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ON THE CIRCLE OF ORTHOPOLES, CIRCLES OF ORTHOPOLE K-POINTS AND THE RELATED SYSTEM OF CENTER CIRCLES

BY

CHEN-JUNG HSU (許振榮)

1. **The circle of orthopoles.** Let a triangle $\Delta A_1 A_2 A_3$ and a line g be given. Suppose A_1', A_2', A_3' are points on the line g such that $A_1 A_1'$, $A_2 A_2'$, $A_3 A_3'$ are perpendicular to g. Then it is known that the line passing through A_1' and perpendicular to $A_2 A_3$, the line passing through A_2' and perpendicular to $A_3 A_1$, and the line passing through A_3' and perpendicular to $A_1 A_2$ are concurrent. This point of intersection of the three perpendicular lines is called *the orthopole of the line g with respect to the triangle* $\Delta A_1 A_2 A_3$ [1], [2].

PROPOSITION 1. *Let* 1, 2, 3, 4, 5 *be five points on a circle* (*i.e., they are concyclic points*). *Let l, m be any two of these points, and i, j, k be the rest of them. For example, consider the orthopole of the line lm with respect to the triangle Δijk. There are ${}_5C_2 = 10$ of such orthopoles.*

(i) *These ${}_5C_2$ orthopoles are concyclic, say on the circle* [1, 2, 3, 4, 5], *the center of which is denoted as* {1, 2, 3, 4, 5}.

(ii) *Starting with six concyclic points* 1, 2, 3, 4, 5, 6, *omitting each point in turn, six circles are found by* (i); *these are concurrent, say in the point* {1, 2, 3, 4, 5, 6}.

(iii) *Starting with seven concyclic points* 1, 2, 3, 4, 5, 6, 7, *seven points of the kind in* (ii) *are found; these lie on a circle denoted by* [1, 2, 3, 4, 5, 6, 7].

(iv) *Continuing thus infinitely, there is, in each case, finally a unique point or circle according as the number is even or odd.*

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(v) *The six centers* {1, 2, 3, 4, 5} *etc. of the six circles* [1, 2, 3, 4, 5] *etc. are on a circle with center* {1, 2, 3, 4, 5, 6}. *This circle is denoted as* [1, 2, 3, 4, 5, 6]. *The points* {1, 2, 3, 4, 5, 6} *etc. are on the circle* [1, 2, 3, 4, 5, 6, 7] *and so on. In this way, we have a system of center circles.*

Proof. (i) Without loss of generality we can assume that the circle is the unit circle in the Gaussian plane. Let the points 1, 2, 3, 4, 5 on the unit circle be represented by complex numbers t_1, t_2, t_3, t_4, t_5 repectively. Let

$$\frac{z}{a}+\frac{\bar{z}}{\bar{a}}=1 \tag{1}$$

be the equation of the line g with complex variable z and the complex conjugate $\bar{z}$ of z. Then the orthopole of g with respect to the triangle $\Delta 123$ is represented by the complex coordinate ([3], p. 281):

$$z=\frac{1}{2}s_1+\frac{\bar{a}}{2a}s_3+\frac{a}{2}, \tag{2}$$

where $s_1=t_1+t_2+t_3$, $s_3=t_1t_2t_3$. The line 45 has the equation:

$$z+t_4t_5\bar{z}=t_4+t_5. \tag{3}$$

By putting $a=t_4+t_5$, $\bar{a}=(t_4+t_5)/t_4t_5$, and $\bar{a}/a=1/t_4t_5$ in (2), we have

$$\begin{aligned} z&=\frac{1}{2}s_1+\frac{1}{2}\frac{s_3}{t_4t_5}+\frac{1}{2}(t_4+t_5)\\ &=\frac{1}{2}(t_1+t_2+t_3+t_4+t_5)+\frac{1}{2}\frac{t_1t_2t_3}{t_4t_5}, \end{aligned}$$

that is,

$$z=\frac{1}{2}\sigma_1+\frac{1}{2}\frac{\sigma_5}{(t_4t_5)^2}, \tag{4}$$

where $\sigma_1=t_1+t_2+t_3+t_4+t_5$ and $\sigma_5=t_1t_2t_3t_4t_5$. This shows that the orthopole (4) of the line 45 with respect to the triangle $\Delta 123$ is on the circle [1, 2, 3, 4, 5]:

$$z=\frac{1}{2}\sigma_1+\frac{1}{2}t, \qquad |t|=1 \tag{5}$$

corresponding to $t = \sigma_5/(t_4 t_5)^2$. The orthopole of and line lm with respect to Δijk is obviously also on this circle. Thus the circle [1, 2, 3, 4, 5] is the circle centered at the point {1, 2, 3, 4, 5}:

$$z = \frac{1}{2}\sigma_1 = \frac{1}{2}(t_1 + t_2 + t_3 + t_4 + t_5), \tag{6}$$

and having radius $\frac{1}{2}$. This circle is called the ***circle of orthopoles of the five concyclic points*** 1, 2, 3, 4, 5.

(ii) Suppose that the concyclic points 1, 2, 3, 4, 5, 6 have the coordinates t_1, t_2, t_3, t_4, t_5, t_6. Then the circles of orthopoles: [1, 2, 3, 4, 5], $\cdots$, [2, 3, 4, 5, 6] have respectively the equations: $z = \frac{1}{2}(t_1 + t_2 + t_3 + t_4 + t_5) + \frac{1}{2}t, \cdots$, $z = \frac{1}{2}(t_2 + t_3 + t_4 + t_5 + t_6) + \frac{1}{2}t$. These circles are obviously concurrent in the point:

$$z = \frac{1}{2}(t_1 + t_2 + t_3 + t_4 + t_5 + t_6). \tag{7}$$

This is the coordinate of the point {1, 2, 3, 4, 5, 6}.

(iii) Let $t_1, \cdots, t_7$ be respectively coordinates of concyclic points $1, \cdots, 7$. There are seven points of the kind mentioned in (ii): $z = \frac{1}{2}(t_1 + \cdots + t_6), \cdots, z = \frac{1}{2}(t_2 + \cdots + t_7)$. These points obviously lie on the circle:

$$z = \frac{1}{2}(t_1 + t_2 + \cdots + t_7) + \frac{1}{2}t \tag{8}$$

(respectively corresponding to $t = -t_7, \cdots, t = -t_1$). Thus [1, 2, 3, 4, 5, 6, 7] has the equation (8).

(iv) can be shown by mathematical induction.

(v) [1, 2, 3, 4, 5, 6]: $z = \frac{1}{2}(t_1 + t_2 + t_3 + t_4 + t_5 + t_6) + \frac{1}{2}t$ is the circle of the centers: $\frac{1}{2}(t_1 + \cdots + t_5), \cdots, \frac{1}{2}(t_2 + \cdots + t_6)$ (corresponding to $t = -t_6, \cdots, t = -t_1$) of the circles [1, 2, 3, 4, 5], $\cdots$, [2, 3, 4, 5, 6]. Next, [1, 2, 3, 4, 5, 6, 7]; $z = \frac{1}{2}(t_1 + t_2 + \cdots + t_7) + \frac{1}{2}t$ is the circle of the centers $\frac{1}{2}(t_1 + \cdots + t_6), \cdots, \frac{1}{2}(t_2 + \cdots + t_7)$ of the circles [1, 2, 3, 4, 5, 6], $\cdots$, [2, 3, 4, 5, 6, 7] and so on.

2. **Circles of orthopole K-points.** Let A_1, A_2, A_3, A_4 be four concyclic points, and let g be a given line. Then it is known that

the orthopoles of the line g with respect to respectively $\Delta A_1 A_2 A_3$, $\Delta A_1 A_2 A_4$, $\Delta A_1 A_3 A_4$, and $\Delta A_2 A_3 A_4$ are collinear. This line on which the orthopoles lie is called the *orthopole line of g with respect to four concyclic points A_1, A_2, A_3, A_4.*

In a previous paper ([2], proposition) 3), the following theorem is obtained:

Let A_i, $i = 1, 2, 3, 4$ and Q_j, $j = 1, 2, 3$ be points on a circle. Then the three orthopole lines (also called orthopole K-lines) of each of the lines $Q_1 Q_2$, $Q_2 Q_3$, and $Q_3 Q_1$ with respect to the four points A_1, A_2, A_3, and A_4 are concurrent. This point of intersection is called the *orthopole K-point of the points Q_1, Q_2, Q_3 with respect to A_1, A_2, A_3 and A_4.* Next, let A_5 be another point on the circle. Then we have five orthopole K-points which are orthopole K-points of Q_1, Q_2, Q_3 with respect to any set of four points out of the five points $A_1, \cdots, A_5$. These five orthopole K-points are collinear. This line is called *the orthopole K-line of the points Q_1, Q_2, Q_3 with respect to the points $A_1, \cdots, A_5$.* Let Q_4 be still another point on the circle. Then there exist four orthopole K-lines of each set of three points out of the four points Q_1, Q_2, Q_3, Q_4 with respect to the points $A_1, \cdots, A_5$. These four orthopole K-lines are concurrent. The point of intersection is called the *orthopole K-point of the points Q_1, Q_2, Q_3, Q_4 with respect to the points $A_1, \cdots, A_5$,* and so on. We can continue this process infinitely. Generally, to n points $A_1, \cdots, A_n$ and $m = n - 1$ points $Q_1, \cdots, Q_m$ on a circle there corresponds an orthopole K-point (called *the orthopole K-point of the points $Q_1, \cdots, Q_{n-1}$ with respect to the points $A_1, \cdots, A_n$*), and to $(n + 1)$ points $A_1, \cdots, A_{n+1}$ and $m = n - 1$ points $Q_1, \cdots, Q_{n-1}$ on a circle there corresponds an orthopole K-line (called *the orthopole K-line of the points $Q_1, \cdots, Q_{n-1}$ with respect to the points $A_1, \cdots, A_{n+1}$*).

The terms: the orthopole K-point of the three points Q_1, Q_2, Q_3 with respect to the four points A_1, A_2, A_3, A_4; the orthopole K-point of the four points Q_1, Q_2, Q_3, Q_4 with respect to the five points A_1, A_2, A_3, A_4, A_5; and generally, the orthopole K-point of the $(n - 1)$ points $Q_1, \cdots, Q_{n-1}$ with respect to the n points

$A_1, \cdots, A_n$ are all defined in the above statement.

PROPOSITION 2. *Let* 1, 2, $\cdots$, 7 *be seven concyclic points. Consider the orthopole K-point of* i_5, i_6, i_7 *(any three points of the given seven points) with respect to the remaining four points* i_1, i_2, i_3 *and* i_4. *There are* ${}_7C_4$ *orthopole K-points of this kind. These* ${}_7C_4$ *orthopole K-points are concyclic. Generally, let* 1, 2, $\cdots$, $2n-1$ *be* $(2n-1)$ *concyclic points. Then there are* ${}_{2n-1}C_{n-1}$ *orthopole K-points of* $(n-1)$ *points* $i_{n+1}, i_{n+2}, \cdots, i_{2n-1}$ *(any* $(n-1)$ *points out of the given* $(2n-1)$ *point) with respect to the remaining n points* $i_1, i_2, \cdots, i_n$. *These* ${}_{2n-1}C_{n-1}$ *orthopole K-points are concyclic.*

Proof. It is shown in [2] that the orthopole K-point of the points Q_1, Q_2, Q_3 having coordinates τ_1, τ_2, τ_3 with respect to the points A_1, A_2, A_3, A_4 having coordinates t_1, t_2, t_3, t_4 has the coordinate:

$$z = \frac{1}{2} r_1 + \frac{1}{2} \rho_1 - \frac{1}{2} \frac{r_4}{\rho_3}, \tag{9}$$

where $r_1 = t_1 + t_2 + t_3 + t_4$, $r_4 = t_1 t_2 t_3 t_4$ and $\rho_1 = \tau_1 + \tau_2 + \tau_3$, $\rho_3 = \tau_1 \tau_2 \tau_3$.. Therefore, if the seven concyclic points 1, 2, $\cdots$, 7 have the coordinates $t_1, \cdots, t_7$, then the orthopole K-point of the points 5, 6, 7 with respect to the points 1, 2, 3, 4, is represented by

$$z = \frac{1}{2} (t_1 + t_2 + \cdots + t_7) - \frac{1}{2} \frac{t_1 t_2 \cdots t_7}{(t_5 t_6 t_7)^2}. \tag{10}$$

This point is obviously on the circle:

$$z = \frac{1}{2} (t_1 + t_2 + \cdots + t_7) - \frac{1}{2} t. \tag{11}$$

This circle is denoted as [1, 2, 3, 4, 5, 6, 7], which has the radius $\frac{1}{2}$, and is centered at the point:

$$z = \frac{1}{2} (t_1 + t_2 + \cdots + t_7). \tag{12}$$

This point is also denoted as {1, 2, 3, 4, 5, 6, 7}. Similarly all ${}_7C_3$ orthopole K-points of this kind also lie on this circle.

Generally, the orthopole K-point of $(n-1)$ points $Q_1, \cdots, Q_{n-1}$ having coordinates $\tau_1, \cdots, \tau_{n-1}$ with respect to n points $A_1, \cdots, A_n$ having coordinates $t_1, \cdots, t_n$ has the coordinate [2]:

$$z = \frac{1}{2} s_1 + \frac{1}{2} \sigma_1 - (-1)^n \frac{s_n}{\sigma_{n-1}}, \tag{13}$$

where $s_1 = t_1 + \cdots + t_n$, $s_n = t_1 \cdots t_n$, $\sigma_1 = \tau_1 + \cdots + \tau_{n-1}$ and $\sigma_{n-1} = \tau_1 \cdots \tau_{n-1}$. Thus, if the $(2n-1)$ concyclic points $1, 2, \cdots, (2n-1)$ are represented by $t_1, t_2, \cdots, t_{2n-1}$, then the orthopole K-point of the $(n-1)$ points $n+1, n+2, \cdots, 2n-1$ with respect to the points $1, 2, \cdots, n$ is represented by

$$z = \frac{1}{2}(t_1 + \cdots + t_{2n-1}) - (-1)^n \frac{t_1 \cdots t_{2n-1}}{(t_{n+1} \cdots t_{2n-1})^2}. \tag{14}$$

This orthopole K-point is obviously on the circle:

$$z = \frac{1}{2}(t_1 + \cdots + t_{2n-1}) - \frac{1}{2} t, \tag{15}$$

and all ${}_{2n-1}C_{n-1}$ orthopole K-points of this kind also lie on this circle. This circle is denoted as $[1, 2, \cdots, (2n-1)]$ and is called *the circle of orthopole K-points, corresponding to the given* $(2n-1)$ *concyclic points* $1, 2, \cdots, (2n-1)$. This circle has radius $\frac{1}{2}$ and is centered at the point $\{1, 2, \cdots, (2n-1)\}$:

$$z = \frac{1}{2}(t_1 + t_2 + \cdots + t_{2n-1}). \tag{16}$$

Corresponding to the parts (ii)—(v) of Proposition 1, we have generally the following:

PROPOSITION 3.

(i) *Starting with* $(2n)$ *concyclic points* $1, 2, \cdots, (2n)$, *omitting each point in turn,* $(2n)$ *circles of orthopole K-points are found by later part of Proposition* 2; *these are concurrent, say in the point* $\{1, 2, \cdots, (2n)\}$.

(ii) *Starting with* $(2n+1)$ *concyclic points* $1, 2, \cdots, (2n+1)$, $(2n+1)$ *points of the kind in* (i) *are found, these lie on a circle denoted by* $[1, 2, \cdots, (2n+1)]$.

(iii) *Continuing thus infinitely, there is, in each case, finally a unique point or circle according as the number is even or odd.*

(iv) *The* $(2n)$ *centers* $\{1, 2, \cdots, (2n-1)\}$ *etc. of the* $(2n)$ *circles* $[1, 2, \cdots, (2n-1)]$ *etc. of orthopole K-points are on the circle* *of radius* $\frac{1}{2}$ *with the center* $\{1, 2, \cdots, (2n)\}$. *This circle is denoted as* $[1, 2, \cdots, (2n)]$. *The points* $\{1, 2, \cdots, (2n)\}$ *etc.* (*the centers of the circle* $[1, 2, \cdots, (2n)]$ *etc.*) *are on the circle* $[1, 2, \cdots, (2n+1)]$, *and so on. In this way, we have a system of center circles.*

The proof of Proposition 3 is completely the same as the later part of the proof of Proposition 1. The centers of $(2n)$ circles of orthopole K-points are $\frac{1}{2}(t_1 + \cdots + t_{2n-1}), \cdots, \frac{1}{2}(t_2 + \cdots + t_{2n})$. The concurrent point of these $(2n)$ circles of orthopole K-points is represented by $\frac{1}{2}(t_1 + \cdots + t_{2n})$. Starting with $(2n+1)$ concyclic points having coordinates $t_1, \cdots, t_{2n+1}$, $(2n+1)$ points $\frac{1}{2}(t_1 + \cdots + t_{2n}), \cdots, \frac{1}{2}(t_2 + \cdots + t_{2n+1})$ are points of the kind in (i), and these points are on the circle $z = \frac{1}{2}(t_1 + \cdots + t_{2n+1}) + \frac{1}{2}t$, and so on.

REMARK. In the paper [2], it was remarked that if A_1, A_2, A_3, A_4 and Q_1, Q_2 are concyclic points, then the orthopole line of the line $Q_1 Q_2$ with respect to the concyclic points A_1, A_2, A_3, A_4 is actually the *Kantor line* of the points Q_1, Q_2 with respect to the points A_1, A_2, A_3, and A_4. Thus the orthopole K-point of the points Q_1, Q_2, Q_3 with respect to A_1, A_2, A_3, A_4 is actually the *Kantor point* of Q_1, Q_2, Q_3 with respect to A_1, A_2, A_3, and A_4. And so on. By this reason, the Propositions 2 and 3 still hold if we replace each orthopole K-point in their statements by the corresponding Kantor point. For the definitions of Kantor lines and Kantor points, please see [2].

Added in the proof. The present author just found the following theorem of Ariga as an exercise in the book of Kobayashi ([3], page 279, Exercise 12):

THEOREM (Ariga). *Let* A_i, $i = 1, 2, 3, 4, 5$ *be five points on a circle. Take arbitrarily two point* A_h, *and* A_i *from the five points and then consider the point of intersection of the Simson lines of* A_h, *and* A_i *with respect to the triangle* Δjkl *formed by the remaining three points* A_j, A_k *and* A_l. *In this way we can construct* 10 *points of intersection of different Simson lines. These ten points are concyclic.*

Due to the above remark, it is obvious that the part (i) of the Proposition 1 is equivalent to this theorem of Ariga.

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Institute of Mathematics
Academia Sinica
Nankang, Taipei, Taiwan 115
R. O. C.