_I = \tilde{u}_I|_{b\Omega'_r}$. Then

$$(3.8) \quad D_b u = \sum_{k=1}^{n-1} \eta_k \wedge W_k u - \frac{1}{2} \eta_k \wedge \mathcal{R}_{(k)} u.$$

Next we will construct a particular coefficients u_I for u , so that (3.5) and (3.6) will make sense, and we will use this particular coefficients to calculate the formal adjoint D_b^* of D_b . Basically the idea is that we choose the particular extension which has no transversal component, namely, we push out the original q -form u parallelly. Then we prove the following lemma.

LEMMA 3.9. The coefficients of $(q!) u = \sum_I u_{i_1, \dots, i_q} \eta_{i_1} \wedge \dots \wedge \eta_{i_q}$ are unique if they satisfy

$$(3.10) \quad \sum_{i_q=1}^{n-1} u_{i_1, \dots, i_q} \bar{\omega}_{i_q} = 0, \quad \text{for } i_1, \dots, i_{q-1} = 1, \dots, n-1,$$

where the coefficients u_{i_1, \dots, i_q} 's are skew-symmetric.

Proof.

$$\begin{aligned} u &= \sum_I' u_{i_1, \dots, i_q} \eta_{i_1} \wedge \dots \wedge \eta_{i_q} \\ &= \sum_I' u_{i_1, \dots, i_q} \eta_{i_1} \wedge \dots \wedge \eta_{i_q} \\ &\quad + \frac{1}{(q-1)!} \left(\sum_{i_1=1}^{n-1} \dots \sum_{i_{q-1}=1}^{n-1} \right. \\ &\quad \left. \cdot \left(\sum_{i_q=1}^{n-1} u_{i_1, \dots, i_q} \bar{\omega}_{i_q} \right) \eta_{i_1} \wedge \dots \wedge \eta_{i_{q-1}} \right) \wedge \eta_0 \end{aligned}$$

$$\begin{aligned}
&= \sum_I' u_I \eta_I + \sum_I' u_{j_1 \dots j_q} \left(\sum_{k=1}^q \bar{\omega}_{j_k} \eta_{j_1} \wedge \dots \wedge \underset{k\text{-th position}}{\eta_0} \wedge \dots \wedge \eta_{j_q} \right) \\
&= \sum_I' u_{j_1 \dots j_q} \left(\eta_{j_1} \wedge \dots \wedge \eta_{j_q} + \sum_{k=1}^q \bar{\omega}_{j_k} \eta_{j_1} \wedge \dots \wedge \underset{k\text{-th position}}{\eta_0} \wedge \dots \wedge \eta_{j_q} \right) \\
&= \sum_I' u_{j_1 \dots j_q} (\eta_{j_1} + \bar{\omega}_{j_1} \eta_0) \wedge \dots \wedge (\eta_{j_q} + \bar{\omega}_{j_q} \eta_0) \\
&= \sum_I' u_{j_1 \dots j_q} e_{j_1} \wedge \dots \wedge e_{j_q} \\
&= \sum_I' u_I e_I.
\end{aligned}$$

Hence if $u = 0$ and the coefficients satisfy (3.10), then it implies $u_I = 0$ for each multiindices. This proves the lemma.

These particular coefficients will be called admissible, and from now on all of the computations will base on the admissible coefficients. Then we have some easy consequences.

LEMMA 3.11. *If α and β are two q -form defined on $b\Omega'$, expressed in admissible coefficients, then*

$$\begin{aligned}
(3.12) \quad &\langle \alpha, \beta \rangle = \sum_I' \alpha_I \bar{\beta}_I, \quad \text{where} \\
&\alpha = \sum_I' \alpha_I \eta_I, \quad \beta = \sum_I' \beta_I \eta_I,
\end{aligned}$$

$$(3.13) \quad \langle W_k \alpha, \beta \rangle = \sum_I' W_k(\alpha_I) \bar{\beta}_I.$$

Proof.

$$\begin{aligned}
\langle \alpha, \beta \rangle &= \left\langle \sum_I' \alpha_I \eta_I, \sum_J' \beta_J \eta_J \right\rangle \\
&= \left\langle \sum_I' \alpha_I e_I, \sum_J' \beta_J e_J \right\rangle \\
&= \sum_I' \alpha_I \bar{\beta}_I.
\end{aligned}$$

This proves (3.12). To prove (3.13), we have to rewrite $W_k \alpha$ as follows

$$\begin{aligned}
&\left\langle \sum_I' W_k(\alpha_I) \eta_I, \sum_I' \beta_I \eta_I \right\rangle \\
&= \left\langle \sum_I' W_k(\alpha_I) (e_{i_1} - \bar{\omega}_{i_1} \eta_0) \wedge \dots \wedge (e_{i_q} - \bar{\omega}_{i_q} \eta_0), \sum_I' \beta_I e_I \right\rangle
\end{aligned}$$

$$\begin{aligned}
&= \sum_I' W_k(\alpha_I) \bar{\beta}_I \\
&\quad - \left\langle \sum_I' W_k(\alpha_I) \left(\sum_{p \in I} \bar{\omega}_p \epsilon_{p \hat{f}_p}^I \eta_0 \wedge e_{\hat{f}_p} \right), \sum_I' \beta_I e_I \right\rangle \\
&= \sum_I' W_k(\alpha_I) \bar{\beta}_I \\
&\quad - \left\langle \sum_I' W_k(\alpha_I) \left(\sum_{p \in I} \bar{\omega}_p \epsilon_{p \hat{f}_p}^I e_{\hat{f}_p} \right), \left(\sum_I' \beta_I e_I \right) \llcorner \eta_0 \right\rangle,
\end{aligned}$$

where $I = (i_1, \dots, i_q)$. If $J = (j_1, \dots, j_{q-1})$, set $J \cup \{t\} = (j_1, \dots, j_{q-1}, t)$, then

$$\begin{aligned}
\left(\sum_I' \beta_I e_I \right) \llcorner \eta_0 &= \sum_{k=1}^{n-1} \bar{\omega}_k \left(\sum_I' \beta_I e_I \llcorner e_k \right) \\
&= \sum_I' \beta_I \left(\sum_{p \in I} \bar{\omega}_p \epsilon_{p \hat{f}_p}^I e_{\hat{f}_p} \right) \\
&= \sum_{|J|=q-1} \left(\sum_{i \in J} \beta_{\langle i, J \rangle} \epsilon_{iJ}^{\langle i, J \rangle} \bar{\omega}_i \right) e_J \\
&= (-1)^q \sum_{|J|=q-1} \left(\sum_{t=1}^{n-1} \beta_{J \cup \{t\}} \bar{\omega}_t \right) e_J \\
&= 0.
\end{aligned}$$

This is true because β_I 's are admissible. Hence the proof of (3.13) is now complete.

LEMMA 3.14. *If α is a q -form in admissible coefficients, then so is $\alpha \llcorner \eta_k$.*

Proof. $\alpha = \sum_I' \alpha_I \eta_I = 1/q! \sum_I \alpha_I \eta_I$, where α_I is skew-symmetric. If $I = (i_1, \dots, i_q)$, then

$$\begin{aligned}
\alpha \llcorner \eta_k &= \frac{1}{q!} \sum_I (\alpha_{i_1, \dots, i_q} \eta_{i_1} \wedge \dots \wedge \eta_{i_q}) \llcorner \eta_k \\
&= \frac{1}{q!} \sum_I \left(\sum_{p=1}^q \alpha_{i_1, \dots, i_{p-1} i_{p+1} \dots i_q i_p} (-1)^{p+1} Q_{i_p k} \right. \\
&\quad \left. \cdot \eta_{i_1} \wedge \dots \wedge \hat{\eta}_{i_p} \wedge \dots \wedge \eta_{i_q} \right) \\
&= \frac{1}{q!} \sum_I \left(\sum_{p=1}^q \alpha_{i_1 \dots i_{p-1} i_{p+1} \dots i_q i_p} (-1)^{q+1} Q_{i_p k} \right. \\
&\quad \left. \cdot \eta_{i_1} \wedge \dots \wedge \hat{\eta}_{i_p} \wedge \dots \wedge \eta_{i_q} \right)
\end{aligned}$$

$$\begin{aligned}
 &= \frac{1}{q!} \sum_{J=(j_1, \dots, j_{q-1})} \left(q \left((-1)^{q+1} \right. \right. \\
 &\quad \cdot \sum_{j_q=1}^{n-1} \alpha_{j_1, \dots, j_{q-1}, j_q} Q_{j_q k} \left. \eta_{j_1} \wedge \dots \wedge \eta_{j_{q-1}} \right) \\
 &= \sum'_{J=(j_1, \dots, j_{q-1})} \beta_{j_1, \dots, j_{q-1}} \eta_{j_1} \wedge \dots \wedge \eta_{j_{q-1}},
 \end{aligned}$$

where

$$\begin{aligned}
 \beta_{j_1, \dots, j_{q-1}} &= (-1)^{q+1} \sum_{j_q=1}^{n-1} \alpha_{j_1, \dots, j_{q-1}, j_q} Q_{j_q k} \\
 &= (-1)^{q+1} \alpha_{j_1, \dots, j_{q-1}, k}.
 \end{aligned}$$

It is easy to see that $\sum_{j_{q-1}=1}^{n-1} \beta_{j_1, \dots, j_{q-1}} \bar{\omega}_{j_{q-1}} = 0$. So we are done.

LEMMA 3.15. If α is a q -form, then $q\|\alpha\|^2 = \sum_{k=1}^{n-1} \|\alpha \llcorner \eta_k\|^2$.

Proof. Write α in admissible coefficients, $\alpha = \sum_I \alpha_I \eta_I$. By Lemma 3.14, $\alpha \llcorner \eta_k = \sum_J \beta_J \eta_J$, where $\beta_J = (-1)^{q+1} \alpha_{Jk}$. Hence

$$\begin{aligned}
 \sum_{k=1}^{n-1} \|\alpha \llcorner \eta_k\|^2 &= \sum_{k=1}^{n-1} \sum'_{|J|=q-1} \|\alpha_{Jk}\|^2 \\
 &= q \sum'_{|I|=q} \|\alpha_I\|^2 = q\|\alpha\|^2.
 \end{aligned}$$

This proves the lemma.

In contrast with Lemma 3.11, the next lemma shows how to compute the inner product of two q -forms when they are not expressed in terms of the admissible coefficients.

LEMMA 3.16. Let α and β be two q -forms, namely,

$$\alpha = \sum' I \alpha_I \eta_I, \text{ and } \beta = \sum' J \beta_J \eta_J.$$

Then $\langle \alpha, \beta \rangle = \sum_{I, J} \alpha_I \bar{\beta}_J \det(I, J)$, where $\det(I, J) = \det(Q_{ij})$ for $i \in I$ and $j \in J$.

Proof. We only have to calculate the inner product of the form,

$$\begin{aligned}
 \langle \eta_I, \eta_J \rangle &= \langle \eta_{i_1} \wedge \dots \wedge \eta_{i_q}, \eta_{j_1} \wedge \dots \wedge \eta_{j_q} \rangle \\
 &= \langle (\eta_{i_1} \wedge \dots \wedge \eta_{i_q}) \llcorner \eta_{j_1}, \eta_{j_2} \wedge \dots \wedge \eta_{j_q} \rangle \\
 &= \left\langle \sum_{k=1}^q (-1)^{k+1} Q_{i_k j_1} \eta_{i_1} \wedge \dots \wedge \hat{\eta}_{i_k} \wedge \dots \wedge \eta_{i_q}, \right. \\
 &\quad \left. \cdot \eta_{j_2} \wedge \dots \wedge \eta_{j_q} \right\rangle \\
 &= \det(Q_{ij}),
 \end{aligned}$$

where $i \in \{i_1, \dots, i_q\}$ and $j \in \{j_1, \dots, j_q\}$. The last equality is obtained by induction hypothesis and the expansion formula for the determinant. This completes the proof of the lemma.

LEMMA 3.17. $\sum' \alpha_{jk} \det(J, J') = \alpha_{j,k}$ for some fixed

$$J' = (1, \dots, q).$$

Proof.

$$\begin{aligned} \sum' \alpha_{jk} \det(J, J') &= \sum'_{J=(A_1, \dots, A_q)} \alpha_{jk} \det((A_1, \dots, A_q), (1, \dots, q)) \\ &= \frac{1}{q!} \sum_{A_1, \dots, A_q=1}^{n-1} \alpha_{A_1 \dots A_q k} \\ &\quad \cdot \left(\sum_{\pi} \varepsilon_{\pi} Q_{A_{\pi(1)} 1} \dots Q_{A_{\pi(q)} q} \right) \\ &= \frac{1}{q!} \sum_{\pi} \sum_{A_1, \dots, A_q=1}^{n-1} \alpha_{A_{\pi(1)} \dots A_{\pi(q)} k} \\ &\quad \cdot Q_{A_{\pi(1)} 1} \dots Q_{A_{\pi(q)} q} \\ &= \frac{1}{q!} \sum_{\pi} \alpha_{1, \dots, q, k} \\ &= \alpha_{1, \dots, q, k} \\ &= \alpha_{j,k}. \end{aligned}$$

This completes the proof of the lemma.

Now we go back to prove the a priori estimate on $b\Omega_r^*$. By using the admissible coefficients, we can define the formal adjoint D_b^* of D_b by duality:

$$(3.18) \quad D_b^* u = \sum_{k=1}^{n-1} W_k^* (u \lfloor \eta_k) - \frac{1}{2} r_{(k)}^*(u \lfloor \eta_k).$$

Let g be a real smooth function defined on \mathbf{R} which is zero if $x \leq \frac{1}{2}$ and is one if $x \geq 1$. Then define

$$(3.19) \quad \chi^2(b^2) = g(\lambda b^2), \quad \text{for } \lambda > 0.$$

This function χ will serve as the cut-off function in our calculations and χ tends to the characteristic function on $b\Omega_r^*$ if λ tends to infinity. Let u be a q -form defined on $b\Omega_r^*$. Consider $\chi D_b u$ and $\chi D_b^* u$, we have

$$(3.20) \quad \|xD_b u\|^2 + \|xD_b^* u\|^2 = A - B + \frac{1}{4} \left(\left\| \sum_k \chi \eta_k \wedge r_{(k)} u \right\|^2 + \left\| \sum_k \chi r_{(k)}^*(u \llcorner \eta_k) \right\|^2 \right),$$

where

$$A = \left\| \sum_k \chi \eta_k \wedge W_k u \right\|^2 + \left\| \sum_k \chi W_k^*(u \llcorner \eta_k) \right\|^2,$$

and

$$B = \operatorname{Re} \left(\sum_{j,k} \langle \chi \eta_j \wedge W_j u, \chi \eta_k \wedge r_{(k)} u \rangle + \langle \chi W_k^*(u \llcorner \eta_k), \chi r_{(k)}^*(u \llcorner \eta_k) \rangle \right).$$

The terms of the following form

$$(3.21) \quad \begin{aligned} \langle \theta_1 u, u \rangle, \langle \theta_1 \chi u, \chi u \rangle, \langle \theta_0 W u, u \rangle &= \sum_m \langle \theta_0 W_m u, u \rangle, \\ \langle \theta_0 \chi W u, \chi u \rangle &= \sum_m \langle \theta_0 \chi W_m u, \chi u \rangle, \end{aligned}$$

will be called admissible errors which eventually will be absorbed by the main term of our estimate. First we examine term B .

LEMMA 3.22. *B is an admissible error.*

Proof. Note that the coefficients of each term in $r_{(k)}^*(u \llcorner \eta_k)$ is the linear combinations of $u_I \bar{r}_{ijm}$. Hence

$$\langle \chi W_k^*(u \llcorner \eta_k), \chi r_{(k)}^*(u \llcorner \eta_k) \rangle = \langle W_k^*(u \llcorner \eta_k), \chi^2 r_{(k)}^*(u \llcorner \eta_k) \rangle.$$

After we write the right hand side out, we get terms like

$$\langle u_{I_1}, W_k(\chi^2) \bar{r}_{ijm} u_{I_2} \det(J_2, J_1) \rangle, \langle u_{I_1}, \chi^2 W_k(\bar{r}_{ijm}) u_{I_2} \det(J_2, J_1) \rangle,$$

$$\langle u_{I_1}, \chi^2 \bar{r}_{ijm} W_k(u_{I_2}) \det(J_2, J_1) \rangle, \langle u_{I_1}, \chi^2 \bar{r}_{ijm} u_{I_2} W_k \det(J_2, J_1) \rangle,$$

where $|I_1| = |I_2| = q$, and $|J_1| = |J_2| = q - 1$. It is not hard to see that

$$W_k(\chi^2) \bar{r}_{ijm} \det(J_2, J_1) = \theta_1, \quad W_k(\bar{r}_{ijm}) \det(J_2, J_1) = \theta_0,$$

$$\bar{r}_{ijm} \det(J_2, J_1) = \theta_0, \quad \bar{r}_{ijm} W_k(\det(J_2, J_1)) = \theta_1.$$

Hence they are of the following form

$$\langle \theta_1 u, u \rangle, \langle \theta_1 \chi u, \chi u \rangle, \langle \theta_0 \chi W u, \chi u \rangle.$$

It follows that this term is an admissible error, and it is easy to see that by the same arguments the term $\langle \chi \eta_j \wedge W_j u, \chi \eta_k \wedge r_{(k)} u \rangle$ is an admissible error. This proves the lemma.

In order to get an a priori estimate, we have to make use of theorem A. Before doing so, we first list some properties which we need in the sequel.

$$(P1) \quad [\bar{Y}_p, Q_{jk}] = -b^{-1} \bar{\omega}_j Q_{pk} \bar{r} + \theta_0.$$

$$(P1)' \quad [Y_p, Q_{jk}] = -b^{-1} \omega_k Q_{jp} r + \theta_0.$$

$$(P2) \quad [\bar{W}_j, Q_{pq}] = -b^{-1} \bar{\omega}_p \bar{r} Q_{jq} + \theta_0.$$

$$(P2)' \quad [W_j, Q_{pq}] = -b^{-1} \omega_q r Q_{pj} + \theta_0.$$

$$(P3) \quad [W_k, [\bar{W}_j, Q_{pq}]] = \frac{|r|^2}{b^2} (Q_{jk} \bar{\omega}_p \omega_q - Q_{pk} Q_{jq}) + \theta_1.$$

$$(P3)' \quad \sum_{k=1}^{n-1} [W_j, [\bar{W}_k, Q_{jk}]] = -(n-2) \frac{|r|^2}{b^2} Q_{jj} + \theta_1.$$

(P4) If $J_1 = (i_1, \dots, i_{q-1})$ and $J_2 = (m_1, \dots, m_{q-1})$, then

$$\bar{W}_j(\det(J_1, J_2))$$

$$= \sum_{t=1}^{q-1} \sum_{\pi} \epsilon_{\pi} (-b^{-1} \bar{r} \bar{\omega}_{i_{\pi(t)}}) Q_{i_{\pi(1)} m_1} \cdots Q_{i_{\pi(t-1)} m_{t-1}} Q_{j m_t} \\ \cdot Q_{i_{\pi(t+1)} m_{t+1}} \cdots Q_{i_{\pi(q-1)} m_{q-1}} + \theta_0.$$

$$W_j((\det(J_1, J_2)))$$

$$(P4)' \quad = \sum_{t=1}^{q-1} \sum_{\pi} \epsilon_{\pi} (-b^{-1} r \omega_{m_t}) Q_{i_{\pi(1)} m_1} \cdots Q_{i_{\pi(t-1)} m_{t-1}} Q_{i_{\pi(t)} j} \\ \cdot Q_{i_{\pi(t+1)} m_{t+1}} \cdots Q_{i_{\pi(q-1)} m_{q-1}} + \theta_0.$$

$$W_k(\bar{W}_j(\det(J_1, J_2)))$$

$$(P5) \quad = \frac{|r|^2}{b^2} \sum_{t=1}^{q-1} \sum_{s=1}^{q-1} \sum_{\pi} \epsilon_{\pi} \bar{\omega}_{i_{\pi(t)}} \omega_{m_s} Q_{i_{\pi(1)} m_1} \cdots Q_{i_{\pi(s)} k} \cdots \\ \cdot Q_{j m_t} \cdots Q_{i_{\pi(q-1)} m_{q-1}} \\ - \frac{|r|^2}{b^2} \sum_{t=1}^{q-1} \sum_{\pi} \epsilon_{\pi} Q_{i_{\pi(1)} m_1} \cdots \\ \cdot (Q_{i_{\pi(t)} k} Q_{j m_t}) \cdots Q_{i_{\pi(q-1)} m_{q-1}} + \theta_1.$$

$$(P6) \quad W_j(\chi^2) = \theta_0.$$

$$(P6)' \quad W_j \bar{W}_k(\chi^2) = \theta_1, \quad W_j \bar{W}_j(\chi^2) = \theta_1.$$

$$(P7) \quad W_j(\bar{\omega}_p) = \frac{r}{b} Q_{pj} + \theta_0.$$

(P8) If we set $X^0 = ibS + \bar{r} Y^0 - r \bar{Y}^0$, then

$$[W_j, \bar{W}_k] = b^{-1} Q_{kj} X^0 + \sum_m \theta_0 W_m + \sum_m \theta_0 \bar{W}_m.$$

$$(P9) \quad W_j^* = -\bar{W}_j + (n-2) \frac{\bar{r}\bar{\sigma}_j}{b^2} + \theta_0, \text{ on } b\Omega_r.$$

$$(P10) \quad W_k \left((n-2) \frac{\bar{r}\bar{\sigma}_j}{b^2} \right) = (n-2) \frac{|\tau|^2}{b^2} Q_{jk} + \theta_1.$$

Proof of (P1).

$$\begin{aligned} [\bar{Y}_p, Q_{jk}] &= -\sum_m b^{-1} \bar{\omega}_j Q_{mk} [\bar{Y}_p, \sigma_m] - \sum_m b^{-1} \omega_k Q_{jm} [\bar{Y}_p, \bar{\sigma}_m] \\ &= -\sum_m b^{-1} \bar{\omega}_j Q_{mk} \bar{r} \delta_{pm} + \theta_0 \\ &= -b^{-1} \bar{\omega}_j Q_{pk} \bar{r} + \theta_0. \end{aligned}$$

(P1)' can be proved by the same arguments.

Proof of (P2).

$$\begin{aligned} [\bar{W}_j, Q_{pq}] &= \sum_{s=1}^{n-1} Q_{js} Y_s(Q_{pq}) \\ &= \sum_{s=1}^{n-1} Q_{js} (-b^{-1} \bar{\omega}_p Q_{sq} \bar{r}) + \theta_0 \\ &= -b^{-1} \bar{\omega}_p Q_{jq} \bar{r} + \theta_0. \end{aligned}$$

We prove (P2)' similarly.

Proof of (P3).

$$\begin{aligned} W_k(\bar{W}_j) Q_{pq}) &= \sum_{s=1}^{n-1} Q_{sk} Y_s(-b^{-1} \bar{\omega}_p Q_{jq} \bar{r} + \theta_0) \\ &= \sum_{s=1}^{n-1} Q_{sk} Q_{jq} Y_s \left(\frac{-\bar{\sigma}_p \bar{r}}{b^2} \right) \\ &\quad - \sum_{s=1}^{n-1} Q_{sk} b^{-1} \bar{\omega}_p \bar{r} Y_s(Q_{jq}) + \theta_1 \\ &= \sum_{s=1}^{n-1} Q_{sk} Q_{jq} \left(-\frac{|\tau|^2 \delta_{sp}}{b^2} \right) \\ &\quad + \sum_{s=1}^{n-1} Q_{sk} Q_{jq} \frac{|\tau|^2 \bar{\sigma}_p \sigma_s}{b^4} \\ &\quad + \frac{|\tau|^2}{b^2} \sum_{s=1}^{n-1} Q_{sk} Q_{js} \bar{\omega}_p \omega_q + \theta_1 \\ &= \frac{|\tau|^2}{b^2} (Q_{jk} \bar{\omega}_p \omega_q - Q_{pk} Q_{jq}) + \theta_1. \end{aligned}$$

Proof of (P3)'.

$$\begin{aligned} \sum_{k=1}^{n-1} W_j(\bar{W}_k(Q_{jk})) &= \sum_{k=1}^{n-1} \frac{|r|^2}{b^2} (Q_{kj} \bar{\omega}_j \omega_k - Q_{jj} Q_{kk}) + \theta_1 \\ &= -\frac{|r|^2}{b^2} Q_{jj} \sum_{k=1}^{n-1} Q_{kk} + \theta_1 \\ &= -(n-2) \frac{|r|^2}{b^2} Q_{jj} + \theta_1. \end{aligned}$$

Proof of (P4). If $J_1 = (i_1, \dots, i_{q-1})$ and $J_2 = (m_1, \dots, m_{q-1})$, then

$$\begin{aligned} \bar{W}_j(\det(J_1, J_2)) &= \sum_{s=1}^{n-1} Q_{js} \bar{Y}_s \left(\sum_{\pi} \epsilon_{\pi} Q_{i_{\pi(1)} m_1} \cdots Q_{i_{\pi(q-1)} m_{q-1}} \right) \\ &= \sum_{s=1}^{n-1} \sum_{\pi} \sum_{t=1}^{q-1} Q_{js} \epsilon_{\pi} Q_{i_{\pi(1)} m_1} \cdots \bar{Y}_s(Q_{i_{\pi(t)} m_t}) \cdots Q_{i_{\pi(q-1)} m_{q-1}} \\ &= \sum_{s=1}^{n-1} \sum_{\pi} \sum_{t=1}^{q-1} Q_{js} \epsilon_{\pi} Q_{i_{\pi(1)} m_1} \cdots \\ &\quad \cdot (-b^{-1} \bar{\omega}_{i_{\pi(t)}} Q_{s m_t} \bar{r}) \cdots Q_{i_{\pi(q-1)} m_{q-1}} + \theta_0 \\ &= \sum_{t=1}^{q-1} \sum_{\pi} \epsilon_{\pi} (-b^{-1} \bar{r} \bar{\omega}_{i_{\pi(t)}}) Q_{i_{\pi(1)} m_1} \cdots \\ &\quad \cdot Q_{j m_t} \cdots Q_{i_{\pi(q-1)} m_{q-1}} + \theta_0. \end{aligned}$$

Proof of (P4)'.

$$\begin{aligned} W_j(\det(J_1, J_2)) &= \sum_{s=1}^{n-1} \sum_{\pi} \sum_{t=1}^{q-1} Q_{sj} \epsilon_{\pi} Q_{i_{\pi(1)} m_1} \cdots Y_s(Q_{i_{\pi(t)} m_t}) \cdots Q_{i_{\pi(q-1)} m_{q-1}} \\ &= \sum_{s=1}^{n-1} \sum_{\pi} \sum_{t=1}^{q-1} Q_{sj} \epsilon_{\pi} Q_{i_{\pi(1)} m_1} \cdots \\ &\quad \cdot (-b^{-1} \omega_{m_t} Q_{i_{\pi(t)} s} r) \cdots Q_{i_{\pi(q-1)} m_{q-1}} + \theta_0 \\ &= \sum_{t=1}^{q-1} \sum_{\pi} \epsilon_{\pi} (-b^{-1} r \omega_{m_t}) Q_{i_{\pi(1)} m_1} \cdots \\ &\quad \cdot Q_{i_{\pi(t)} j} \cdots Q_{i_{\pi(q-1)} m_{q-1}} + \theta_0. \end{aligned}$$

Proof of (P5).

$$\begin{aligned} W_k(\bar{W}_j(\det(J_1, J_2))) &= \sum_{d=1}^{n-1} Q_{dk} Y_d(\bar{W}_j(\det(J_1, J_2))) \end{aligned}$$

$$\begin{aligned}
&= \sum_{d=1}^{n-1} \sum_{t=1}^{q-1} \sum_{\pi} \varepsilon_{\pi} Q_{dk}(Y_d(-b^{-1} \bar{r} \bar{\omega}_{i_{\pi(t)}})) \\
&\quad \cdot Q_{i_{\pi(1)} m_1} \cdots Q_{j_m t} \cdots Q_{i_{\pi(q-1)} m_{q-1}} \\
&\quad + \sum_{d=1}^{n-1} \sum_{t=1}^{q-1} \sum_{\pi} \varepsilon_{\pi} Q_{dk}(-b^{-1} \bar{r} \bar{\omega}_{i_{\pi(t)}}) \\
&\quad \cdot \left(\sum_{s=1}^{q-1} Q_{i_{\pi(1)} m_1} \cdots (Y_d Q_{i_{\pi(s)} m_s}) \cdots \right. \\
&\quad \left. \cdot Q_{j_m t} \cdots Q_{i_{\pi(q-1)} m_{q-1}} \right) + \theta_1 \\
&= \sum_{d=1}^{n-1} \sum_{t=1}^{q-1} \sum_{\pi} \varepsilon_{\pi} Q_{dk} \left(-\frac{|\bar{r}|^2}{b^2} \delta_{d, i_{\pi(t)}} + \frac{|\bar{r}|^2}{b^4} \bar{\sigma}_{i_{\pi(t)}} \sigma_d \right) \\
&\quad \cdot Q_{i_{\pi(1)} m_1} \cdots Q_{j_m t} \cdots Q_{i_{\pi(q-1)} m_{q-1}} \\
&\quad + \sum_{d=1}^{n-1} \sum_{t=1}^{q-1} \sum_{\pi} \varepsilon_{\pi} Q_{dk}(-b^{-1} \bar{r} \bar{\omega}_{i_{\pi(t)}}) \\
&\quad \cdot \left(\sum_{s=1}^{q-1} Q_{i_{\pi(1)} m_1} \cdots (-b^{-1} \omega_{m_s} r Q_{i_{\pi(s)} d}) \cdots \right. \\
&\quad \left. \cdot Q_{j_m t} \cdots Q_{i_{\pi(q-1)} m_{q-1}} \right) + \theta_1 \\
&= -\frac{|\bar{r}|^2}{b^2} \sum_{t=1}^{q-1} \sum_{\pi} \varepsilon_{\pi} Q_{i_{\pi(t)} k} Q_{i_{\pi(1)} m_1} \cdots Q_{j_m t} \cdots Q_{i_{\pi(q-1)} m_{q-1}} \\
&\quad + \frac{|\bar{r}|^2}{b^2} \sum_{t=1}^{q-1} \sum_{s=1}^{q-1} \sum_{\pi} \varepsilon_{\pi} \bar{\omega}_{i_{\pi(t)}} \omega_{m_s} Q_{i_{\pi(1)} m_1} \cdots Q_{i_{\pi(s)} k} \cdots \\
&\quad \cdot Q_{j_m t} \cdots Q_{i_{\pi(q-1)} m_{q-1}} + \theta_1.
\end{aligned}$$

Proof of (P6). Recall that $\chi^2 = g(tb^2)$, for $t > 0$, then

$$\begin{aligned}
W_j(\chi^2) &= W_j(g(tb^2)) \\
&= g'(tb^2) t W_j(b^2) = t g'(tb^2) \sum_{k=1}^{n-1} Q_{kj} Y_k(b^2) \\
&= t g'(tb^2) \sum_{k=1}^{n-1} Q_{kj} \left(\sum_{m=1}^{n-1} \sigma_m Y_k(\bar{\sigma}_m) + \bar{\sigma}_m Y_k(\sigma_m) \right) \\
&= t g'(tb^2) \sum_{k=1}^{n-1} Q_{kj} r \sigma_k + \theta_0 \\
&= t g'(tb^2) r \left(\sum_{k=1}^{n-1} Q_{kj} \sigma_k \right) + \theta_0 \\
&= \theta_0.
\end{aligned}$$

The last equality holds because $\sum_{k=1}^{n-1} Q_{kj} \sigma_k = 0$. Also from the computations we get $W_j(b^2) = b^2 \theta_0$, $\bar{W}_j(b^2) = b^2 \theta_0$, for

$j = 1, \dots, n - 1$.

Proof of (P6)'. By (P6), we obtain $W_j(x^2) = tb^2 g'(tb^2) \Theta_0$, hence

$$\begin{aligned} \bar{W}_k(W_j(x^2)) &= t^2 g''(tb^2) \bar{W}_k(b^2) b^2 \Theta_0 \\ &\quad + tg'(tb^2) \bar{W}_k(b^2) \Theta_0 + tb^2 g'(tb^2) \bar{W}_k(\Theta_0) \\ &= t^2 g''(tb^2)(b^2 \Theta_0) b^2 \Theta_0 + tg'(tb^2)(b^2 \Theta_0) \Theta_0 + \Theta_1 \\ &= \Theta_1 + \Theta_1 + \Theta_1 \\ &= \Theta_1. \end{aligned}$$

Proof of (P7).

$$\begin{aligned} W_j(\bar{\omega}_p) &= \sum_{k=1}^{n-1} Q_{kj} Y_k \left(\frac{\bar{\sigma}_p}{b} \right) \\ &= \sum_{k=1}^{n-1} Q_{kj} \frac{Y_k(\bar{\sigma}_p)}{b} - \frac{\bar{\sigma}_p}{b^2} \left(\frac{1}{2b} \right) W_j(b^2) \\ &= \frac{\tau}{b} Q_{pj} + \Theta_0. \end{aligned}$$

Proof of (P8). First we recall that

$$[Y_j, \bar{Y}_k] = i\delta_{jk} S - \sum_m \bar{q}_{kj}{}_m Y_m + \sum_m q_{jk}{}_m \bar{Y}_m,$$

then we have

$$\begin{aligned} [Y_j, \bar{W}_k] &= iQ_{kj} S - \sum_{m,p} Q_{km} \bar{q}_{mj}{}_p \omega_p Y^0 \\ &\quad - b^{-1} \sum_p Q_{kp} \left([Y_j, \bar{\sigma}_p] - \sum_m q_{jp}{}_m \bar{\sigma}_m \right) \bar{Y}^0 \\ &\quad + \sum_m \Theta_0 W_m + \sum_m \Theta_0 \bar{W}_m \\ &= Q_{kj}(iS - b^{-1} \tau \bar{Y}^0) - \sum_{m,p} b^{-1} Q_{km} \bar{q}_{mj}{}_p \sigma_p Y^0 \\ &\quad + \sum_m \Theta_0 W_m + \sum_m \Theta_0 \bar{W}_m. \end{aligned}$$

On the other hand, we have

$$\begin{aligned} &- \sum_{m,p} b^{-1} Q_{km} \bar{q}_{mj}{}_p \sigma_p \\ &= Q_{kj} b^{-1} \bar{\tau} - b^{-1} \sum_m Q_{km} \left((\bar{\tau} \delta_{mj} + \sum_p \bar{q}_{mj}{}_p \sigma_p) \right) \\ &= Q_{kj} b^{-1} \bar{\tau} - b^{-1} \sum_m Q_{km} [\bar{Y}_m, \sigma_j]. \end{aligned}$$

It follows that

$$\begin{aligned}
 [Y_j, \bar{W}_k] &= Q_{kj}(iS - b^{-1} \gamma \bar{Y}^0 + b^{-1} \bar{\gamma} Y^0) \\
 &\quad - b^{-1} [\bar{W}_k, \sigma_j] Y^0 + \sum_m \theta_0 W_m + \sum_m \theta_0 \bar{W}_m \\
 &= b^{-1} Q_{kj} X^0 - b^{-1} [\bar{W}_k, \sigma_j] Y^0 + \sum_m \theta_0 W_m + \sum_m \theta_0 \bar{W}_m.
 \end{aligned}$$

Hence we obtain

$$\begin{aligned}
 [W_j, \bar{W}_k] &= \left[\sum_{m=1}^{n-1} Q_{mj} Y_m, \bar{W}_k \right] \\
 &= \sum_{m=1}^{n-1} Q_{mj} [Y_m, \bar{W}_k] - \sum_{m=1}^{n-1} \bar{W}_k (Q_{mj}) Y_m \\
 &= \sum_{m=1}^{n-1} Q_{mj} \left(b^{-1} Q_{km} X^0 - b^{-1} [\bar{W}_k, \sigma_m] Y^0 \right. \\
 &\quad \left. + \sum_p \theta_0 W_p + \sum_p \theta_0 \bar{W}_p \right) \\
 &\quad - \sum_{m=1}^{n-1} \bar{W}_k (Q_{mj}) (\omega_m Y^0 + W_m) \\
 &= b^{-1} Q_{kj} X^0 + \sum_p \theta_0 W_p + \sum_p \theta_0 \bar{W}_p \\
 &\quad - b^{-1} \sum_{m=1}^{n-1} (Q_{mj} \bar{W}_k (\sigma_m) + \bar{W}_k (Q_{mj}) \sigma_m) Y^0 \\
 &= b^{-1} Q_{kj} X^0 + \sum_p \theta_0 W_p + \sum_p \theta_0 \bar{W}_p,
 \end{aligned}$$

since $\sum_{m=1}^{n-1} Q_{mj} \sigma_m = 0$. This completes the proof of (P8).

Proof of (P9). Since W_j is a vector field tangent to the level set of t , so it is not hard to see that the formal adjoint W_j^* of W_j on $b\Omega'$ is the restriction of the formal adjoint W_j^* of W_j on the whole domain. Hence we simply compute the formal adjoint W_j^* of W_j on the whole domain, and we have

$$\begin{aligned}
 W_j^* &= \left(\sum_{k=1}^{n-1} Q_{kj} Y_k \right)^* \\
 &= \sum_{k=1}^{n-1} \bar{Q}_{kj} Y_k^* - \sum_{k=1}^{n-1} \bar{Y}_k (\bar{Q}_{kj}) \\
 &= - \sum_{k=1}^{n-1} \bar{Q}_{kj} \bar{Y}_k + \theta_0 - \sum_{k=1}^{n-1} (-b^{-1} \omega_j Q_{kk} \bar{\gamma}) \\
 &= - \bar{W}_j + (n-2) \frac{\bar{\sigma}_j \bar{\gamma}}{b^2} + \theta_0.
 \end{aligned}$$

Proof of (P10).

$$\begin{aligned}
 & W_b \left((n-2) \frac{\bar{r}\bar{\sigma}_j}{b^2} \right) \\
 &= (n-2) \sum_{p=1}^{n-1} Q_{pb} Y_p(\bar{r}) \frac{\bar{\sigma}_j}{b^2} \\
 &\quad + (n-2) \sum_{p=1}^{n-1} Q_{pb} \bar{r} \left(\frac{Y_p(\bar{\sigma}_j)}{b^2} - \frac{\bar{\sigma}_j}{b^4} \left(\sum_{m=1}^{n-1} \sigma_m Y_p \bar{\sigma}_m + \bar{\sigma}_m Y_p \sigma_m \right) \right) \\
 &= (n-2) \frac{|\bar{r}|^2}{b^2} Q_{jb} - (n-2) \frac{|\bar{r}|^2}{b^4} \sum_{p=1}^{n-1} Q_{pb} \bar{\sigma}_j \sigma_p + \theta_1 \\
 &= (n-2) \frac{|\bar{r}|^2}{b^2} Q_{jb} + \theta_1.
 \end{aligned}$$

We now go back to estimate term A , where A is given by

$$A = \left\| \sum_k \chi \eta_k \wedge W_k u \right\|^2 + \left\| \sum_k \chi W_k^*(u \llcorner \eta_k) \right\|^2.$$

THEOREM 3.23.

$$\begin{aligned}
 A &= \sum_{j=1}^{n-1} \|\chi W_j u\|^2 + \sum_{j,k} \sum'_{\{J_1\}=q-1} \langle [W_k, W_j^*] u_{J_1 j}, \chi^2 u_{J_1 k} \rangle \\
 &\quad - (q-1) q \left\| \chi \left(\frac{ru}{b} \right) \right\|^2 + \text{admissible errors}.
 \end{aligned}$$

Proof.

$$\begin{aligned}
 A &= \sum_{j,k} \langle \chi^2 \eta_j \wedge W_j u, \eta_k \wedge W_k u \rangle \\
 &\quad + \sum_{j,k} \langle \chi^2 W_j^*(u \llcorner \eta_j), W_k^*(u \llcorner \eta_k) \rangle \\
 &= \sum_{j,k} (\langle \chi^2 (\eta_j \llcorner \eta_k) W_j u, W_k u \rangle - \langle \chi^2 \eta_j \wedge (W_j u \llcorner \eta_k), W_k u \rangle) \\
 &\quad + \sum_{j,k} \langle \chi^2 W_j^*(u \llcorner \eta_j), W_k^*(u \llcorner \eta_k) \rangle \\
 &= \sum_{j,k} (\langle \chi^2 Q_{jk} W_j u, W_k u \rangle - \langle \chi^2 W_j u \llcorner \eta_k, W_k u \llcorner \eta_j \rangle) \\
 &\quad + \sum_{j,k} \langle \chi^2 W_j^*(u \llcorner \eta_j), W_k^*(u \llcorner \eta_k) \rangle \\
 &= \sum_{j=1}^{n-1} \|\chi W_j u\|^2 - \sum_{j,k} \langle \chi^2 W_j u \llcorner \eta_k, W_k u \llcorner \eta_j \rangle \\
 &\quad + \sum_{j,k} \langle \chi^2 W_j^*(u \llcorner \eta_j), W_k^*(u \llcorner \eta_k) \rangle \\
 &= \sum_{j=1}^{n-1} \|\chi W_j u\|^2 - I + II.
 \end{aligned}$$

Estimate for II .

$$\begin{aligned}
 \text{II} &= \sum_{j,k} \sum'_{J_1, J_2} \langle \chi^2 W_j^*(u_{J_1 j}), W_k^*(u_{J_2 k}) \det(J_2, J_1) \rangle \\
 &= \sum_{j,k} \sum'_{J_1} (\langle W_k(\chi^2) W_j^*(u_{J_1 j}), u_{J_1 k} \rangle + \langle \chi^2 W_k W_j^*(u_{J_1 j}), u_{J_1 k} \rangle) \\
 &\quad + \sum_{j,k} \sum'_{J_1, J_2} \langle W_j^*(u_{J_1 j}), \chi^2 u_{J_2 k} \bar{W}_k(\det((J_2, J_1))) \rangle \\
 &= \sum_{j,k} \sum'_{J_1} \{ \langle u_{J_1 j}, \bar{W}_k(\chi^2) W_j(u_{J_1 k}) \rangle + \langle u_{J_1 j}, W_j \bar{W}_k(\chi^2) u_{J_1 k} \rangle \\
 &\quad + \langle W_k u_{J_1 j}, W_j(\chi^2) u_{J_1 k} \rangle + \langle W_k u_{J_1 j}, \chi^2 W_j u_{J_1 k} \rangle \\
 &\quad + \langle [W_k, W_j^*] u_{J_1 j}, \chi^2 u_{J_1 k} \rangle \} \\
 &\quad + \sum_{j,k} \sum'_{J_1, J_2} \{ \langle u_{J_1 j}, W_j(\chi^2) u_{J_2 k} \bar{W}_k \det(J_2, J_1) \rangle \\
 &\quad + \langle u_{J_1 j}, \chi^2 W_j(u_{J_2 k}) \bar{W}_k \det(J_2, J_1) \rangle \\
 &\quad + \langle u_{J_1 j}, \chi^2 u_{J_2 k} W_j \bar{W}_k \det(J_2, J_1) \rangle \}.
 \end{aligned}$$

Estimate for I .

$$\begin{aligned}
 \text{I} &= \sum_{j,k} \langle \chi^2 W_j u \lfloor \eta_k, W_k u \lfloor \eta_j \rangle \rangle \\
 &= \sum_{j,k} \sum'_{J_1, J_2} \langle \chi^2 W_j(u_{J_1 k}), W_k(u_{J_2 j}) \det(J_2, J_1) \rangle \\
 &= \sum_{j,k} \sum'_{J_1} \langle \chi^2 W_j(u_{J_1 k}), W_k(u_{J_1 j}) \rangle \\
 &\quad - \sum_{j,k} \sum'_{J_1, J_2} \langle \chi^2 W_j(u_{J_1 k}), u_{J_2 j} W_k \det(J_2, J_1) \rangle.
 \end{aligned}$$

Hence we obtain

$$\begin{aligned}
 A &= \sum_j \| \chi W_j u \|^2 + \sum_{j,k} \sum'_{J_1} \{ \langle u_{J_1 j}, \bar{W}_k(\chi^2) W_j(u_{J_1 k}) \rangle \\
 &\quad + \langle u_{J_1 j}, W_j \bar{W}_k(\chi^2) u_{J_1 k} \rangle \\
 &\quad + \langle W_k(u_{J_1 j}), W_j(\chi^2) u_{J_1 k} \rangle + \langle [W_k, W_j^*] u_{J_1 j}, \chi^2 u_{J_1 k} \rangle \} \\
 &\quad + \sum_{j,k} \sum'_{J_1, J_2} \{ \langle u_{J_1 j}, W_j(\chi^2) u_{J_2 k} \bar{W}_k \det(J_2, J_1) \rangle \\
 &\quad + \langle u_{J_1 j}, \chi^2 u_{J_2 k} W_j \bar{W}_k \det(J_2, J_1) \rangle \\
 &\quad + 2\operatorname{Re} \langle u_{J_1 j}, \chi^2 W_j(u_{J_2 k}) \bar{W}_k \det(J_2, J_1) \rangle \}.
 \end{aligned}$$

Estimate for

$$E_1 = \sum_{j,k} \sum'_{J_1, J_2} \langle u_{J_1 j}, \chi^2 u_{J_2 k} W_j \bar{W}_k \det(J_2, J_1) \rangle.$$

$$\begin{aligned}
E_1 &= \left(\frac{1}{(q-1)!} \right)^2 \sum_{j,k} \sum_{J_1, J_2} \langle \chi^2 u_{J_2 k}, u_{J_1 j} W_k \bar{W}_j \det(J_1, J_2) \rangle \\
&= \left(\frac{1}{(q-1)!} \right)^2 \frac{|r|^2}{b^2} \\
&\quad \cdot \sum_{j,k} \sum_{J_1, J_2} \left\langle \chi^2 u_{J_2 k}, u_{J_1 j} \left(\sum_{t=1}^{q-1} \sum_{s=1}^{q-1} \sum_{\pi} \varepsilon_{\pi} \bar{\omega}_{i_{\pi(t)}} \omega_{m_s} Q_{i_{\pi(1)} m_1} \cdots \right. \right. \\
&\quad \left. \left. \cdots Q_{i_{\pi(s)} k} \cdots Q_{j m_t} \cdots Q_{i_{\pi(q-1)} m_{q-1}} \right) \right\rangle \\
&\quad - \sum_{t=1}^{q-1} \sum_{\pi} \varepsilon_{\pi} Q_{i_{\pi(1)} m_1} \cdots Q_{i_{\pi(t)} k} \cdots Q_{i_{\pi(q-1)} m_{q-1}} Q_{j m_t} \rangle \\
&\quad + \langle \theta_1 \chi u, \chi u \rangle \\
&= - \left(\frac{1}{(q-1)!} \right)^2 \frac{|r|^2}{b^2} \\
&\quad \cdot \sum_{j,k} \sum_{J_1, J_2} \sum_{t=1}^{q-1} \sum_{\pi} \varepsilon_{\pi} \langle \chi^2 u_{J_2 k}, u_{J_1 j} Q_{i_{\pi(1)} m_1} \cdots Q_{i_{\pi(t)} k} \cdots \right. \\
&\quad \left. \cdots Q_{i_{\pi(q-1)} m_{q-1}} Q_{j m_t} \rangle \\
&\quad + \langle \theta_1 \chi u, \chi u \rangle \\
&= - \left(\frac{1}{(q-1)!} \right)^2 \frac{|r|^2}{b^2} \\
&\quad \cdot \sum_{k=1}^{n-1} \sum_{J_2} \sum_{t=1}^{q-1} \sum_{\pi} \langle \chi^2 u_{J_2 k}, u_{m_1 \cdots k \cdots m_{q-1} m_t} \rangle + \langle \theta_1 \chi u, \chi u \rangle \\
&\quad t\text{-th position} \\
&= \frac{|r|^2}{b^2} \sum_{k=1}^{n-1} \sum_{J_2} (q-1) \langle \chi^2 u_{J_2 k}, u_{J_2 k} \rangle + \langle \theta_1 \chi u, \chi u \rangle \\
&= q(q-1) \left\| \chi \left(\frac{ru}{b} \right) \right\|^2 + \langle \theta_1 \chi u, \chi u \rangle.
\end{aligned}$$

Estimate for

$$\begin{aligned}
E_2 &= 2 \sum_{j,k} \sum'_{J_1, J_2} \langle u_{J_1 j}, \chi^2 W_j(u_{J_2 k}) \bar{W}_k \det(J_2, J_1) \rangle \\
E_2 &= -2b^{-1} r \sum_{t=1}^{q-1} \sum_{\pi} \sum_{j,k} \sum_{J_1, J_2} \left(\frac{1}{(q-1)!} \right)^2 \\
&\quad \cdot \langle u_{J_1 j} \varepsilon_{\pi} \omega_{m_t} Q_{i_{\pi(1)} m_1} \cdots Q_{i_{\pi(t)} k} \cdots Q_{i_{\pi(q-1)} m_{q-1}}, \chi^2 W_j(u_{J_2 k}) \rangle \\
&\quad + \langle \theta_0 \chi \bar{W} u, \chi u \rangle \\
&= -2b^{-1} r \sum_{t=1}^{q-1} \sum_{\pi} \sum_{j,k} \sum_{J_2} \left(\frac{1}{(q-1)!} \right)^2 \\
&\quad \cdot \langle u_{m_1 \cdots k \cdots m_{q-1} j}, \chi^2 W_j(u_{J_2 k}) \bar{\omega}_{m_t} \rangle \\
&\quad t\text{-th position} \\
&\quad + \langle \theta_0 \chi \bar{W} u, \chi u \rangle
\end{aligned}$$

$$\begin{aligned}
&= -2b^{-1} r \sum_{t=1}^{q-1} \sum_{\pi} \sum_{j,k} \sum_{J_2} \left(\frac{1}{(q-1)!} \right)^2 \\
&\quad \cdot \langle u_{m_1 \dots k \dots m_{q-1} j}, -x^2 u_{J_2 k} W_j(\bar{\omega}_{m_t}) \rangle \\
&\quad + \langle \theta_0 x W u, x u \rangle \\
&= 2b^{-1} r \sum_{t=1}^{q-1} \sum_{\pi} \sum_{j,k} \sum_{J_2} \left(\frac{1}{(q-1)!} \right)^2 \\
&\quad \langle u_{m_1 \dots k \dots m_{q-1} j}, x^2 u_{J_2 k} r b^{-1} Q_{m_t j} \rangle \\
&\quad + \langle \theta_0 x W u, x u \rangle + \langle \theta_1 x u, x u \rangle \\
&= -2b^{-1} r (q-1) \sum_{k=1}^{n-1} \sum'_{J_2} \langle u_{J_2 k}, x^2 b^{-1} r u_{J_2 k} \rangle \\
&\quad + \langle \theta_0 x W u, x u \rangle + \langle \theta_1 x u, x u \rangle \\
&= -2q(q-1) \left\| x \left(\frac{ru}{b} \right) \right\|^2 + \langle \theta_1 x u, x u \rangle + \langle \theta_0 x W u, x u \rangle.
\end{aligned}$$

Estimate for

$$\begin{aligned}
E_3 &= \sum_{j,k} \sum'_{J_1, J_2} \langle u_{J_1 j}, W_j(x^2) u_{J_2 k} \bar{W}_k \det(J_2, J_1) \rangle. \\
E_3 &= -b^{-1} r \sum_{t=1}^{q-1} \sum_{\pi} \sum_{j,k} \sum_{J_1, J_2} \sum' \varepsilon_{\pi} \langle u_{J_1 j} \bar{\omega}_{m_t} Q_{i_{\pi(1)} m_1} \dots Q_{i_{\pi(t)} k} \dots \\
&\quad \cdot Q_{i_{\pi(q-1)} m_{q-1}}, W_j(x^2) u_{J_2 k} \rangle + \langle \theta_1 u, u \rangle \\
&= \langle \theta_1 u, u \rangle.
\end{aligned}$$

Since $W_j(x^2) = \theta_0$, and we observe that

$$\begin{aligned}
&b^{-1} r \bar{\omega}_{m_t} \bar{Q}_{i_{\pi(1)} m_1} \dots \bar{Q}_{i_{\pi(t)} k} \dots \bar{Q}_{i_{\pi(q-1)} m_{q-1}} W_j(x^2) \\
&= b^{-1} (r \bar{\omega}_{m_t} \bar{Q}_{i_{\pi(1)} m_1} \dots \bar{Q}_{i_{\pi(t)} k} \dots \bar{Q}_{i_{\pi(q-1)} m_{q-1}} W_j(x^2)) \\
&= b^{-1} \theta_0 \\
&= \theta_1.
\end{aligned}$$

By combining these estimates and applying (P6), (P6)', we obtain

$$\begin{aligned}
A &= \sum_j \|x W_j u\|^2 + \sum_{j,k} \sum'_{\{J_1\}=q-1} \langle [W_k, W_j^*] u_{J_1 j}, x^2 u_{J_1 k} \rangle \\
&\quad - q(q-1) \left\| x \left(\frac{ru}{b} \right) \right\|^2 + \text{admissible errors}.
\end{aligned}$$

This completes the proof of Theorem 3.23.

The next step is to apply integration by parts to the first term $\sum_j \|x W_j u\|^2$ in A to cancel the bad term and get the desired a priori estimate.

THEOREM 3.24.

$$\begin{aligned} q \sum_{j=1}^{n-1} \|x W_j u\|^2 &= q \sum_{j=1}^{n-1} \|x \tilde{W}_j u\|^2 + (q(n-2)^2 - q^2) \left\| x \left(\frac{ru}{b} \right) \right\|^2 \\ &\quad + \sum_{j,k,m} \sum_{J_2}^{\prime} \langle u_{J_2 m}, x^2 Q_{kj} [W_j^*, W_k] u_{J_2 m} \rangle \\ &\quad + \text{admissible errors,} \end{aligned}$$

where $\tilde{W}_s = \sum_{j=1}^{n-1} Q_{sj} W_j^*$.

Proof.

$$\begin{aligned} q \sum_{j=1}^{n-1} \|x W_j u\|^2 &= \sum_{j,k} \|x W_j u \lceil \eta_k\|^2 = \sum_{j,k,p,m} \sum_{J_1, J_2}^{\prime} \\ &\quad \cdot \langle x^2 W_j(u_{J_1 p}), W_j(u_{J_2 m}) Q_{mp} \det(J_2, J_1) \rangle \\ &= \sum_{j,k,p,m} \sum_{J_1, J_2}^{\prime} \langle x^2 W_j(u_{J_1 p}), Q_{kj} W_k(u_{J_2 m}) Q_{mp} \det(J_2, J_1) \rangle \\ &= \sum_{j,k,p,m} \sum_{J_1, J_2}^{\prime} \langle u_{J_1 p}, x^2 Q_{kj} W_j^* W_k \\ &\quad \cdot (u_{J_2 m}) Q_{mp} \det(J_2, J_1) \rangle - R', \end{aligned}$$

where

$$\begin{aligned} R' &= \sum_{j,k,m,p} \sum_{J_1, J_2}^{\prime} \{ \langle u_{J_1 p}, \bar{W}_j(x^2) Q_{kj} W_k(u_{J_2 m}) Q_{mp} \det(J_2, J_1) \rangle \\ &\quad + \langle u_{J_1 p}, x^2 \bar{W}_j(Q_{kj}) W_k(u_{J_2 m}) Q_{mp} \det(J_2, J_1) \rangle \\ &\quad + \langle u_{J_1 p}, x^2 Q_{kj} W_k(u_{J_2 m}) \bar{W}_j(Q_{mp}) \det(J_2, J_1) \rangle \\ &\quad + \langle u_{J_1 p}, x^2 Q_{kj} W_k(u_{J_2 m}) Q_{mp} \bar{W}_j((\det J_2, J_1)) \rangle \} \\ &= R'_1 + R'_2 + R'_3 + R'_4. \end{aligned}$$

Estimate for R' .

$$\begin{aligned} R'_1 &= \langle \theta_0 W u, u \rangle. \\ R'_2 &= \sum_{j,k,m,p} \sum_{J_1, J_2}^{\prime} \langle u_{J_1 p}, x^2 (-b^{-1} \bar{\omega}_{kj} \bar{r} Q_{jj}) W_k(u_{J_2 m}) \\ &\quad \cdot Q_{mp} \det(J_2, J_1) \rangle + \langle \theta_0 x W u, x u \rangle \\ &= \langle \theta_0 x W u, x u \rangle. \end{aligned}$$

$$\begin{aligned} R'_3 &= \sum_{j,m,p} \sum_{J_2}^{\prime} \langle u_{J_2 p}, x^2 W_j(u_{J_2 m}) \bar{W}_j(Q_{mp}) \rangle \\ &= \sum_{j,m,p} \sum_{J_2}^{\prime} \langle u_{J_2 p}, x^2 W_j(u_{J_2 m}) (-b^{-1} \bar{\omega}_m \bar{r} Q_{jp}) \rangle \\ &\quad + \langle \theta_0 x W u, x u \rangle \end{aligned}$$

$$\begin{aligned}
&= \sum_{m,p} \sum'_{J_2} \langle u_{J_2 p}, x^2 b^{-1} \bar{r} u_{J_2 m} W_p(\bar{\omega}_m) \rangle + \langle \theta_0 x W u, x u \rangle \\
&= \sum_{m,p} \sum'_{J_2} \langle u_{J_2 p}, x^2 b^{-1} \bar{r} u_{J_2 m} (b^{-1} r Q_{m,p}) \rangle \\
&\quad + \langle \theta_1 x u, x u \rangle + \langle \theta_0 x W u, x u \rangle \\
&= \sum_m \sum'_{J_2} \langle b^{-1} \bar{r} u_{J_2 m}, x^2 b^{-1} r u_{J_2 m} \rangle \\
&\quad + \langle \theta_1 x u, x u \rangle + \langle \theta_0 x W u, x u \rangle \\
&= q \left\| x \left(\frac{ru}{b} \right) \right\|^2 + \langle \theta_1 x u, x u \rangle + \langle \theta_0 x W u, x u \rangle. \\
R'_4 &= \sum_{j,m} \sum'_{J_1, J_2} \langle u_{J_1 m} W_j \det(J_1, J_2), x^2 W_j(u_{J_2 m}) \rangle \\
&= -r b^{-1} \sum_{j,k} \sum'_{J_1, J_2} \sum_{t=1}^{q-1} \sum_{\pi} \epsilon_{\pi} \langle u_{J_1 k} \omega_{m_t} Q_{i_{\pi(1)} m_1} \cdots Q_{i_{\pi(t)} j} \cdots \\
&\quad \cdot Q_{i_{\pi(q-1)} m_{q-1}}, x^2 W_j(u_{J_2 k}) \rangle + \langle \theta_0 x W u, x u \rangle \\
&= -r b^{-1} \left(\frac{1}{(q-1)!} \right)^2 \\
&\quad \cdot \sum_{j,k} \sum'_{J_2} \sum_{t=1}^{q-1} \sum_{\pi} \underset{t\text{-th position}}{\uparrow} \langle u_{m_1 \dots j \dots m_{q-1} k}, x^2 W_j(u_{J_2 k}) \bar{\omega}_{m_t} \rangle \\
&\quad + \langle \theta_0 x W u, x u \rangle \\
&= \frac{|r|^2}{b^2} \left(\frac{1}{(q-1)!} \right)^2 \\
&\quad \cdot \sum_{j,k} \sum'_{J_2} \sum_{t=1}^{q-1} \sum_{\pi} \langle u_{m_1 \dots j \dots m_{q-1} k}, x^2 u_{J_2 k} (Q_{m_t j}) \rangle \\
&\quad + \langle \theta_1 x u, x u \rangle + \langle \theta_0 x W u, x u \rangle \\
&= \frac{|r|^2}{b^2} \sum_{k=1}^{n-1} \sum'_{J_2} (q-1) \langle u_{J_2 k}, x^2 u_{J_2 k} \rangle \\
&\quad + \langle \theta_1 x u, x u \rangle + \langle \theta_0 x W u, x u \rangle \\
&= q(q-1) \left\| x \left(\frac{ru}{b} \right) \right\|^2 + \langle \theta_1 x u, x u \rangle + \langle \theta_0 x W u, x u \rangle.
\end{aligned}$$

From these estimates, it follows that

$$R' = q^2 \left\| x \left(\frac{ru}{b} \right) \right\|^2 + \text{admissible errors.}$$

Therefore we obtain

$$\begin{aligned}
&q \sum_{j=1}^{n-1} \|x W_j u\|^2 \\
&= \sum_{j,k,p,m} \sum'_{J_1, J_2} \langle u_{J_1 p}, x^2 Q_{kj} W_j^* W_k(u_{J_2 m}) Q_{m,p} \det(J_2, J_1) \rangle
\end{aligned}$$

$$\begin{aligned}
& -q^2 \left\| \chi \left(\frac{ru}{b} \right) \right\|^2 + \text{admissible errors} \\
& = \sum_{j,k,p,m} \sum'_{J_1, J_2} \langle u_{J_1 p}, \chi^2 Q_{kj} ([W_j^*, W_k] u_{J_2 m}) Q_{pm} \det(J_2, J_1) \rangle \\
& \quad + \sum_{j,k,p,m} \sum'_{J_1, J_2} \langle u_{J_1 p}, \chi^2 Q_{kj} (W_k W_j^* u_{J_2 m}) Q_{pm} \det(J_2, J_1) \rangle \\
& \quad - q^2 \left\| \chi \left(\frac{ru}{b} \right) \right\|^2 + \text{admissible errors} \\
& = \sum_{j,k,m} \sum'_{J_2} \langle u_{J_2 m}, \chi^2 Q_{kj} [W_j^*, W_k] u_{J_2 m} \rangle - q^2 \left\| \chi \left(\frac{ru}{b} \right) \right\|^2 \\
& \quad + \sum_{j,k,p,m} \sum'_{J_1, J_2} \langle W_k^* (u_{J_1 p}) \chi^2 \\
& \quad \cdot Q_{jk} Q_{pm} \det(J_1, J_2), W_j^* (u_{J_2 m}) \rangle \\
& \quad - \sum_{j,k,p,m} \sum'_{J_1, J_2} \langle u_{J_1 p} \chi^2 \bar{W}_k (Q_{jk}) Q_{pm} \det(J_1, J_2), W_j^* (u_{J_2 m}) \rangle \\
& \quad - R'' + \text{admissible errors}
\end{aligned}$$

where

$$\begin{aligned}
R'' &= \sum_{j,k,p,m} \sum'_{J_1, J_2} \{ \langle u_{J_1 p} \bar{W}_k (\chi^2) Q_{jk} Q_{pm} \det(J_1, J_2), W_j^* u_{J_2 m} \rangle \\
&\quad + \langle u_{J_1 p} \chi^2 Q_{jk} \bar{W}_k (Q_{pm}) \det(J_1, J_2), W_j^* u_{J_2 m} \rangle \\
&\quad + \langle u_{J_1 p} \chi^2 Q_{jk} Q_{pm} \bar{W}_k \det(J_1, J_2), W_j^* u_{J_2 m} \rangle \} \\
&= \sum_{j,m} \sum'_{J_2} \{ \langle W_j (u_{J_2 m}) \bar{W}_j (\chi^2), u_{J_2 m} \rangle \\
&\quad + \langle u_{J_2 m} W_j \bar{W}_j (\chi^2), u_{J_2 m} \rangle \} \\
&\quad + \sum_{j,p,m} \sum'_{J_2} \{ \langle W_j (u_{J_2 p}) \chi^2 \bar{W}_j (Q_{pm}), u_{J_2 m} \rangle \\
&\quad + \langle u_{J_2 p} W_j (\chi^2) \bar{W}_j (Q_{pm}), u_{J_2 m} \rangle \\
&\quad + \langle u_{J_2 p} \chi^2 W_j \bar{W}_j (Q_{pm}), u_{J_2 m} \rangle \} \\
&\quad + \sum_{j,m} \sum'_{J_1, J_2} \{ \langle W_j (u_{J_1 m}) \chi^2 \bar{W}_j \det(J_1, J_2), u_{J_2 m} \rangle \\
&\quad + \langle u_{J_1 m} W_j (\chi^2) \bar{W}_j \det(J_1, J_2), u_{J_2 m} \rangle \\
&\quad + \langle u_{J_1 m} \chi^2 W_j \bar{W}_j \det(J_1, J_2), u_{J_2 m} \rangle \} \\
&= (R'_1 + R'_2) + (R'_3 + R'_4 + R'_5) + (R'_6 + R'_7 + R'_8).
\end{aligned}$$

Estimate for R'' .

$$R'_1 = \langle \Theta_0 W u, u \rangle.$$

$$R'_2 = \langle \Theta_1 u, u \rangle.$$

$$R'_3 = \bar{R}_3 = q \left\| \chi \left(\frac{ru}{b} \right) \right\|^2 + \langle \Theta_1 x u, x u \rangle + \langle \Theta_0 x W u, x u \rangle.$$

$$R''_4 = \sum_{j,p,m} \sum'_{J_2} \langle u_{J_2 b} W_j(\chi^2)(-b^{-1} \bar{r} \bar{\omega}_p Q_{jm}), u_{J_2 m} \rangle + \langle \theta_1 u, u \rangle$$

$$= \langle \theta_1 u, u \rangle.$$

$$\begin{aligned} R''_5 &= \sum_{j,p,m} \sum'_{J_2} \frac{|\tau|^2}{b^2} \langle u_{J_2 p} \chi^2(Q_{jj} \bar{\omega}_p \omega_m - Q_{pj} Q_{jm}), u_{J_2 m} \rangle \\ &\quad + \langle \theta_1 \chi u, \chi u \rangle \\ &= - \sum_m \sum'_{J_2} \frac{|\tau|^2}{b^2} \langle u_{J_2 m}, \chi^2 u_{J_2 m} \rangle + \langle \theta_1 \chi u, \chi u \rangle \\ &= -q \left\| \chi \left(\frac{ru}{b} \right) \right\|^2 + \langle \theta_1 \chi u, \chi u \rangle. \end{aligned}$$

$$R''_6 = \overline{R}_4 = q(q-1) \left\| \chi \left(\frac{ru}{b} \right) \right\|^2 + \langle \theta_1 \chi u, \chi u \rangle + \langle \theta_0 \chi W u, \chi u \rangle.$$

R''_7 can be estimated similarly as we did for E_3 , and we get

$$R''_7 = \langle \theta_1 u, u \rangle.$$

$$\begin{aligned} R''_8 &= -\frac{|\tau|^2}{b^2} \left(\frac{1}{(q-1)!} \right)^2 \sum_{j,k} \sum_{J_1, J_2} \sum_{t=1}^{q-1} \sum_{\pi} \\ &\quad \cdot \epsilon_{\pi} \langle u_{J_1 k} \chi^2 Q_{i_{\pi(1)} m_1} \cdots (Q_{i_{\pi(t)} j} Q_{j m_t}) \cdots \\ &\quad \cdot Q_{i_{\pi(q-1)} m_{q-1}}, u_{J_2 k} \rangle + \langle \theta_1 \chi u, \chi u \rangle \\ &= -q(q-1) \left\| \chi \left(\frac{ru}{b} \right) \right\|^2 + \langle \theta_1 \chi u, \chi u \rangle. \end{aligned}$$

By combining all of these estimates and applying (P6) and (P6)', we get

$$R'' = \text{admissible errors.}$$

This shows that

$$\begin{aligned} q \sum_{j=1}^{n-1} \| \chi W_j u \|^2 &= \sum_{j,k,m} \sum'_{J_2} \langle u_{J_2 m}, \chi^2 Q_{kj} [W_j^*, W_k] u_{J_2 m} \rangle \\ &\quad - q^2 \left\| \chi \left(\frac{ru}{b} \right) \right\|^2 + q \sum_{j=1}^{n-1} \| \chi \tilde{W}_j u \|^2 \\ &\quad - \sum_{j,k,p,m} \sum'_{J_1, J_2} \langle u_{J_1 p} \chi^2 \bar{W}_k (Q_{jk}) \\ &\quad \cdot Q_{pm} \det(J_1, J_2), W_j^* u_{J_2 m} \rangle \\ &\quad + \text{admissible errors} \end{aligned}$$

$$\begin{aligned}
&= q \sum_j \| \chi \tilde{W}_j u \|^2 - q^2 \left\| \chi \left(\frac{ru}{b} \right) \right\|^2 \\
&\quad + \sum_{j,k,m} \sum'_{J_2} \langle u_{J_2m}, \chi^2 Q_{kj} [W_j^*, W_k] u_{J_2m} \rangle \\
&\quad - \sum_{j,k,m} \sum'_{J_2} \{ \langle W_j(u_{J_2m}) \chi^2 \bar{W}_k Q_{jk}, u_{J_2m} \rangle \\
&\quad + \langle u_{J_2m} W_j(\chi^2) \bar{W}_k(Q_{jk}), u_{J_2m} \rangle \\
&\quad + \langle u_{J_2m} \chi^2 W_j \bar{W}_k Q_{jk}, u_{J_2m} \rangle \} \\
&\quad + \text{admissible errors.}
\end{aligned} \tag{10}$$

The last three terms can be estimated as follows.

$$\begin{aligned}
&\sum_{j,k,m} \sum'_{J_2} \langle W_j(u_{J_2m}) \chi^2 \bar{W}_k Q_{jk}, u_{J_2m} \rangle \\
&= \sum_{j,k,m} \sum'_{J_2} \langle W_j(u_{J_2m}) \chi^2 (-b^{-1} \bar{\omega}_j \bar{r} Q_{kk}), u_{J_2m} \rangle \\
&\quad + \langle \Theta_0 \chi W u, \chi u \rangle \\
&= \langle \Theta_0 \chi W u, \chi u \rangle.
\end{aligned}$$

Similarly we have

$$\sum_{j,k,m} \sum'_{J_2} \langle u_{J_2m} W_j(\chi^2) \bar{W}_k(Q_{jk}), u_{J_2m} \rangle = \langle \Theta_1 u, u \rangle.$$

For the last term, we estimate as follows.

$$\begin{aligned}
&\sum_{j,k,m} \sum'_{J_2} \langle u_{J_2m} \chi^2 W_j \bar{W}_k Q_{jk}, u_{J_2m} \rangle \\
&= \sum_{j,k,m} \sum'_{J_2} \langle u_{J_2m} \chi^2 \left(\frac{|r|^2}{b^2} \right) (Q_{kj} \bar{\omega}_j \omega_k - Q_{jj} Q_{kk}), u_{J_2m} \rangle \\
&\quad + \langle \Theta_1 \chi u, \chi u \rangle \\
&= -(n-2)^2 \sum_m \sum'_{J_2} \langle \chi^2 u_{J_2m}, u_{J_2m} \rangle + \langle \Theta_1 \chi u, \chi u \rangle \\
&= -(n-2)^2 q \left\| \chi \left(\frac{ru}{b} \right) \right\|^2 + \langle \Theta_1 \chi u, \chi u \rangle.
\end{aligned}$$

Put everything together, we obtain the following estimate.

$$\begin{aligned}
q \sum_{j=1}^{n-1} \| \chi W u \|^2 &= q \sum_{j=1}^{n-1} \| \chi \tilde{W}_j u \|^2 + (q(n-2)^2 - q^2) \left\| \chi \left(\frac{ru}{b} \right) \right\|^2 \\
&\quad + \sum_{j,k,m} \sum'_{J_2} \langle u_{J_2m}, \chi^2 Q_{kj} [W_j^*, W_k] u_{J_2m} \rangle \\
&\quad + \text{admissible errors.}
\end{aligned}$$

This completes the proof of Theorem 3.24.

The last step is to calculate these two commutator terms which appear in Theorem 3.23 and Theorem 3.24.

$$\begin{aligned}
 \text{(i)} \quad & \sum_{j,k} \sum'_{J_1} \langle [W_k, W_j^*] u_{J_1 j}, \chi^2 u_{J_1 k} \rangle \\
 &= - \sum_{j,k} \sum'_{J_1} \langle [W_k, \bar{W}] u_{J_1 j}, \chi^2 u_{J_1 k} \rangle \\
 &\quad + (n-2) \sum_{j,k} \sum'_{J_1} \left\langle W_k \left(\frac{\bar{r}\bar{\sigma}_j}{b^2} \right) u_{J_1 j}, \chi^2 u_{J_1 k} \right\rangle + \langle \Theta_1 \chi u, \chi u \rangle \\
 &= - \sum_{j,k} \sum'_{J_1} \left\langle \left(b^{-1} Q_{jk} X^0 \right. \right. \\
 &\quad \left. \left. + \sum_m \Theta_0 W_m + \sum_m \Theta_0 \bar{W}_m \right) u_{J_1 j}, \chi^2 u_{J_1 k} \right\rangle \\
 &\quad + (n-2) \frac{|r|^2}{b^2} \sum_{j,k} \sum'_{J_1} \langle Q_{jk} u_{J_1 j}, \chi^2 u_{J_1 k} \rangle + \langle \Theta_1 \chi u, \chi u \rangle \\
 &= - \sum_k \sum'_{J_1} \left\langle b^{-1} X^0 u_{J_1 k}, \chi^2 u_{J_1 k} \right\rangle \\
 &\quad + q(n-2) \left\| \chi \left(\frac{ru}{b} \right) \right\|^2 + \langle \Theta_0 \chi \bar{W} u, \chi u \rangle \\
 &\quad + \langle \Theta_0 \chi \bar{W} u, \chi u \rangle + \langle \Theta_1 \chi u, \chi u \rangle \\
 &= - \sum_k \sum'_{J_1} \left\langle b^{-1} X^0 u_{J_1 k}, \chi^2 u_{J_1 k} \right\rangle \\
 &\quad + q(n-2) \left\| \chi \left(\frac{ru}{b} \right) \right\|^2 \\
 &\quad + \text{admissible errors.}
 \end{aligned}$$

$$\begin{aligned}
 \text{(ii)} \quad & \sum_{j,k,m} \sum'_{J_1} \langle u_{J_1 m}, \chi^2 Q_{kj} [W_j^*, W_k] u_{J_1 m} \rangle \\
 &= \sum_{j,k,m} \sum'_{J_1} \langle u_{J_1 m}, \chi^2 Q_{kj} [W_k, \bar{W}_j] u_{J_1 m} \rangle \\
 &\quad - \sum_{j,k,m} \sum'_{J_1} \left\langle u_{J_1 m}, \chi^2 Q_{kj} W_k \left((n-2) \frac{\bar{r}\bar{\sigma}_j}{b^2} \right) u_{J_1 m} \right\rangle \\
 &\quad + \langle \Theta_1 \chi u, \chi u \rangle \\
 &= \sum_{j,k,m} \sum'_{J_1} \left\langle u_{J_1 m}, \chi^2 Q_{kj} \left(b^{-1} Q_{jk} X^0 \right. \right. \\
 &\quad \left. \left. + \sum_p \Theta_0 W_p + \sum_p \Theta_0 \bar{W}_p \right) u_{J_1 m} \right\rangle \\
 &\quad - (n-2) \frac{|r|^2}{b^2} \sum_{j,m,k} \sum'_{J_1} \langle u_{J_1 m}, \chi^2 Q_{kj} Q_{jk} u_{J_1 m} \rangle \\
 &\quad + \langle \Theta_1 \chi u, \chi u \rangle
 \end{aligned}$$

$$\begin{aligned}
 &= (n-2) \sum_m \sum'_{J_1} \langle x^2 u_{J_1 m}, b^{-1} X^0 u_{J_1 m} \rangle \\
 &\quad - q(n-2)^2 \left\| x \left(\frac{ru}{b} \right) \right\|^2 \\
 &\quad + \text{admissible errors.}
 \end{aligned}$$

Now we go back to the equation 3.20.

$$\begin{aligned}
 &\|x D_b u\|^2 + \|x D_b^* u\|^2 \\
 &= A - B + \frac{1}{4} \left(\left\| \sum_k x \eta_k \wedge r_{(k)} u \right\|^2 + \left\| \sum_k x r_{(k)}^*(u \llcorner \eta_k) \right\|^2 \right) \\
 &= \sum_j \|x W_j u\|^2 + \sum_{j,k} \sum'_{J_1} \langle [W_k, W_j^*] u_{J_1 j}, x^2 u_{J_1 k} \rangle \\
 &\quad - q(q-1) \left\| x \left(\frac{ru}{b} \right) \right\|^2 \\
 &\quad + \frac{1}{4} \left(\left\| \sum_k x \eta_k \wedge r_{(k)} u \right\|^2 + \left\| \sum_k x r_{(k)}^*(u \llcorner \eta_k) \right\|^2 \right) \\
 &\quad + \text{admissible errors} \\
 &= \frac{n-2-q}{n-2} \sum_j \|x W_j u\|^2 \\
 &\quad + \frac{1}{n-2} \left\{ q \sum_{j=1}^{n-1} \|x \tilde{W}_j u\|^2 + (q(n-2)^2 - q^2) \left\| x \left(\frac{ru}{b} \right) \right\|^2 \right. \\
 &\quad \left. + \sum_{j,k,m} \sum'_{J_1} \langle u_{J_1 m}, x^2 Q_{kj}[W_j^*, W_k] u_{J_2 m} \rangle \right\} \\
 &\quad + \sum_{j,k} \sum'_{J_1} \langle [W_k, W_j^*] u_{J_1 j}, x^2 u_{J_1 k} \rangle \\
 &\quad - q(q-1) \left\| x \left(\frac{ru}{b} \right) \right\|^2 \\
 &\quad + \frac{1}{4} \left(\left\| \sum_k x \eta_k \wedge r_{(k)} u \right\|^2 + \left\| \sum_k x r_{(k)}^*(u \llcorner \eta_k) \right\|^2 \right) \\
 &\quad + \text{admissible errors} \\
 &= \frac{n-2-q}{n-2} \sum_j \|x W_j u\|^2 + \frac{q}{n-2} \sum_j \|x \tilde{W}_j u\|^2 \\
 &\quad + \left(\frac{q(n-2)^2 - q^2}{n-2} - q(q-1) \right) \left\| x \left(\frac{ru}{b} \right) \right\|^2 \\
 &\quad + \frac{1}{n-2} \left\{ (n-2) \sum_m \sum'_{J_1} \langle x^2 u_{J_1 m}, b^{-1} X^0 u_{J_1 m} \rangle \right. \\
 &\quad \left. - q(n-2)^2 \left\| x \left(\frac{ru}{b} \right) \right\|^2 \right\} \\
 &\quad - \sum_k \sum'_{J_1} \langle b^{-1} X^0 u_{J_1 k}, x^2 u_{J_1 k} \rangle + q(n-2) \left\| x \left(\frac{ru}{b} \right) \right\|^2
 \end{aligned}$$

$$\begin{aligned}
& + \frac{1}{4} \left(\left\| \sum_k \chi_{\eta_k} \wedge r_{(k)} u \right\|^2 + \left\| \sum_k \chi r_{(k)}^* (u \llcorner \eta_k) \right\|^2 \right) \\
& \quad + \text{admissible errors} \\
& = \frac{n-2-q}{n-2} \sum_j \| \chi W_j u \|^2 + \frac{q}{n-2} \sum_j \| \chi \tilde{W}_j u \|^2 \\
& \quad + \left(\frac{q(n-2)^2 - q^2}{n-2} - q(q-1) \right) \left\| \chi \left(\frac{ru}{b} \right) \right\|^2 \\
& \quad + \frac{1}{4} \left(\left\| \sum_k \chi_{\eta_k} \wedge r_{(k)} u \right\|^2 + \left\| \sum_k \chi r_{(k)}^* (u \llcorner \eta_k) \right\|^2 \right) \\
& \quad + \text{admissible errors}.
\end{aligned}$$

Finally we state and prove our main theorem.

THEOREM 3.25 *Let ${}^0T''$ be a strongly pseudoconvex CR-structure on a manifold M of real dimension $2n-1$. Pick a fixed supplementary real vector field S so that the Levi-form is positive definite. Use this Levi-form as the metric on ${}^0T''$ and pick a smooth orthonormal base Y_1, \dots, Y_{n-1} of ${}^0T''$. Let t be an admissible distance function to a reference point p_0 . Let W_j and b be defined as before. Set*

$$b\Omega_r = \{p \in M \mid t(p) = r\}, \quad b\Omega'_r = b\Omega_r - \{p \in M \mid b(p) = 0\}.$$

Consider $b\Omega'_r$ as a CR-manifold with the induced CR-structure. Let u be a q -form on $b\Omega'_r$ such that

- (i) u is C^1 on $b\Omega'_r$, and
- (ii) $D_b u, D_b^* u, b^{-1} u$ and $W_j u$ are in L^2 .

Then $\tilde{W}_j u$ is also in L^2 and for any small $\delta > 0$, the following estimate holds

$$\begin{aligned}
& \frac{1}{1-\delta} (\|D_b u\|^2 + \|D_b^* u\|^2) \\
& \geq \frac{n-2-q}{n-2} \sum_j \|W_j u\|^2 + \frac{q}{n-2} \sum_j \|\tilde{W}_j u\|^2 \\
& \quad + \left(\frac{q(n-2)^2 - q^2}{n-2} - q(q-1) \right) \left\| \frac{ru}{b} \right\|^2
\end{aligned}$$

provided r is sufficiently small and $n-2-q > 0$ and

$$\left(\frac{(q(n-2)^2 - q^2)}{n-2} - q(q-1) \right) > 0,$$

where r appears in (iii) of Definition 2.23.

Proof. Just recall that $x^2 = g(tb^2)$, and x tends to the characteristic function when t tends to infinity. By the positivity of the coefficients, the admissible errors can be absorbed by the main terms, provided r is sufficiently small. This completes the proof of the theorem.

DEFINITION 3.26. Denote by $Q^q(b\Omega'_r, {}^0T'')$ the pre-Hilbert space of all ${}^0T''$ q -forms u on $b\Omega'_r$ satisfying conditions (i) and (ii) in Theorem 3.25, where the norm is given by

$$(3.27) \quad \|u\|_q^2 = \|D_b u\|^2 + \|D_b^* u\|^2.$$

Also denote by $L_{(a)}^q(b\Omega'_r, {}^0T'')$ the Hilbert space of all ${}^0T''$ q -forms u on $b\Omega'_r$ such that $|b^a u|^2$ are integrable, i.e., the norm is given by

$$(3.28) \quad \|u\|_a = \|b^a u\|.$$

$Q^q(b\Omega'_r, {}^0T'')$ and $L_{(a)}^q(b\Omega'_r, {}^0T'')$ will be abbreviated to Q and $L_{(a)}$ respectively, when there is no possibility of confusion.

We denote by $[Q^q(b\Omega'_r, {}^0T'')]$ or $[Q]$ the completion of $Q^q(b\Omega'_r, {}^0T'')$. When r is sufficiently small, then we see from Theorem 3.25 that for any $u \in [Q]$, $D_b u$, $D_b^* u$, $b^{-1} u$, $W_j u$ and $\tilde{W}_j u$ are all in L^2 , and their L^2 -norms are bounded by $\|u\|_Q$.

PROPOSITION 3.29. *For any $\alpha \in L_{(1)}$, there is an unique u in $[Q]$ such that*

$$\langle D_b u, D_b v \rangle + \langle D_b^* u, D_b^* v \rangle = \langle \alpha, v \rangle$$

holds for all v in $[Q]$. Moreover

$$\|u\|_Q \leq \left((1 - \delta) \left(\frac{q(n-2)^2 - q^2}{n-2} - q(q-1) \right) \right)^{-1/2} \|r^{-1} \alpha\|_1.$$

Proof. Define the map

$$\Lambda : [Q] \longrightarrow \mathbf{C}$$

by $\Lambda(v) = \langle \alpha, v \rangle$ for v in $[Q]$. Since

$$\begin{aligned} |\langle \alpha, v \rangle| &\leq \|b r^{-1} \alpha\| \cdot \|b^{-1} r v\| \\ &\leq \left((1 - \delta) \left(\frac{q(n-2)^2 - q^2}{n-2} - q(q-1) \right) \right)^{-1/2} \|b r^{-1} \alpha\| \cdot \|v\|_Q, \end{aligned}$$

hence the conclusion follows from the representation theorem of a bounded linear functional on Hilbert space.

REMARK 3.30. If we set $u = N_b \alpha$ for the unique u given by Proposition 3.29, then N_b , called the Neumann operator associated with ${}^0T''$ on $b\Omega'_r$, is a bounded linear map of $L_{(1)}$ into $[Q]$, and

$$\|N_b \alpha\|_Q \leq \left((1 - \delta) \left(\frac{q(n-2)^2 - q^2}{n-2} - q(q-1) \right) \right)^{-1/2} \|r^{-1} \alpha\|_1.$$

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