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SOME EXPONENTIAL DIOPHANTINE EQUATIONS

BY

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(i) For the equation $2^a + k^b = 2^c + m^d$, $k \neq m$, and $k, m = 3, 5, 7, 11, 13, 17$, or 19, we find all nonnegative integral solutions.

(ii) We find all integral solutions of $1 + 3^a = 2^b + 2^c 3^d$ and $5^a = 1 + 2^b + 2^c$.

(iii) For the equations $3^a + 7^b = 1 + 2^c + 5^d$ and $1 + 3^a = 2^b 5^c + 2 \cdot 3^d 5^e$, we find all nonnegative integral solutions.

By an exponential Diophantine equation (eDe) we mean an equation in which the bases are (given or unknown) integers; the exponents are unknown integers. More recently, J. L. Brenner and L. L. Foster [2] solved many eDe's. In the paper, we shall solve some similar eDe's by using a finite number of moduli.

THEOREM 1. *The only solutions of* $2^a + 3^b = 2^c + 7^d$ *in nonnegative integers are given in Table* 1. *In this table,* t *denotes an arbitrary nonnegative integer.*

Table 1.

a	b	c	d
1	1	2	0
3	2	4	0
3	0	1	1
6	0	4	2
t	0	t	0
1	2	2	1
3	1	2	1
5	4	6	2

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Proof. Let (a, b, c, d) be another solution. Clearly $a > 0$, $c > 0$, $a \neq c$. From [2], $b > 0$, $d > 0$. Using mod 3, c is even, a is odd. We consider four cases.

Case 1.1. $a = 1$. Then $2 + 3^b = 2^c + 7^d$. By [2], $c > 2$. Hence we have a contradiction mod 8.

Case 1.2. $c = 2$. Then $2^a + 3^b = 4 + 7^d$. From Case 1.1, $a > 2$. Using mod 8, b is odd, d is odd. Thus using mod 5 we have $(a, b, d) \equiv (3, 1, 1) \pmod 4$. Suppose $a > 3$. This yields a contradication mod 16. Hence $a = 3, 4 + 3^b = 7^d$. Clearly $b > 1$. Using mod 9, $d \equiv 2 \pmod 3$ so that in fact $d \equiv 5 \pmod{12}$. Thus we have a contradiction mod 13.

Case 1.3. $a = 3$. Then $8 + 3^b = 2^c + 7^d$. By Case 1.2, $c > 3$. Using mod 16, $(b, d) \equiv (2, 0) \pmod{(4,2)}$. Thus using mod 5, $(c, d) \equiv (0, 0) \pmod 4$. Combining these results and using mod 7 we get $(b, c) \equiv (0, 1) \pmod{(6, 3)}$ so that in fact $(b, c) \equiv (6, 4) \pmod{12}$. Hence using mod 13 we have $d \equiv 7 \pmod{12}$, a contradiction.

Case 1.4. $a \geq 5$ and $c \geq 4$. Using mod 16, $(b, d) \equiv (0, 0) \pmod{(4, 2)}$. Thus using mod 5, $(a, b, c, d) \equiv (1, 0, 2, 2) \pmod 4$. Hence using mod 7 we have $(a, b, c) \equiv (0, 0, 1), (1, 2, 2)$ or $(2, 4, 0) \pmod{(3, 6, 3)}$ so that in fact $(a, b, c) \equiv (9, 0, 10), (1, 8, 2)$ or $(5, 4, 6) \pmod{12}$. Combining the above results and using mod 13 we get $(a, b, c, d) \equiv (5, 4, 6, 2) \pmod{12}$. Thus using mod 19, $(a, b, c) \equiv (5, 4, 6)$ or $(11, 10, 0) \pmod{18}$. Hence using moduli 27, 37 successively we have $(a, b, c, d) \equiv (5, 4, 6, 2) \pmod{(36, 18, 36, 9)}$ so that in fact $(a, b, c, d) \equiv (5, 4, 6, 2) \pmod{36}$. Using mod 72, $d \equiv 2 \pmod{24}$, so that using mod 32, $b \equiv 4 \pmod 8$. Applying these results and using mod 193 we conclude that $(a, b, c) \equiv (5, 4, 6) \pmod{(96, 16, 96)}$. Combining the above results and using mod 97, $d \equiv 2 \pmod{96}$ so that in fact $(a, b, c, d) \equiv (5, 4, 6, 2) \pmod{(96, 48, 96, 96)}$. Suppose $a > 5$. We have a contradiction mod 64. Thus $a = 5$, $32 + 3^b = 2^c + 7^d$. Suppose $c > 6$. Using mod 256, $b \equiv 52 \pmod{64}$. We consider our equation modulo the prime 257. Since the order of 2 is 16 and the order of 3 is 256, we have the following possibilities: $7^d \equiv -32 + 3^{52+64n} \pmod{257}$, $n = 0, 1, 2, 3$. Hence $d \equiv 141, 231, 251$ or $75 \pmod{256}$, each of these congruences is a contradiction. Thus $c=6$, $3^b = 32 + 7^d$. Clearly $b > 4$. Using mod 243, $d \equiv 29 \pmod{81}$ so that in fact $d \equiv 110 \pmod{162}$. Hence using mod 163 we conclude that $3^b \equiv 32 + 7^{110} \equiv -60 \pmod{163}$, $b \equiv 95 \pmod{162}$, a contradiction. □

THEOREM 2. *The only solutions to* $2^a + 11^b = 2^c + 13^d$ *in nonnegative integers are given in Table 2. In this table, t denotes an arbitrary nonnegative integer.*

Table 2.

a	b	c	d
3	2	7	0
4	0	2	1
t	0	t	0
2	1	1	1
6	2	4	2

Proof. Let (a, b, c, d) be another solution.

We will first show that $b > 0$. Assume the contrary. Then $2^a + 1 = 2^c + 13^d$. Clearly $d > 0$. Using mod 3, $a \equiv c \pmod 2$. Thus using mod 13 we have $(a, c) \equiv (4, 2), (5, 11), (8, 10), (9, 5)$ or $(11, 3) \pmod{12}$. Hence using mod 7, $(a, c, d) \equiv (4, 2, 1)$ or $(5, 11, 0)$ (mod $(12, 12, 2)$), so that using mod 5, $(a, c, d) \equiv (4, 2, 1)$ (mod $(12, 12, 4)$). Suppose $c>2$. This yields a contradiction mod 8. Thus $c = 2$, $2^a = 3 + 13^d$. Clearly $d > 1$. Using mod 169, $a \equiv 124 \pmod{156}$. We have a contradiction mod 157. Hence $b > 0$.

We will next show that $d > 0$. Assume the contrary. Then $2^a + 11^b = 2^c + 1$. Clearly $b > 0$, $c > 3$. Using mod 8, $a \geq 3$, b is even. Suppose $a > 3$. Using mod 16, $b \equiv 0 \pmod 4$. Thus using mod 61, $a \equiv c \pmod{60}$. This yields a contradiction mod 11. Hence $a = 3$, $2^c = 7 + 11^b$. Clearly $c > 7$. Using mod 256, $d \equiv 34 \pmod{64}$, so that we have a contradiction mod 193. Thus $d > 0$.

Clearly $a > 0$, $c > 0$, $a \neq c$. We consider five cases.

Case 2.1. $a = 1$. Then $2 + 11^b = 2^c + 13^d$. Using mod 3, b is even, c is odd. We have a contradiction mod 8.

Case 2.2. $c = 1$. Then $2^a + 11^b = 2 + 13^d$. Using mod 4, b is odd. Thus using mod 3, a is even. Applying these results and using mod 5 we have $(a, b, d) \equiv (2, 1, 1)$ (mod $(4, 2, 4)$). Suppose $a > 2$. This yields a contradiction mod 8. Hence $a = 2$, $2 + 11^b = 13^d$. Clearly $b > 1$. Using mod 121, $d \equiv 61$ (mod 110). Thus using mod 23, $b \equiv 20$ (mod 22), a contradiction.

Case 2.3. $a = 2$. Then $4 + 11^b = 2^c + 13^d$. From Case 2.2, $c > 2$. Using mod 4, b is even, Thus using mod 3, c is even. Hence using mod 5 we get $(c, d) \equiv (0, 2)$ or $(2, 0)$ (mod 4). In each case we have a contradiction mod 8.

Case 2.4. $c = 2$. Then $2^a + 11^b = 4 + 13^d$ By Case 2.1, $a > 2$. Using mod 8, d is odd. b is even, so that using mod 3, a is even, Thus using mod 5 we get $(a, d) \equiv (0, 1)$ (mod 4), so that using mod 16, $b \equiv 0$ (mod 4). Further, using mod 11, $(a, d) \equiv (0, 3)$ or $(8, 5)$ (mod 10). Combining these results and using mod 41 we have the following possibilities: $11^b \equiv 4 + 13^{13+20n} - 1$ or $4 + 13^{5+20m} - 10$ (mod 41), $n, m = 0, 1$. Hence $b \equiv 35, 6, 30$ or 25 (mod 40), each of these congruences is a contradiction.

Case 2.5. $a \geq 3$ and $c \geq 3$. Using mod 8, b is even, d is even. Thus using mod 3, $a \equiv c$ (mod 2). We consider the following possibilities: $d \equiv 0$ or 2 (mod 4).

Suppose that $d \equiv 0$ (mod 4). Using mod 7, $(a, b, c) \equiv (0, 1, 2)$, $(0, 2, 1)$ or $(t, 0, t)$ (mod 3). Hence using mod 5, $a \equiv c$ (mod 4). Applying the above results, we have the following table of possibilities mod 12:

Table A. (a, b, c) mod 12.

a	b	c
9	2	1
9	8	1
6	4	2
6	10	2
0	2	4
0	8	4
9	4	5
9	10	5
3	2	7
3	8	7
0	4	8
0	10	8
6	2	10
6	8	10
3	4	11
3	10	11
t	0	t
t	6	t

In each case we have a contradiction mod 13.

Thus $d \equiv 2 \pmod 4$. Applying the above results and using mod 5, $(a, c) \equiv (2, 0) \pmod 4$, so that using mod 16, $b \equiv 2 \pmod 4$. Hence using moduli 13, 7 successively we get $(a, b, c) \equiv (6, 2, 4)$, $(2, 6, 4)$ or $(6, 10, 8) \pmod{12}$. Thus using mod 9, $(a, b, c, d) \equiv (6, 2, 4, 2)$, $(2, 6, 4, 2)$ or $(6, 10, 8, 1)$ $(\text{mod } (12, 12, 12, 3))$. Combining these results and using moduli 37, 19 successively we conclude that $(a, b, c, d) \equiv (6, 2, 4, 2)$ $(\text{mod } (36, 12, 36, 36))$. Hence using mod 73, $(b, d) \equiv (2, 2) \pmod{72}$. Therefore using mod 32, $c = 4$. Thus $2^a + 11^b = 16 + 13^d$. Combining the above results and using mod 17, $(a, b) \equiv (6, 2)$ $(\text{mod } (8, 16))$ so that in fact $(a, b, d) \equiv (6, 2, 2)$ $(\text{mod } (72, 144, 72))$. Using mod 64, $d \equiv 2 \pmod{16}$. Hence using mod 97, $(a, d) \equiv (6, 2)$ $(\text{mod } (48, 96))$. Applying these results and using mod 193 we conclude that $(a, b, d) \equiv (6, 2, 2)$ $(\text{mod } (96, 64, 64))$. Using mod 128, $a = 6$. Thus $48 + 11^b = 13^d$, $b > 2$. Using mod 1331, $d \equiv 332 \pmod{1210}$ so that in fact $d \equiv 332 \pmod{3630}$. Hence we have a contradiction mod 3631.

THEOREM 3. *The only solutions of* $2^a + 17^b = 2^c + 19^d$ *in nonnegative integers are given in Table 3. In this table, t denotes an arbitrary nonnegative integer.*

Table 3.

a	b	c	d
4	1	5	0
t	0	t	0
2	1	1	1

Proof. Let (a, b, c, d) be another solution.

We will first show that $b > 0$. Assume the contrary. Then $2^a + 1 = 2^c + 19^d$. Using mod 9, $a \equiv c \pmod 6$, so that using mod 7, $d \equiv 0 \pmod 6$. Thus using mod 27, $a \equiv c \pmod{18}$. Hence we have a contradiction mod 19. Therefore $b > 0$.

We will next show that $d > 0$. Assume the contrary. Then $2^a + 17^b = 2^c + 1$. Clearly $b > 0$, $c > 4$. Using moduli 17, 5 successively we get $(a, b, c) \equiv (7, 0, 3)$ or $(4, 1, 5)$ $(\text{mod } (8, 4, 8))$.

Suppose $(a, b, c) \equiv (4, 1, 5) \pmod{(8, 4, 8)}$. Using mod 32, $a = 4$, so that using mod 64, $c = 5$, a contradiction. Thus $(a, b, c) \equiv (7, 0, 3) \pmod{(8, 4, 8)}$. Using moduli 9. 7 successively we have $(a, b, c) \equiv (t, 0, t) \pmod 6$. Suppose $a > 7$. Using mod 512, $b \equiv 0 \pmod{32}$ so that in fact $b \equiv 0 \pmod{96}$. Hence using mod 97, $a \equiv c \pmod{48}$, a contradiction. Thus $a = 7$, $2^c = 127 + 17^d$. Clearly $d > 1$. Using mod 289, $c \equiv 59 \pmod{136}$. We have a contradiction mod 137. Hence $d > 0$.

Clearly $a > 0$, $c > 0$, $a \neq c$. We consider five cases.

Case 3.1. $a = 1$. Then $2 + 17^b = 2^c + 19^d$. Using mod 9, b is even, $c \equiv 1 \pmod 6$. Thus using mod 16, $d \equiv 1 \pmod 4$, so that using mod 17 we have the following possibilities: $2^c \equiv 2 - 19^{1+4n} \pmod{17}$, $n = 0, 1$. Hence $c \equiv 2 \pmod 8$, a contradiction.

Case 3.2. $c = 1$. Then $2^a + 17^b = 2 + 19^d$. Using mod 8 we conclude that d is odd, $a = 2$. Thus $2 + 17^b = 19^d$, so that using mod 9, b is odd. Clearly $b > 1$. Using mod 289, $d \equiv 33 \pmod{136}$. Hence using mod 137, $b \equiv 12 \pmod{68}$, a contradiction.

Case 3.3. $a = 2$. Then $4 + 17^b = 2^c + 19^d$. From Case 3.2, $c > 2$. We have a contradiction mod 8.

Case 3.4. $c = 2$. Then $2^a + 17^b = 4 + 19^d$. By Case 3.1, $a > 2$. We have a contradiction mod 8.

Case 3.5. $a \geq 3$ and $c \geq 3$. Using mod 8, d is even.

Suppose that b is even. Using mod 9, $a \equiv c \pmod 6$, so that using mod 5, $(a, b, c) \equiv (0, 2, 2)$ or $(t, 0, t) \pmod 4$, Thus using mod 17 we conclude that $(a, b, c) \equiv (t, 0, t) \pmod 4$ so that in fact $(a, b, c) \equiv (t, 0, t) \pmod{(12, 4, 12)}$. Hence using moduli 13, 7 successively we get $(b, d) \equiv (0, 0) \pmod{(6, 12)}$. Thus using mod 27 we have $a \equiv c \pmod{18}$, so that we have a contradiction mod 19.

Hence b is odd. Using mod 3, c is odd, a is even, so that using mod 5, $(a, b, c) \equiv (0, 1, 1)$ or $(0, 3, 3) \pmod 4$. Hence using mod 17 we conclude that $(a, b, c, d) \equiv (0, 1, 1, 0) \pmod 4$. Thus using moduli 9, 7 successively we get $(a, b, c, d) \equiv (2, 3, 1, 0)$, $(4, 1, 5, 0)$ or $(4, 3, 5, 2) \pmod 6$. Combining the above results and using mod 13 we have $(a, b, c, d) \equiv (4, 1, 5, 0) \pmod{(12, 6, 12, 12)}$. Hence using moduli 27, 19 successively we get (a, b, c)

$\equiv (4, 4, 5) \pmod{(18, 9, 18)}$ so that in fact $(a, b, c) \equiv (4, 13, 5) \pmod{36}$. Thus using mod 37, $d \equiv 1 \pmod{36}$, a contradiction.

By [2], Theorems 1, 2 and 3, it suffices to solve the rest of the eDe's of (i) in positive integers. Clearly $a \neq c$. We let t denote an arbitrary positive integer.

THEOREM 4. *The only solutions to* $2^a + 3^b = 2^c + 5^d$ *in positive integers are given in Table 4.*

Table 4.

a	b	c	d
1	3	2	2
2	1	1	1
2	2	3	1
3	4	6	2
5	2	4	2

Proof. Let (a, b, c, d) be another solution. We consider five cases.

Case 4.1. $a = 1$. Then $2 + 3^b = 2^c + 5^d$. From [2], $c > 2$. Using mod 8, b is odd, d is odd. Hence we have a contradiction mod 3.

Case 4.2. $c = 1$. Then $2^a + 3^b = 2 + 5^d$. Suppose $a > 2$. Using mod 8, b is odd, d is even, so that we have a contradiction mod 3. Thus $a = 2$, $2 + 3^b = 5^d$. Clearly $b > 1$. Using mod 9, $d \equiv 5 \pmod 6$, so that using mod 7, $b \equiv 0 \pmod 6$. We have a contradiction mod 13.

Case 4.3. $a = 2$. Then $4 + 3^b = 2^c + 5^d$. By Case 4.2, $c > 2$. Using mod 8, d is odd, b is even, so that using mod 5, $(b, c) \equiv (2, 3) \pmod 4$. Further, using moduli 9, 7 successively we have $(b, c, d) \equiv (0, 1, 5)$ or $(2, 3, 1) \pmod 6$. Combining these results and using mod 13 we get $(b, c, d) \equiv (2, 3, 1) \pmod{12}$. Hence using mod 16 we conclude that $c = 3$, $3^b = 4 + 5^d$, $b > 2$. Using mod 27, $d \equiv 13 \pmod{18}$. Thus using mod 19, $b \equiv 7 \pmod{18}$, a contradiction.

Case 4.4. $c = 2$. Then $2^a + 3^b = 4 + 5^d$. From Case 4.1, $a > 2$.

Using mod 8, d is odd, b is even. Hence we have a contradiction mod 3.

Case 4.5. $a \geq 3$ and $c \geq 3$. Using mod 8, b is even, d is even. Further, using mod 3, a is add, c is even. Thus using mod 5 we have the following possibilities: $(a, b, c) \equiv (3, 0, 2)$ or $(1, 2, 0)$ (mod 4).

We will show that $a > 3$. Assume the contrary. Then $8 + 3^b = 2^c + 5^d$. Thus $(b, c) \equiv (0, 2)$ (mod 4). Further, using moduli 64, 17 successively we conclude that $(b, c, d) \equiv (4, 6, 2)$ (mod (16, 8, 16)). Hence using moduli 13, 7 successively we have $(b, c, d) \equiv (4, 6, 2)$ (mod (6, 12, 12)) so that in fact $(b, c, d) \equiv (4, 6, 2)$ (mod (48, 24, 48)). Thus using mod 97, $(c, d) \equiv (6, 2)$ (mod (48, 96)). Further, using mod 193 we get $(c, d) \equiv (6, 2)$ (mod (96, 192)). Combining these results and using mod 257 we conclude that $(b, d) \equiv (4, 2)$ (mod 256). Hence using mod 256, $c = 6$. Thus $3^b = 56 + 5^d$, $d > 2$. Using mod 125, $b \equiv 84$ (mod 100). We have a contradiction mod 101. Hence $a > 3$.

Suppose that $(a, b, c) \equiv (3, 0, 2)$ (mod 4). Using mod 16, $d \equiv 0$ (mod 4), so that using mod 13 we have $(a, b, c) \equiv (3, 1, 10)$ (mod (12, 3, 12)). Combining these results and using mod 9 we get $d \equiv 0$ (mod 6) so that in fact $(a, b, c, d) \equiv (3, 4, 10, 0)$ (mod 12). This yields a contradiction mod 7.

Hence $(a, b, c) \equiv (1, 2, 0)$ (mod 4). Using mod 16, $d \equiv 2$ (mod 4). Thus using mod 17 we get $(a, b, c, d) \equiv (5, 2, 4, 2)$ or $(1, 10, 0, 10)$ (mod (8, 16, 8, 16)). Hence using mod 64 we conclude that $(a, b, c, d) \equiv (5, 2, 4, 2)$ (mod (8, 16, 8, 16)). Using mod 32, $c = 4$, so that using mod 64, $a = 5$. Thus $16 + 3^b = 5^d$, $b > 2$. Using mod 27, $d \equiv 8$ (mod 18) so that in fact $d \equiv 26$ (mod 36). Hence we have a contradiction mod 37.

THEOREM 5. *The only solutions of* $2^a + 3^b = 2^c + 11^d$ *in positive integers are* $(a, b, c, d) = (2, 2, 1, 1)$, $(4, 1, 3, 1)$, $(4, 3, 5, 1)$ *and* $(7, 2, 4, 2)$.

Proof. Let (a, b, c, d) be another solution. We consider seven cases.

Case 5.1. $a = 1$. Then $2 + 3^b = 2^c + 11^d$. Using mod 3, d is even, c is even. Thus using mod 8 we conclude that b is odd, $c=2$. Hence $3^b = 2 + 11^d$, so that we have a contradiction mod 11.

Case 5.2. $c=1$. Then $2^a + 3^b = 2 + 11^d$. Using mod 3, d is odd, a is even. Thus using mod 8 we conclude that b is even, $a=2$. Hence $2+3^b = 11^d$, $d>1$. We have a contradiction mod 121.

Case 5.3. $a=2$. Then $4 + 3^b = 2^c + 11^d$. From Case 5.2, $c>2$. We have a contradiction mod 8.

Case 5.4. $c=2$. Then $2^a + 3^b = 4 + 11^d$. By Case 5.1, $a>2$. We have a contradiction mod 8.

Case 5.5. $a=3$. Then $8 + 3^b = 2^c + 11^d$. Using mod 3, c is even, d is even. Thus using mod 4, b is even. Further, using mod 5 we have $(b, c) \equiv (2, 0) \pmod 4$, so that using mod 16, $d \equiv 0 \pmod 4$. Hence using mod 17 we conclude that $(b, c, d) \equiv (2, 4, 0)$ or $(2, 0, 8) \pmod{(16, 8, 16)}$. Therefore using mod 64 we conclude that $(b, d) \equiv (2, 0) \pmod{16}$, $c=4$. Thus $3^b = 8 + 11^d$, so that we have a contradiction mod 11.

Case 5.6. $c=3$. Then $2^a + 3^b = 8 + 11^d$. Using mod 3, d is odd, a is even. Thus using mod 5 we have $(a, b) \equiv (0, 1) \pmod 4$, so that using mod 16, $d \equiv 1 \pmod 4$. Hence using mod 17 we get $(a, b, d) \equiv (4, 1, 1)$ or $(0, 9, 13) \pmod{(8, 16, 16)}$. Therefore using mod 64 we conclude that $(b, d) \equiv (1, 1) \pmod{16}$, $a=4$. Thus $8 + 3^b = 11^d$, $d>1$, so that we have a contradiction mod 121.

Case 5.7. $a \geq 4$ and $c \geq 4$. Using mod 8 we have the following subcases: $(b, d) \equiv (0, 0)$ or $(1, 1) \pmod 2$.

Subcase 5.7.1. $(b, d) \equiv (0, 0) \pmod 2$. Using mod 3, c is even, a is odd. Hence using mod 5 we get $(a, b, c) \equiv (3, 2, 0) \pmod 4$, so that using mod 16, $d \equiv 2 \pmod 4$. Thus using mod 17 we have $(a, b, c, d) \equiv (3, 10, 0, 10)$, $(3, 14, 0, 14)$, $(7, 2, 4, 2)$ or $(7, 6, 4, 6)$ $\pmod{(8, 16, 8, 16)}$. Therefore using mod 64 we conclude that $(a, b, d) \equiv (7, 2, 2) \pmod{(8, 16, 16)}$, $c=4$. Hence $2^a + 3^b = 16 + 11^d$, so that using mod 193 we get $(a, d) \equiv (7, 2) \pmod{(96, 64)}$. Thus using mod 257, $b \equiv 2 \pmod{256}$, so that using mod 256 we conclude that $a=7$. Hence $112 + 3^b = 11^d$, $b>2$. Using mod 27, $d \equiv 14 \pmod{18}$ so that in fact $d \equiv 50 \pmod{72}$. Thus we have a contradiction mod 73,

Subcase 5.7.2. $(b, d) \equiv (1, 1) \pmod 2$. Using mod 3, c is odd,

a is even. Hence using mod 5, 16 successively we have $(a, b, c, d) \equiv (0, 3, 1, 1)$ or $(0, 1, 3, 3) \pmod 4$.

We will first show that $a > 4$. Assume the contrary. Then $16 + 3^b = 2^c + 11^d$, $b > 1$. Using moduli 9, 7 successively we get $(b, c, d) \equiv (3, 5, 1) \pmod 6$. Thus using mod 37, $(b, c) \equiv (3, 5)$ (mod (18, 36)), so that using mod 27, $d \equiv 1 \pmod{18}$. Hence using mod 73 we have $(b, d) \equiv (3, 1)$ (mod (12, 72)), so that using mod 13, $c \equiv 5 \pmod{12}$. Thus using mod 17 we get $(b, c, d) \equiv (3, 5, 1)$ (mod (16, 8, 16)). Therefore using mod 64 we conclude that $c = 5, 3^b = 16 + 11^d$. Clearly $d > 1$, so that we have a contradiction mod 121. Hence $a > 4$.

We will next show that $c > 5$. Assume the contrary. Then $2^a + 3^b = 32 + 11^d$. Thus $(a, b, d) \equiv (0, 3, 1) \pmod 4$. Hence using moduli 9, 7 successively we have $(a, b, d) \equiv (0, 5, 5)$ or $(4, 3, 1) \pmod 6$ so that in fact $(a, b, d) \equiv (0, 11, 5)$ or $(4, 3, 1) \pmod{12}$. Therefore using mod 13 we get $(a, b, d) \equiv (4, 3, 1) \pmod{12}$. Thus using mod 73, $(a, d) \equiv (4, 1)$ (mod (9, 72)). Combining these results and using mod 17 we conclude that $(a, b, d) \equiv (4, 3, 1)$ (mod (8, 16, 16)). This yields a contradiction mod 64. Hence $c > 5$.

Applying the above results and using mod 17 we have the following table of possibilities mod (8, 16, 8, 16):

Table B. (a, b, c, d) mod (8, 16, 8, 16).

a	b	c	d
0	13	3	3
0	1	7	11
0	5	7	15
0	7	1	5
0	11	1	9
0	11	5	5
4	9	3	3
4	13	3	7
4	5	7	11
4	3	1	13
4	3	5	1
4	15	5	13

In each case we have a contradiction mod 64.

THEOREM 6. *The only solutions to* $2^a + 3^b = 2^c + 13^d$ *in positive integers are* $(a, b, c, d) = (1, 3, 4, 1)$ *and* $(3, 2, 2, 1)$.

Proof. Let (a, b, c, d) be another solution. Using mod 3, a is odd, c is even. We consider four cases.

Case 6.1. $a = 1$. Then $2 + 3^b = 2^c + 13^d$. Using mod 13, $(b, c) \equiv (0, 4) \pmod{(3, 12)}$. Thus using mod 8, b is odd, d is odd. Hence using mod 9, $d \equiv 1 \pmod 3$ so that in fact $(a, b, d) \equiv (3, 4, 1) \pmod{(6, 12, 6)}$. Therefore using mod 5, $(b, d) \equiv (3, 1) \pmod 4$, so that using mod 17, $(b, c) \equiv (3, 4) \pmod{(16, 8)}$. Combining these results and using mod 97 we have $(c, d) \equiv (4, 1) \pmod{(48, 96)}$ Thus using mod 32 we conclude that $c = 4$, $3^b = 14 + 13^d$, $b > 3$. Using mod 81, $d \equiv 19 \pmod{27}$ so that in fact $d \equiv 73 \pmod{108}$. Hence we have a contradiction mod 109.

Case 6.2. $c = 2$. Then $2^a + 3^b = 4 + 13^d$. From Case 6.1, $a > 2$. Using mod 8, b is even, d is odd. Thus using moduli 9, 7 successively we have $(a, b, d) \equiv (3, 2, 1)$ or $(1, 0, 5) \pmod 6$. Hence using mod 13, $(a, b, d) \equiv (3, 2, 1) \pmod{(12, 6, 6)}$. Therefore using mod 5, $(b, d) \equiv (2, 1) \pmod 4$ so that in fact $(a, b, d) \equiv (3, 2, 1) \pmod{12}$. Thus using mod 16 we conclude that $a = 3$, $4 + 3^b = 13^d$, $b > 2$. Using mod 27, $d \equiv 7 \pmod 9$ so that in fact $d \equiv 25 \pmod{36}$. Hence using mod 37, $b \equiv 0 \pmod{18}$, a contradiction.

Case 6.3. $a = 3$. Then $8 + 3^b = 2^c + 13^d$. By Case 6.2, $c > 2$. Using mod 13, $(b, c) \equiv (0, 8)$ or $(2, 2) \pmod{(3, 12)}$. Thus using moduli 9, 7 successively we conclude that $(b, c, d) \equiv (2, 2, 1) \pmod{(6, 12, 6)}$. This yields a contradiction mod 8.

Case 6.4. $a \geq 5$ and $c \geq 4$. Using mod 16, $(b, d) \equiv (0, 0)$ or $(2, 2) \pmod 4$. Hence using mod 5 we have $a \equiv c \pmod 4$, a contradiction.

THEOREM 7. *The only solutions of* $2^a + 3^b = 2^c + 17^d$ *in positive integers are* $(a, b, c, d) = (4, 1, 1, 1)$, $(4, 2, 3, 1)$ *or* $(6, 4, 7, 1)$.

Proof. Let (a, b, c, d) be another solution. We consider five cases.

Case 7.1. $a=1$. Then $2+3^b=2^c+17^d$. Using mod 8 we conclude that b is odd, $c=2$. Thus $3^b=2+17^d$, $b>1$. We have a contradiction mod 9.

Case 7.2. $c=1$. Then $2^a+3^b=2+17^d$. Using mod 3, d is odd, a is even. Hence using mod 16 we get $b\equiv 1 \pmod 4$, $a>2$. Thus using mod 5, $(a, d)\equiv(0, 1) \pmod 4$, so that using mod 17, $(a, b)\equiv(4, 1)$ (mod (8, 16)). Therefore using mod 32 we conclude that $a=4$, $14+3^b=17^d$, $b>1$. We have a contradiction mod 9.

Case 7.3. $a=2$. Then $4+3^b=2^c+17^d$. From Case 7.2, $c>2$. We have a contradiction mod 8.

Case 7.4. $c=2$. Then $2^a+3^b=4+17^d$. By Case 7.1, $a>2$. We have a contradiction mod 8.

Case 7.5. $a\geq 3$ and $c\geq 3$. Using mod 8, b is even. We consider the following subcases: $d\equiv 0$ or $1 \pmod 2$.

Subcase 7.5.1. $d\equiv 0 \pmod 2$. Using mod 3, a is odd, c is even. Thus using mod 5 we get $(a, b, c, d)\equiv(1, 0, 2, 2)$ or $(3, 2, 0, 0) \pmod 4$. Hence using mod 17 we conclude that (a, b, c, d) $\equiv(7, 2, 0, 0)$ or $(3, 10, 4, 0)$ (mod (8, 16, 8, 4)). In each case we have a contradiction mod 64.

Subcase 7.5.2. $d\equiv 1 \pmod 2$. Using mod 3, c is odd, a is even. Therefore using moduli 9, 7 successively we have (a, b, c, d) $\equiv(0, 4, 1, 1)$, $(4, 2, 3, 1)$ or $(4, 4, 3, 5) \pmod 6$. Thus using mod 13 we get $(a, b, c, d)\equiv(4, 2, 3, 1)$ or $(6, 4, 7, 1)$ (mod (12, 6, 12, 6)), so that using mod 5, $(a, b, c, d)\equiv(4, 2, 3, 1)$ or $(6, 4, 7, 1)$ (mod 12).

Suppose that $(a, b, c, d)\equiv(4, 2, 3, 1) \pmod{12}$. Using mod 16 we have $c=3$, $2^a+3^b=8+17^d$. Hence using mod 17, $(a, b)\equiv(4, 2)$ (mod (8, 16)). Applying these results and using mod 32 we conclude that $a=4$, $8+3^b=17^d$, $b>2$. Using mod 27, $d\equiv 5$ (mod 6), a contradiction.

Therefore $(a, b, c, d)\equiv(6, 4, 7, 1) \pmod{12}$. Using mod 17 we get $(a, b, c)\equiv(6, 4, 7)$ or $(2, 12, 3)$ (mod (8, 16, 8)). Applying these results and using mod 64 we have $(a, b, c, d)\equiv(6, 4, 7, 1)$ (mod (8, 16, 8, 4)) so that in fact $(a, b, c, d)\equiv(6, 4, 7, 1)$ (mod (24, 48, 24, 12)). Hence using mod 97, $(a, c, d)\equiv(6, 7, 1)$ or $(30, 7, 25)$ (mod (48, 48, 96)). Thus using mod 193 we get

$(a, c, d) \equiv (6, 7, 1) \pmod{(96, 96, 192)}$, so that using mod 257, $b \equiv 4 \pmod{256}$. Therefore using mod 128 we have $a = 6$, so that using mod 256, $c = 7$. Thus $3^b = 64 + 17^d$, $d > 1$. Using mod 289, $b \equiv 52 \pmod{272}$. Hence using mod 137, $d \equiv 6 \pmod{68}$, a contradiction.

THEOREM 8. *The only solutions to* $2^a + 3^b = 2^c + 19^d$ *in positive integers are* $(a, b, c, d) = (1, 4, 6, 1), (3, 3, 4, 1), (5, 1, 4, 1)$ *or* $(5, 5, 8, 1)$.

Proof. Let (a, b, c, d) be another solution. Using mod 3, a is odd, c is even. We consider four cases.

Case 8.1. $a = 1$. Then $2 + 3^b = 2^c + 19^d$. Clearly $b > 2$. Using mod 9, $c \equiv 0 \pmod 6$. Thus using mod 8, d is odd, b is even, so that using mod 5, $(b, c) \equiv (0, 2) \pmod 4$. Combining these results and using mod 13 we get $(b, c, d) \equiv (1, 6, 1) \pmod{(3, 12, 12)}$ so that in fact $(b, c, d) \equiv (4, 6, 1) \pmod{12}$. Hence using mod 17, $(b, c, d) \equiv (4, 6, 1)$ or $(12, 2, 1) \pmod{(16, 8, 8)}$ so that in fact $(b, c, d) \equiv (4, 6, 1)$ or $(28, 18, 1) \pmod{(48, 24, 24)}$. Therefore using mod 769 we conclude that $(b, c, d) \equiv (4, 6, 1) \pmod{(48, 384, 24)}$, Thus using mod 193 we have $d \equiv 1 \pmod{192}$, so that using mod 257, $(b, d) \equiv (4, 1) \pmod{256}$. Hence using mod 128 we conclude that $c = 6$, $3^b = 62 + 19^d$, $d > 1$. Using mod 361, $b \equiv 76 \pmod{342}$, Thus we have a contradiction mod 229.

Case 8.2. $c = 2$. Then $2^a + 3^b = 4 + 19^d$. From Case 8.1, $a > 2$. We have a contradiction mod 8.

Case 8.3. $a = 3$. Then $8 + 3^b = 2^c + 19^d$. By Case 8.2, $c > 2$. Using mod 8, $b \equiv d \pmod 2$. Clearly $b > 1$. Hence using mod 9 we have $c \equiv 4 \pmod 6$, so that using mod 5, $(b, c, d) \equiv (2, 0, 0)$ or $(3, 0, 1) \pmod{(4, 4, 2)}$. Applying these results and using moduli 13, 7 successively we get $(b, c, d) \equiv (3, 4, 1)$ or $(2, 4, 0) \pmod{(6, 12, 12)}$. Thus using moduli 19, 27 successively we have $(b, c, d) \equiv (3, 4, 1) \pmod{(18, 18, 3)}$ so that in fact $(b, c, d) \equiv (3, 4, 1) \pmod{(36, 36, 12)}$. Therefore using mod 17 we get $(b, c, d) \equiv (3, 4, 1)$ or $(15, 4, 5) \pmod{(16, 8, 8)}$. Combining these results and using mod 97 we have $(b, c, d) \equiv (3, 4, 1)$ or $(15, 4, 5) \pmod{(48, 48, 32)}$. Hence using mod 64 we conclude that $c = 4$,

$(b, d) \equiv (3, 1) \pmod{(48, 32)}$. Thus $(b, d) \equiv (3, 1) \pmod{(144, 96)}$ $3^b = 8 + 19^d$, $b > 3$. Using mod 81, $d \equiv 4 \pmod 9$ so that in fact $d \equiv 13 \pmod{36}$. We have a contradiction mod 37.

Case 8.4. $a \geq 5$ and $c \geq 4$. Using mod 8 we have the following subcases: $(b, d) \equiv (0, 0)$ or $(1, 1) \pmod 2$.

Subcase 8.4.1. $(b, d) \equiv (0, 0) \pmod 2$. Applying the above results and using moduli 5, 16 successively we have $(a, b, c, d) \equiv (3, 2, 0, 2) \pmod 4$. Hence using mod 17 we have the following possibilities: $3^b \equiv 2^{4k} + 19^{2+4m} - 2^{3+4n} \pmod{17}$, $k, m, n = 0, 1$. Thus $b \equiv 9, 13, 15, 12, 4, 1, 5$ or $7 \pmod{16}$, each of these congruences is a contradiction.

Subcase 8.4.2. $(b, d) \equiv (1, 1) \pmod 2$. Applying the above results and using moduli 5, 16 successively we have $(a, b, c, d) \equiv (1, 1, 0, 1)$ or $(3, 3, 0, 3) \pmod 4$. Hence using mod 17 we conclude that $(a, b, c, d) \equiv (5, 1, 4, 1)$, $(5, 5, 0, 1)$, $(1, 9, 0, 5)$ or $(1, 13, 4, 5) \pmod{(8, 16, 8, 8)}$.

We will show that $c > 4$. Assume the contrary. Then $2^a + 3^b = 16 + 19^d$. Suppose $b > 1$. Using mod 9, $a \equiv 3 \pmod 6$. Applying the above results and using mod 13 we have $(a, b, d) \equiv (9, 1, 9)$ or $(9, 2, 11) \pmod{(12, 3, 12)}$ so that in fact $(a, b, d) \equiv (9, 1, 9)$ or $(9, 5, 11) \pmod{(12, 6, 12)}$. In each case we have a contradiction mod 7. Thus $b = 1$, $2^a = 13 + 19^d$, $a > 5$. Applying the above results and using moduli 9, 7 successively we get $(a, d) \equiv (5, 1) \pmod 6$ so that in fact $(a, d) \equiv (5, 1) \pmod{24}$. Hence using mod 97, $(a, d) \equiv (5, 1) \pmod{(48, 32)}$, so that we have a contradiction mod 64. Therefore $c > 4$.

Applying the above results and using mod 32 we conclude that $(a, b, c, d) \equiv (5, 5, 0, 1)$ or $(1, 9, 0, 5) \pmod{(8, 16, 8, 8)}$. Thus using moduli 13, 7 successively we get $(a, b, c, d) \equiv (9, 3, 0, 9)$, $(1, 1, 4, 5)$, $(5, 1, 4, 5)$, $(1, 5, 8, 5)$ or $(5, 5, 8, 1) \pmod{(12, 6, 12, 12)}$. Combining the above results and using mod 9 we have $(a, b, c, d) \equiv (5, 5, 8, 1) \pmod{(12, 6, 12, 12)}$ so that in fact $(a, b, c, d) \equiv (5, 5, 8, 1)$ or $(17, 41, 8, 13) \pmod{(24, 48, 24, 24)}$. Hence using mod 97 we conclude that $(a, b, c, d) \equiv (5, 5, 8, 1) \pmod{(48, 48, 48, 32)}$. Thus $(a, b, c, d) \equiv (5, 5, 8, 1) \pmod{(48, 48, 48, 96)}$, so that using mod 64 we conclude that $a = 5$, $32 + 3^b = 2^c + 19^d$. Hence using mod 193, $(c, d) \equiv (8, 1) \pmod{(96, 192)}$. Applying these results and using mod 257 we conclude that $(b, d) \equiv (5, 1)$ or $(133, 193) \pmod{256}$. Suppose $c > 8$. In each case we have a contradiction mod 1024. Therefore $c = 8$, $3^b = 224 + 19^d$, $b > 5$. Using mod 729, $d \equiv 28 \pmod{81}$ so that in fact $d \equiv 109 \pmod{162}$. Hence using mod 163, $b \equiv 4 \pmod{162}$, a contradiction.

THEOREM 9. *The only solutions of* $2^a + 5^b = 2^c + 7^d$ *in positive integers are* $(a, b, c, d) = (2, 1, 1, 1)$ *and* $(5, 2, 3, 2)$.

Proof. Let (a, b, c, d) be another solution. We consider five cases.

Case 9.1. $a = 1$. Then $2 + 5^b = 2^c + 7^d$. Using mod 3, b is even, c is odd. Thus we have a contradiction mod 8.

Case 9.2. $c = 1$. Then $2^a + 5^b = 2 + 7^d$. From [2], $a > 2$. Using mod 8, d is odd, b is even. Hence using mod 5, $(a, d) \equiv (2, 1) \pmod 4$, so that we have a contradiction mod 3.

Case 9.3. $a = 2$. Then $4 + 5^b = 2^c + 7^d$. By Case 9.2, $c > 2$. Using mod 8, b is odd, d is even. Thus using mod 5, $(c, d) \equiv (3, 0) \pmod 4$. Further, using mod 7, $(b, c) \equiv (1, 1) \pmod{(6, 3)}$. Combining these results and using mod 9 we get $(c, d) \equiv (1, 1) \pmod{(6, 3)}$ so that in fact $(b, c, d) \equiv (1, 7, 4)$ or $(7, 7, 4)$ (md 12). In each case we have a contradiction mod 13.

Case 9.4. $c = 2$. Then $2^a + 5^b = 4 + 7^d$. From Case 9.1, $a > 2$. Using mod 8, d is even, b is odd. Hence we have a contradiction mod 3.

Case 9.5. $a \geq 3$ and $c \geq 3$. Using mod 8, b is even, d is even. Thus using mod 3, $a \equiv c \pmod 2$. Therefore using mod 5 we have the following subcases: $(a, c, d) \equiv (3, 1, 0)$ or $(1, 3, 2) \pmod 4$.

Subcase 9.5.1. $(a, c, d) \equiv (3, 1, 0) \pmod 4$.

We will show that $a > 3$. Assume the contrary. Then $8 + 5^b = 2^c + 7^d$. Using mod 16, $b \equiv 2 \pmod 4$. Combining the above results and using mod 7 we have $(b, c) \equiv (0, 1) \pmod{(6, 3)}$ so that in fact $(b, c) \equiv (6, 1) \pmod{12}$. Hence using mod 13, $d \equiv 3 \pmod{12}$, a contradiction. Therefore $a > 3$.

Applying the above results and using mod 32, $b \equiv 0 \pmod 8$, so that using mod 17 we have the following possibilities:

$7^d \equiv 2^{3+4k} + 5^{8n} - 2^{1+4m}$ (mod 17), $k, n, m = 0, 1$. Hence $d \equiv 1, 14, 5, 15, 6, 13, 9$ or 7 (mod 16), each of these congruences is a contradiction.

Subcase 9.5.2. $(a, c, d) \equiv (1, 3, 2)$ (mod 4).

We will show that $c > 3$. Assume the contrary. Then $2^a + 5^b = 8 + 7^d$. Applying the above results and using mod 32 we have $b \equiv 2$ (mod 8), so that using mod 17, $(a, b, d) \equiv (5, 2, 2)$ or $(1, 2, 10)$ (mod (8, 16, 16)). Thus using mod 64 we conclude that $a = 5$, $(b, d) \equiv (2, 2)$ (mod 16). Hence $24 + 5^b = 7^d$, $b > 2$. Using mod 125, $d \equiv 6$ (mod 20), so that we have a contradiction mod 11. Therefore $c > 3$.

Applying the above results and using mod 32 we get $b \equiv 4$ (mod 8), so that using mod 13, $(a, c, d) \equiv (5, 3, 6)$ or $(9, 11, 6)$ (mod 12). Thus using mod 7 we conclude that $(a, b, c, d) \equiv (5, 2, 3, 6)$ (mod (12, 6, 12, 12)). This yields a contradiction mod 9.

THEOREM 10. *The only solutions to* $2^a + 5^b = 2^c + 11^d$ *in positive integers are* $(a, b, c, d) = (1, 2, 4, 1), (3, 1, 1, 1), (2, 3, 3, 2), (3, 4, 9, 2)$ *and* $(7, 2, 5, 2)$.

Proof. Let (a, b, c, d) be another solution. We consider five cases.

Case 10.1. $a = 1$. Then $2 + 5^b = 2^c + 11^d$. Using mod 5, $c \equiv 0$ (mod 4), so that using mod 3, b is even, d is odd. Thus using mod 16, $(b, d) \equiv (2, 1)$ or $(0, 3)$ (mod 4). Hence using mod 13 we get $(b, c, d) \equiv (0, 0, 7)$ or $(2, 4, 1)$ (mod (4, 12, 12)). Therefore using moduli 9, 7 successively we conclude that $(b, c, d) \equiv (2, 4, 1)$ (mod (6, 12, 12)). Combining these results and using mod 17 we have $(b, c, d) \equiv (2, 4, 1)$ or $(10, 0, 5)$ (mod (16, 8, 16)). Hence using mod 64 we conclude that $c = 4$, $(b, d) \equiv (2, 1)$ (mod 16). Thus $5^b = 14 + 11^d$, $b > 2$. Using mod 125, $d \equiv 11$ (mod 25) so that in fact $d \equiv 61$ (mod 100). Therefore we have a contradiction mod 101.

Case 10.2. $c = 1$. Then $2^a + 5^b = 2 + 11^d$. Using mod 5, $a \equiv 3$ (mod 4). Using mod 8, b is odd, d is odd. Thus using mod 13 we conclude that $(a, b, d) \equiv (3, 1, 1)$ (mod (12, 4, 12)). Therefore using mod 16 we have $a = 3$, $6 + 5^b = 11^d$, $b > 1$. Using mod 31 we get $(b, d) \equiv (1, 1) \pmod{(3, 30)}$. This yields a contradiction mod 25.

Case 10.3. $a = 2$. Then $4 + 5^b = 2^c + 11^d$. By Case 10.2, $c > 2$. Using mod 8, b is odd, d is even. Using mod 5, $c \equiv 3 \pmod 4$. Suppose $c > 3$. Using mod 16, $(b, d) \equiv (1, 2)$ or $(3, 0) \pmod 4$. Applying these results and using mod 13 we conclude that $(b, c, d) \equiv (3, 7, 0) \pmod{(4, 12, 12)}$. Hence using mod 61 we have $(b, c) \equiv (3, 7) \pmod{(30, 60)}$. This yields a contradiction mod 11. Thus $c = 3$, $5^b = 4 + 11^d$, $b > 3$. Using mod 625, $d \equiv 52 \pmod{125}$ so that in fact $d \equiv 52 \pmod{250}$. Therefore we have a contradiction mod 251.

Case 10.4. $c = 2$. Then $2^a + 5^b = 4 + 11^d$. We have a contradiction mod 5.

Case 10.5. $a \geq 3$ and $c \geq 3$. Using mod 8, b is even, d is even. Thus using mod 3, $a \equiv c \pmod 2$, so that using mod 5, $(a, c) \equiv (3, 1) \pmod 4$.

We will show that $a > 3$. Assume the contrary. Then $8 + 5^b = 2^c + 11^d$. Combining the above results and using mod 16 we have $(b, c, d) \equiv (0, 1, 2)$ or $(2, 1, 0) \pmod 4$. Hence using mod 13 we get $(b, c, d) \equiv (0, 9, 2)$ or $(2, 5, 0) \pmod{(4, 12, 12)}$. Therefore using mod 9 we have $(b, c, d) \equiv (4, 9, 2)$ or $(2, 5, 0) \pmod{(6, 12, 12)}$ so that in fact $(b, c, d) \equiv (4, 9, 2)$ or $(2, 5, 0) \pmod{12}$. Suppose that $(b, c, d) \equiv (2, 5, 0) \pmod{12}$. Thus using mod 61 we get $(b, c) \equiv (2, 5) \pmod{(30, 60)}$. This yields a contradiction mod 11. Hence $(b, c, d) \equiv (4, 9, 2) \pmod{12}$. Thus using mod 17 we have $(b, c, d) \equiv (4, 1, 2)$ or $(8, 5, 14) \pmod{(16, 8, 16)}$ so that in fact $(b, c, d) \equiv (4, 9, 2)$ or $(40, 21, 14) \pmod{(48, 24, 48)}$. Hence using mod 97 we conclude that $(b, c, d) \equiv (4, 9, 2) \pmod{(96, 48, 48)}$. Thus using mod 193 we get $(b, c, d) \equiv (4, 9, 2) \pmod{(192, 96, 64)}$. Applying these results and using mod 769 we conclude that $(b, c, d) \equiv (4, 9, 2) \pmod{(128, 384, 768)}$. Hence using mod 257 we have $b \equiv 4 \pmod{256}$ so that in fact $(b, c, d) \equiv (4, 9, 2) \pmod{(768, 384, 768)}$. Therefore using mod 1024 we conclude that $c = 9$, $5^b = 504 + 11^d$, $d > 2$. Using mod 121, $b \equiv 4 \pmod{55}$ so that in fact $b \equiv 4 \pmod{66}$. Thus using mod 23, $d \equiv 2 \pmod{22}$, so the using mod 727 we conclude that $(b, d) \equiv (4, 2)$

(mod (726, 242)). Hence using mod 1331, $b \equiv 444 \pmod{605}$, a contradiction. Thus $a > 3$.

Applying the above results and using mod 16 we get $(a, b, c, d) \equiv (3, 0, 1, 0)$ or $(3, 2, 1, 2) \pmod 4$. Therefore using mod 17 we conclude that $(a, b, c, d) \equiv (3, 2, 1, 2) \pmod 4$. Hence using mod 13 we have $(a, b, c, d) \equiv (7, 2, 5, 2)$ or $(11, 2, 1, 2)$ (mod (12, 4, 12, 12)), so that using mod 7, $(a, b, c, d) \equiv (7, 2, 5, 2)$ (mod (12, 6, 12, 12)). Therefore using mod 61 we conclude that $(a, b, c) \equiv (7, 2, 5)$, $(19, 26, 53)$ or $(43, 20, 29)$ (mod (60, 30, 60)). Combining the above results and using mod 31 we have $(a, b, c, d) \equiv (7, 2, 5, 2)$ (mod (60, 30, 60, 30)) so that in fact $(a, b, c, d) \equiv (7, 2, 5, 2)$ (mod 60). Hence using mod 241 we conclude that $(a, b, c, d) \equiv (7, 2, 5, 2)$ or $(19, 22, 17, 26)$ (mod (24, 40, 24, 48)). Combining these results and using mod 17 we have $(a, b, c, d) \equiv (7, 2, 5, 2)$ (mod (24, 40, 24, 48)), $b \equiv 2 \pmod{16}$ so that in fact $(a, b, c, d) \equiv (7, 2, 5, 2)$ (mod (24, 48, 24, 48)). Therefore using mod 64 we conclude that $c = 5$, $2^a + 5^b = 32 + 11^d$. Thus using mod 97, $(a, b) \equiv (7, 2)$ (mod (48, 96)), so that using mod 193 we conclude that $(a, b, d) \equiv (7, 2, 2)$ (mod (96, 192, 64)). Hence using mod 256 we conclude that $a = 7$, $96 + 5^b = 11^d$, $b > 2$. Using mod 125, $d \equiv 12 \pmod{25}$ so that in fact $d \equiv 62 \pmod{100}$. Therefore we have a contradiction mod 101.

THEOREM 11. *The only solutions of* $2^a + 5^b = 2^c + 13^d$ *in positive integers are* $(a, b, c, d) = (2, 2, 4, 1)$, $(4, 1, 3, 1)$ *and* $(4, 3, 7, 1)$.

Proof. Let (a, b, c, d) be another solution. Using mod 4, $a \geq 2$, $c \geq 2$. We consider five cases.

Case 11.1. $a = 2$. Then $4 + 5^b = 2^c + 13^d$. Using mod 3, $b \equiv c \pmod 2$. Thus using mod 13 we get $(b, c) \equiv (2, 4)$ (mod (4, 12)), so that using mod 5, $d \equiv 1 \pmod 4$. Hence using mod 7, $b \equiv 2 \pmod 6$, so that using mod 9, $d \equiv 1 \pmod 3$. Combining these results and using mod 17 we have $(b, c, d) \equiv (2, 4, 1)$ (mod (16, 8, 4)) so that in fact $(b, c, d) = (2, 4, 1)$ (mod (48, 24, 12)). Thus using mod 97, $(b, c, d) \equiv (2, 4, 1)$ or $(50, 28, 25)$ (mod (96, 48, 96)). Hence using mod 193 we conclude that $(b, c, d) \equiv (2, 4, 1)$ (mod (192, 96, 64)). Therefore using mod 32 we have $c = 4$, 5^b

$= 12 + 13^d$, $b > 2$. Using mod 125, $d \equiv 81 \pmod{100}$. This yields a contradiction mod 101.

Case 11.2. $c = 2$. Then $2^a + 5^b = 4 + 13^d$. Using mod 3, a is even, b is even. Thus using mod 13, $(a, b) \equiv (4, 0) \pmod{(12, 4)}$, so that using mod 16, $d \equiv 1 \pmod 4$. Hence we have a contradiction mod 5.

Case 11.3. $a = 3$. Then $8 + 5^b = 2^c + 13^d$. Using mod 3, b is even, c is odd. Thus using mod 5, $(c, d) \equiv (1, 0) \pmod 4$, so that using mod 16, $b \equiv 2 \pmod 4$. Hence using mod 13 we have $(b, c) \equiv (2, 11) \pmod{(4, 12)}$, a contradiction.

Case 11.4. $c = 3$. Then $2^a + 5^b = 8 + 13^d$. Using mod 3, $a - b$ is odd, so that using mod 13, $(a, b) \equiv (11, 0)$ or $(4, 1) \pmod{(12, 4)}$. Thus using mod 5 we have $(b, c, d) \equiv (4, 1, 1) \pmod{(12, 4, 4)}$. Hence using moduli 7, 9 successively we get $(b, d) \equiv (1, 1) \pmod{(6, 3)}$. Applying these results and using mod 17 we have $(a, b) \equiv (4, 1)$ or $(0, 13) \pmod{(8, 16)}$ so that in fact $(a, b, d) \equiv (4, 1, 1)$ or $(16, 13, 1) \pmod{(24, 48, 12)}$. Thus using mod 97 we conclude that $(a, b, d) \equiv (4, 1, 1)$ or $(28, 49, 37) \pmod{(48, 96, 96)}$. Thus using mod 193 we have $(a, b, d) \equiv (4, 1, 1) \pmod{(96, 192, 64)}$. Therefore using mod 32 we conclude that $a = 4$, $8 + 5^b = 13^d$, $b > 1$. Using mod 25, $d \equiv 17 \pmod{20}$ so that in fact $d \equiv 7 \pmod{30}$. Hence we have a contradiction mod 31.

Case 11.5. $a \geq 4$ and $c \geq 4$. Using mod 16, $(b, d) \equiv (1, 3), (3, 1), (0, 0)$ or $(2, 2) \pmod 4$. Thus using moduli 3, 5 successively we have $(a, b, c, d) \equiv (0, 3, 3, 1), (2, 1, 1, 3), (3, 0, 1, 0)$ or $(1, 2, 3, 2) \pmod 4$. Therefore using mod 13 we conclude that $(a, b, c, d) \equiv (10, 1, 1, 3)$ or $(4, 3, 7, 1) \pmod{(12, 4, 12, 4)}$. Hence using mod 17 we get $(a, b, c, d) \equiv (4, 3, 7, 1) \pmod{(8, 16, 8, 4)}$ so that in fact $(a, b, c, d) \equiv (4, 3, 7, 1) \pmod{(24, 16, 24, 4)}$. Thus using moduli 7, 9 successively, $(b, d) \equiv (3, 1) \pmod{(6, 3)}$ so that in fact $(b, d) \equiv (3, 1) \pmod{(48, 12)}$. Applying these results and using mod 97 we have $(a, b, c, d) \equiv (4, 3, 7, 1)$ or $(28, 51, 31, 49) \pmod{(48, 96, 48, 96)}$. Therefore using mod 32 we conclude that $a = 4$, $16 + 5^b = 2^c + 13^d$, $(b, c, d) \equiv (3, 7, 1) \pmod{(96, 48, 96)}$. Hence using mod 193 we get $(b, c, d) \equiv (3, 7, 1) \pmod{(192, 96, 64)}$. Thus using mod 256 we get $c = 7$, $5^b = 112 + 13^d$, $d > 1$. Using mod 169, $b \equiv 47$

(mod 52), This yields a contradiction mod 53.

THEOREM 12. *The only solutions to* $2^a + 5^b = 2^c + 17^d$ *in positive integers are* $(a, b, c, d) = (4, 1, 2, 1)$ *and* $(3, 2, 4, 1)$.

Proof. Let (a, b, c, d) be another solution. Using mod 4, $a \geq 2$, $c \geq 2$. We consider five cases.

Case 12.1. $a = 2$. Then $4 + 5^b = 2^c + 17^d$. Using mod 8, b is odd, so that using mod 3, $c - d$ is odd. Thus using mod 5, $(c, d) \equiv (0, 3)$ or $(3, 0) \pmod 4$. Hence using mod 17 we conclude that $(b, c, d) \equiv (9, 4, 3), (5, 0, 3)$ or $(1, 7, 0) \pmod{(16, 8, 4)}$. In each case we have a contradiction mod 16.

Case 12.2. $c = 2$. Then $2^a + 5^b = 4 + 17^d$. Using mod 8, b is odd, so that using mod 3, d is odd, a is even. Thus using mod 5, $(a, d) \equiv (0, 1) \pmod 4$. Hence using mod 17 we get $(a, b, d) \equiv (4, 1, 1)$ or $(0, 13, 1) \pmod{(8, 16, 4)}$. Therefore using moduli 13, 7 successively we have $(a, b, d) \equiv (4, 1, 1) \pmod{(12, 12, 6)}$ so that in fact $(a, b, d) \equiv (4, 1, 1)$ or $(16, 13, 1) \pmod{(24, 48, 12)}$. Thus using mod 97 we conclude, that $(a, b, d) \equiv (4, 1, 1) \pmod{(48, 96, 96)}$. Hence using mod 32 we get $a = 4$, $12 + 5^b = 17^d$, $b > 1$. Using mod 25, $d \equiv 13 \pmod{20}$. Therefore we have a contradiction mod 11.

Case 12.3. $a = 3$. Then $8 + 5^b = 2^c + 17^d$. From Case 12.2, $c > 3$. Using mod 16, $b \equiv 2 \pmod 4$, so that using mod 3, $c - d$ is odd. Hence using mod 5 we have $(c, d) \equiv (0, 1)$ or $(1, 0) \pmod 4$. Combining these results and using mod 17 we conclude that $(b, c, d) \equiv (2, 4, 1) \pmod{(16, 8, 4)}$. Thus using mod 32 we get $c = 4$, $5^b = 8 + 17^d$, $b > 2$. Using mod 13, $d \equiv 1 \pmod 6$. Hence using mod 125, $d \equiv 61 \pmod{100}$ so that in fact $d \equiv 61 \pmod{600}$. This yields a contradiction mod 601.

Case 12.4. $c = 3$. Then $2^a + 5^b = 8 + 17^d$. By Case 12.1, $a > 3$. Using mod 16, $b \equiv 2 \pmod 4$, so that using mod 3, d is even, a is odd. Thus using mod 5, $(a, d) \equiv (1, 2) \pmod 4$. Hence using mod 17 we have the following possibilities: $5^b \equiv 8 - 2^{1+4k} \pmod{17}$, $k = 0, 1$. Therefore $b \equiv 3$ or $7 \pmod{16}$, each of these congruences is a contradiction.

Case 12.5. $a \geq 4$ and $c \geq 4$. Using mod 16, $b \equiv 0 \pmod 4$, Thus using moduli 3, 5 successively we have $(a, c, d) \equiv (3, 1, 0), (1, 3, 2), (3, 0, 1)$ or $(1, 2, 3) \pmod 4$. Hence using mod 32 we conclude that $(a, b, c, d) \equiv (3, 0, 1, 0), (1, 0, 3, 2), (3, 4, 0, 1)$ or $(1, 4, 2, 3)$ (mod $(4, 8, 4, 4)$). Consideration of our equation mod 17, in each case we have a contradiction.

THEOREM 13. *The only solutions of* $2^a + 5^b = 2^c + 19^d$ *in positive integers are* $(a, b, c, d) = (1, 2, 3, 1)$ *and* $(4, 1, 1, 1)$.

Proof. Let (a, b, c, d) be another solution. We consider five cases.

Case 13.1. $a = 1$. Then $2 + 5^b = 2^c + 19^d$. Using mod 3, c is odd, b is even, so that using mod 8, d is odd. Thus using mod 5, $c \equiv 3 \pmod 4$. Combining these results and using moduli 13, 7 successively we conclude that $(b, c, d) \equiv (2, 3, 1) \pmod{12}$. Therefore using mod 16 we have $c = 3$, $5^b = 6 + 19^d$, $b > 2$. Using mod 125, $d \equiv 31 \pmod{50}$ so that in fact $d \equiv 181$ or $481 \pmod{600}$. In each case we have a contradiction mod 601.

Case 13.2. $c = 1$. Then $2^a + 5^b = 2 + 19^d$. Using mod 3, $a - b$ is odd. Using mod 8, d is odd, so that using mod 5, $a \equiv 0 \pmod 4$. Hence using moduli 13, 7 successively we get $(a, b, d) \equiv (4, 1, 1)$ or $(0, 3, 7) \pmod{12}$. Thus using mod 9, $(a, b, d) \equiv (4, 1, 1) \pmod{12}$. Hence using mod 17 we have $(a, b, d) \equiv (4, 1, 1)$ or $(0, 13, 1)$ (mod $(8, 16, 8)$) so that in fact $(a, b, d) \equiv (4, 1, 1)$ or $(16, 13, 1)$ (mod $(24, 48, 24)$). Thus using mod 97 we get $(a, b, d) \equiv (4, 1, 1)$ or $(16, 61, 25)$ (mod $(48, 96, 32)$). Applying these results and using mod 193 we conclude that $(a, b, d) \equiv (4, 1, 1)$ (mod $(96, 192, 192)$). Therefore using mod 32 we have $a = 4$, $14 + 5^b = 19^d$, $b > 1$. Using mod 25, $d \equiv 7 \pmod{10}$ so that in fact $d \equiv 7 \pmod{30}$. This yields a contradiction mod 31.

Case 13.3. $a = 2$. Then $4 + 5^b = 2^c + 19^d$. From Case 13.2 $c > 2$. Using mod 8, b is odd, d is even, so that using mod 5, $c \equiv 3 \pmod 4$. Hence using moduli 13, 7 successively, $(b, c, d) \equiv (3, 7, 0), (3, 3, 10)$ or $(1, 3, 0) \pmod{12}$, so that using mod 9, $(b, c, d) \equiv (3, 7, 0)$ or $(1, 3, 0) \pmod{12}$. Thus using moduli 27, 37 successively we have $(b, c, d) \equiv (3, 7, 0)$ or $(1, 3, 0) \pmod{36}$. In each case we have a contradiction mod 19.

Case 13.4. $c=2$. Then $2^a+5^b=4+19^d$. Using mod 5, $(a, d) \equiv (3, 1) \pmod{(4, 2)}$. Hence we have a contradiction mod 3.

Case 13.5. $a \geq 3$ and $c \geq 3$. Using mod 8, b is even, d is even. Thus using mod 3, $a \equiv c \pmod 2$, so that using mod 5, $(a, c) \equiv (3, 1) \pmod 4$.

We will show that $a > 3$. Assume the contrary. Then $8+5^b=2^c+19^d$. Applying the above results and using mod 16 we have $(b, c, d) \equiv (0, 1, 2)$ or $(2, 1, 0) \pmod 4$. Hence using mod 17 we conclude that $(b, c, d) \equiv (2, 5, 0) \pmod{(16, 8, 8)}$. Thus using mod 13, $(c, d) \equiv (5, 0) \pmod{12}$. Combining these results and using mod 7, $b \equiv 2 \pmod 6$ so that in fact $(b, c, d) \equiv (2, 5, 0) \pmod{(48, 24, 24)}$. Therefore using mod 97 we have $(b, c, d) \equiv (2, 5, 0) \pmod{(96, 48, 32)}$. Hence using mod 64 we conclude that $c=5$, $5^b=24+19^d$. Using mod 19, $b \equiv 1 \pmod 9$, a contradiction. Thus $a > 3$.

Applying the above results and using mod 16 we have $(a, b, c, d) \equiv (3, 0, 1, 0)$ or $(3, 2, 1, 2) \pmod 4$. Hence using mod 17 we conclude that $(a, b, c, d) \equiv (7, 6, 5, 6)$ or $(3, 14, 1, 2) \pmod{(8, 16, 8, 8)}$. Therefore using mod 32 we get $(a, b, c, d) \equiv (3, 14, 1, 2) \pmod{(8, 16, 8, 8)}$. Thus using mod 13 we have $(a, b, c, d) \equiv (7, 2, 5, 10)$ or $(11, 2, 1, 10) \pmod{(12, 4, 12, 12)}$. Hence using mod 7, $(a, b, c, d) \equiv (7, 2, 5, 10) \pmod{(12, 6, 12, 12)}$. This yields a contradiction mod 9.

THEOREM 14. *The only solution to* $2^a+7^b=2^c+11^d$ *in positive integers is* $(a, b, c, d)=(3, 1, 2, 1)$.

Proof. Let (a, b, c, d) be another solution. We consider five cases.

Case 14.1. $a=1$. Then $2+7^b=2^c+11^d$. Using mod 4, $b-d$ is odd. Using mod 7, $(c, d) \equiv (0, 0) \pmod 3$. Thus using mod 9, $(b, c, d) \equiv (1, 3, 0)$ or $(1, 0, 3) \pmod{(3, 6, 6)}$, Hence using mod 13 we have $(b, c, d) \equiv (10, 0, 3)$ or $(1, 3, 0) \pmod{12}$. Therefore using mod 5 we conclude that $(b, c, d) \equiv (1, 3, 0) \pmod{12}$. Thus using mod 16 we get $c=3$, $7^b=6+11^d$. Using mod 11, $b \equiv 7 \pmod{10}$ so that in fact $b \equiv 7 \pmod{30}$. Hence using mod 31, $d \equiv 19 \pmod{30}$, a contradiction.

Case 14.2. $c = 1$. Then $2^a + 7^b = 2 + 11^d$. Using mod 3, d is even, a is odd. Thus we have a contradiction mod 8.

Case 14.3. $a = 2$. Then $4 + 7^b = 2^c + 11^d$. Using mod 3, c is even, d is even. Hence we have a contradiction mod 8.

Case 14.4. $c = 2$. Then $2^a + 7^b = 4 + 11^d$. By Case 14.1, $a > 2$. Using mod 8, d is odd, b is odd, so that using mod 3, a is odd. Thus using mod 7, $(a, d) \equiv (0, 1) \pmod 3$ so that in fact $(a, d) \equiv (3, 1) \pmod 6$. Hence using mod 9, $b \equiv 1 \pmod 6$, so that using mod 13 we have $(a, b, d) \equiv (3, 1, 1) \pmod{12}$. Therefore using mod 16 conclude that $a = 3$, $4 + 7^b = 11^d$, $b > 1$. Using mod 49, $d \equiv 16 \pmod{21}$. This yields a contradiction mod 43.

Case 14.5. $a \geq 3$ and $c \geq 3$. Using mod 8, b is even, d is even. Thus using mod 3, $a \equiv c \pmod 2$. We consider the following subcases: $b \equiv 2$ or $0 \pmod 4$.

Subcase 14.5.1. $b \equiv 2 \pmod 4$. Using mod 5, $(a, c) \equiv (0, 2) \pmod 4$, so that using mod 16, $d \equiv 0 \pmod 4$. Combining these results and using moduli 7, 9 successively we have $(a, b, c, d) \equiv (4, 2, 0, 0)$, $(0, 0, 2, 4)$ or $(2, 1, 4, 2)$ (mod $(6, 3, 6, 6)$) so that in fact $(a, b, c, d) \equiv (4, 2, 6, 0)$, $(0, 6, 2, 4)$ or $(8, 10, 10, 8) \pmod{12}$. Hence using mod 13 we conclude that $(a, b, c, d) \equiv (4, 2, 6, 0) \pmod{12}$. Thus using mod 19, $(a, c) \equiv (4, 6) \pmod{18}$, so that using mod 73 we get $(b, d) \equiv (2, 0)$ (mod $(24, 72)$). Therefore using mod 32 we conclude that $a = 4$, $16 + 7^b = 2^c + 11^d$. Combining the above results and using mod 17 we have $(b, c, d) \equiv (2, 6, 0)$ (mod $(16, 8, 16)$) so that in fact $(b, c, d) \equiv (2, 6, 0)$ (mod $(48, 24, 48)$). Hence using mod 97, $(b, c) \equiv (2, 6)$ (mod $(96, 48)$). Thus using mod 193 we get $(c, d) \equiv (6, 0)$ (mod $(96, 64)$) so that in fact $(b, c, d) \equiv (2, 6, 0)$ (mod $(96, 96, 192)$). Therefore using mod 128 we conclude that $c = 6$, $7^b = 48 + 11^d$. Using mod 11, $b \equiv 6 \pmod{10}$ so that in fact $b \equiv 26 \pmod{40}$. Hence using mod 41, $d \equiv 35 \pmod{40}$, a contradiction.

Subcase 14.5.2. $b \equiv 0 \pmod 4$. Using mod 5, $a \equiv c \pmod 4$.

We will first show that $a > 3$. Assume the contrary. Then $8 + 7^b = 2^c + 11^d$. Using mod 16, $d \equiv 2 \pmod 4$. Using mod 7, $(c, d) \equiv (2, 1) \pmod 3$. Combining the above results and using mod 9 we have $(b, c, d) \equiv (2, 5, 4)$ (mod $(3, 6, 6)$) so that in fact

$(b, c, d) \equiv (8, 11, 10) \pmod{12}$. This yields a contradiction mod 13. Thus $a > 3$.

We will next show that $c > 3$. Assume the contrary. Then $2^a + 7^b = 8 + 11^d$. Using mod 16, $d \equiv 2 \pmod 4$. Using mod 7, $(a, d) \equiv (1, 0) \pmod 3$. Combining the above results and using mod 9 we get $(a, b, d) \equiv (1, 1, 0) \pmod{(6, 3, 6)}$ so that in fact $(a, b, d) \equiv (7, 4, 6) \pmod{12}$. Thus using mod 19, $a \equiv 1 \pmod{18}$, so that using mod 73 we have the following possibilities: $11^d \equiv 2 + 7^{4+12k} - 8 \pmod{73}$, $k = 0, 1$. Hence $d \equiv 59$ or $8 \pmod{72}$, each of these congruences is a contradiction. Therefore $c > 3$.

Using mod 16, $d \equiv 0 \pmod 4$. Applying the above results and using moduli 13, 7 successively we conclude that $(a, b, c, d) \equiv (10, 8, 6, 0)$ or $(8, 8, 4, 8) \pmod{12}$. Hence using mod 9 we get $(a, b, c, d) \equiv (10, 8, 6, 0) \pmod{12}$. Therefore using mod 19 we have $(a, c) \equiv (4, 6) \pmod{18}$ so that in fact $(a, b, c, d) \equiv (22, 8, 6, 0) \pmod{(36, 12, 36, 12)}$. Thus using mod 37 we conclude that $2^{22} + 7^b \equiv 2^6 + 11^0 \pmod{37}$, $b \equiv 1 \pmod 9$, a contradiction.

THEOREM 15. *The only solutions of* $2^a + 7^b = 2^c + 13^d$ *in positive integers are* $(a, b, c, d) = (3, 1, 1, 1)$ *and* $(7, 2, 3, 2)$.

Proof. Let (a, b, c, d) be another solution. Using mod 3, $a \equiv c \pmod 2$. We consider five cases.

Case 15.1. $a = 1$. Then $2 + 7^b = 2^c + 13^d$. Thus $c \equiv 1 \pmod 2$ Suppose $c > 3$. Using mod 16, $(b, d) \equiv (1, 2) \pmod{(2, 4)}$. Hence using mod 5, $(b, c, d) \equiv (3, 0, 2) \pmod 4$, a contradiction. Hence $c = 3$, $7^b = 6 + 13^d$. Using mod 13, $b \equiv 7 \pmod{12}$, so that using mod 5, $d \equiv 3 \pmod 4$. This yields a contradiction mod 7.

Case 15.2. $c = 1$. Then $2^a + 7^b = 2 + 13^d$. Thus $a \equiv 1 \pmod 2$. Using mod 8, b is odd, d is odd, so that using mod 7, $a \equiv 0 \pmod 3$. Therefore using mod 13 we have $(a, b) \equiv (3, 1) \pmod{12}$, so that using mod 5, $d \equiv 1 \pmod 4$. Hence using mod 16 we conclude that $a = 3$, $6 + 7^b = 13^d$, $b > 1$. Using mod 49, $d \equiv 11 \pmod{14}$. This yields a contradiction mod 29.

Case 15.3. $a = 2$. Then $4 + 7^b = 2^c + 13^d$. We have a contradiction mod 7.

Case 15.4. $c = 2$. Then $2^a+7^b=4+13^d$. We have a contradicton mod 7.

Case 15.5. $a \geq 3$ and $c \geq 3$. Using mod 8, b is even, d is even. Thus using mod 7 we have $(a, c) \equiv (1, 0) \pmod 3$ so that in fact $(a, c) \equiv (1, 3)$ or $(4, 0) \pmod 6$. Combining these results and using mod 13 we get $(a, b, c) \equiv (7, 2, 3), (1, 8, 9), (10, 10, 0)$ or $(4, 4, 6) \pmod{12}$. Therefore using mod 5 we conclude that $(a, b, c, d) \equiv (7, 2, 3, 2)$ or $(1, 8, 9, 0)$ (mod (12, 12, 12, 4)).

Suppose that $(a, b, c, d) \equiv (1, 8, 9, 0)$ (mod (12, 12, 12, 4)). Thus using mod 9, $d \equiv 2 \pmod 3$ so that in fact $d \equiv 8 \pmod{12}$. Hence using mod 19 we have $(a, c, d) \equiv (1, 3, 14)$ or $(7, 3, 2)$ (mod 18) so that in fact $(a, b, c, d) \equiv (1, 8, 21, 32)$ or $(25, 8, 21, 20)$ (mod (36, 12, 36, 36)). Consideration of our equation mod 37, in each case we have a contradiction.

Therefore $(a, b, c, d) \equiv (7, 2, 3, 2)$ (mod (12, 12, 12, 4)). Using mod 9, $d \equiv 2 \pmod 3$ so that in fact $(a, b, c, d) \equiv (7, 2, 3, 2) \pmod{12}$. Hence using mod 16 we conclude that $c = 3$, $2^a + 7^b = 8 + 13^d$. Thus using mod 17, $(a, b) \equiv (7, 2)$ (mod (8, 16)) so that in fact $(a, b, d) \equiv (7, 2, 2)$ (mod (24, 48, 12)). Therefore using mod 97 we have $(a, b, d) \equiv (7, 2, 2)$ (mod (48, 96, 96)), so that using mod 193, $(a, b, d) \equiv (7, 2, 2)$ (mod (96, 96, 64)). Hence using mod 256 we conclude that $a = 7$, $120 + 7^b = 13^d$, $b > 2$. Using mod 343, $d \equiv 30 \pmod{98}$ so that in fact $d \equiv 30 \pmod{196}$. Thus using mod 197, $b \equiv 94 \pmod{98}$, so that we have a contradiction mod 29,

THEOREM 16. ***The only solutions to*** $2^a + 7^b = 2^c + 17^d$ ***in positive integers are*** $(a, b, c, d) = (5, 2, 6, 1)$ ***and*** $(8, 2, 4, 2)$.

Proof. Let (a, b, c, d) be another solution. We consider seven cases.

Case 16.1. $a = 1$. Then $2 + 7^b = 2^c + 17^d$. Using mod 4, b is odd. Thus using mod 16 we conclude that $c = 3$, $7^b = 6 + 17^d$. Using mod 7, $d \equiv 0 \pmod 6$, so that using mod 13, $b \equiv 1 \pmod{12}$. Hence using mod 5, $d \equiv 0 \pmod 4$, so that using mod 64 we have $b \equiv 1 \pmod 8$. Therefore using mod 17 we get $b \equiv 13 \pmod{16}$, a contradiction.

Case 16.2. $c=1$. Then $2^a+7^b=2+17^d$. Using mod 3, d is even, a is odd. We have a contradiction mod 8.

Case 16.3. $a=2$. Then $4+7^b=2^c+17^d$. From Case 16.2, $c>2$. We have a contradiction mod 8.

Case 16.4. $c=2$. Then $2^a+7^b=4+17^d$. By Case 16.1, $a>2$. We have a contradiction mod 8.

Case 16.5. $a=3$. Then $8+7^b=2^c+17^d$. We have a contradiction mod 16.

Case 16.6. $c=3$. Then $2^a+7^b=8+17^d$. From Case 16.3, $a>3$. We have a contradiction mod 16.

Case 16.7. $a\geq 4$ and $c\geq 4$. Using mod 16, b is even.

We will first show that $a>4$. Assume the contrary. Then $16+7^b=2^c+17^d$. Using mod 3, c is even, d is even, so that using mod 32, $b\equiv 2 \pmod 4$. Thus using mod 17 we have $(b, c)\equiv(10, 0)$ (mod $(16, 8)$), so that using mod 64, $d\equiv 0 \pmod 4$. Hence we have a contradiction mod 5. Therefore $a>4$.

We will next show that $c>4$. Assume the contrary. Then $2^a+7^b=16+17^d$. Using mod 3, $a\equiv d \pmod 2$, so that using moduli 9, 7 successively we get $(a, b, d)\equiv(2, 2, 2)$ or $(3, 1, 3)$ (mod $(6, 3, 6)$). Thus using mod 13 we have $(a, b, d)\equiv(8, 2, 2)$ or $(3, 1, 3)$ (mod $(12, 12, 6)$), so that using mod 5, $(a, b, d)\equiv(8, 2, 2)$ (mod 12). Hence using mod 17, $(a, b)\equiv(0, 2)$ (mod $(8, 16)$) so that in fact $(a, b, d)\equiv(8, 2, 2)$ (mod $(24, 48, 12)$). Thus using mod 73, $(a, d)\equiv(8, 2)$ (mod $(9, 24)$). Applying these results and using mod 193 we get $(a, d)\equiv(8, 2)$ or $(80, 74)$ (mod $(96, 192)$). Combining the above results and using mod 97 we conclude that $(a, b, d)\equiv(8, 2, 2)$ (mod $(96, 96, 192)$). Therefore using mod 257 we have $b\equiv 2 \pmod{256}$ so that in fact $(a, b, d)\equiv(8, 2, 2)$ (mod $(96, 768, 192)$). Hence using mod 512 we conclude that $a=8$, $240+7^b=17^d$, $b>2$. Using mod 343, $d\equiv 44 \pmod{294}$ so that in fact $d\equiv 142 \pmod{196}$. Thus we have a contradiction mod 197. Hence $c>4$.

Suppose that d is even. Using mod 32, $b\equiv 0 \pmod 4$. Thus using moduli 9, 7 successively we have $(a, b, c, d)\equiv(1, 1, 3, 0)$, $(3, 1, 5, 4)$, $(5, 1, 1, 2)$, $(0, 2, 2, 4)$, $(2, 2, 4, 2)$ or $(4, 2, 0, 0)$ (mod $(6, 3, 6, 6)$). Combining these results and using mod 13 we conclude that $(a, b, c, d) \equiv (10, 8, 6, 0) \pmod{(12, 12, 12, 6)}$. Therefore using mod 17 we have the following possibilities: $7^b \equiv 2^{2+4n} - 2^{2+4m} \pmod{17}$, $n, m=0, 1$. Thus $b \equiv 6$ or $14 \pmod{16}$, each of these congruences is a contradiction.

Therefore d is odd. Using mod 32, $b \equiv 2 \pmod 4$. Thus using mod 3, c is even, a is odd, so that using mod 5, $(a, b, c, d) \equiv (1, 2, 2, 1)$ or $(3, 2, 2, 3) \pmod 4$. Hence using mod 17 we conclude that $(a, b, c, d) \equiv (1, 10, 2, 1)$ or $(5, 2, 6, 1) \pmod{(8, 16, 8, 4)}$. Thus using mod 64 we get $a = 5$, $32 + 7^b = 2^c + 17^d$, $(b, c, d) \equiv (2, 6, 1) \pmod{(16, 8, 4)}$. Hence using moduli 9, 7 successively we get $(b, c, d) \equiv (2, 0, 1) \pmod{(3, 6, 6)}$ so that in fact $(b, c, d) \equiv (2, 6, 1) \pmod{(48, 24, 12)}$. Thus using mod 73 we have $(c, d) \equiv (6, 1) \pmod{(9, 24)}$ so that in fact $(b, c, d) \equiv (2, 6, 1) \pmod{(48, 72, 24)}$. Therefore using mod 128 we conclude that $c = 6$, $7^b = 32 + 17^d$, $d > 1$. Using mod 289, $b \equiv 50 \pmod{272}$. Hence using mod 137, $d \equiv 55 \pmod{68}$, a contradiction.

THEOREM 17. *The only solutions of* $2^a + 7^b = 2^c + 19^d$ *in positive integers are* $(a, b, c, d) = (1, 2, 5, 1)$, $(4, 1, 2, 1)$ *or* $(3, 4, 11, 2)$.

Proof. Let (a, b, c, d) be another solution. Using mod 3, $a \equiv c \pmod 2$. We consider five cases.

Case 17.1. $a = 1$. Then $2 + 7^b = 2^c + 19^d$. Thus c is odd, so that using mod 8, $b - d$ is odd. Hence using mod 19 we have $(b, c) \equiv (2, 5)$ or $(0, 13) \pmod{(3, 18)}$. Therefore using mod 7, $(b, c, d) \equiv (2, 5, 1) \pmod{(3, 18, 6)}$ so that in fact $(b, c, d) \equiv (2, 5, 1) \pmod{(6, 18, 6)}$. Thus using mod 13 we get $(b, c, d) \equiv (2, 5, 1) \pmod{12}$, so that using mod 32, $d \equiv 1 \pmod 8$, Applying these results and using mod 73, $(b, d) \equiv (2, 1) \pmod{(24, 36)}$ so that in fact $(b, c, d) \equiv (2, 5, 1) \pmod{(24, 36, 72)}$. Hence using mod 193 we conclude that $(b, c, d) \equiv (2, 5, 1) \pmod{(24, 96, 192)}$. Therefore using mod 64 we have $c = 5$, $7^b = 30 + 19^d$, $b > 2$. We have a contradiction mod 343.

Case 17.2. $c = 1$. Then $2^a + 7^b = 2 + 19^d$. We have a contradiction mod 8.

Case 17.3. $a = 2$. Then $4 + 7^b = 2^c + 19^d$. Using mod 8, b is odd, d is odd, so that using mod 5, $b \equiv c \pmod 4$. Thus c is odd, a contradiction.

Case 17.4. $c = 2$. Then $2^a + 7^b = 4 + 19^d$. Using mod 8, b is odd, d is odd. Hence using mod 5, $(a, b) \equiv (0, 1) \pmod 4$, so that using mod 16, $d \equiv 1 \pmod 4$. Thus using mod 17 we get $(a, b, d) \equiv (4, 1, 1) \pmod{(8, 16, 8)}$. Therefore using mod 32 we conclude that $a = 4$, $12 + 7^b = 19^d$, $b > 1$. We have a contradiction mod 49.

Case 17.5. $a \geq 3$ and $c \geq 3$. Using mod 8, b is even, d is even. Thus using moduli 9, 7 successively we have $(a, b, c, d) \equiv (1, 1, 3, 0)$, $(3, 1, 5, 2)$, $(5, 1, 1, 4)$, $(0, 2, 2, 2)$, $(2, 2, 4, 4)$ or $(4, 2, 0, 0)$ $\pmod{(6, 3, 6, 6)}$. Hence using mod 19 we conclude that $(a, b, c, d) \equiv (11, 1, 13, 4)$, $(3, 1, 11, 2)$ or $(14, 2, 10, 4) \pmod{(18, 3, 18, 6)}$. We consider the following subcases: $b \equiv 2$ or $0 \pmod 4$.

Subcase 17.5.1. $b \equiv 2 \pmod 4$. Applying the above results and using mod 5, $(a, c) \equiv (0, 2) \pmod 4$. Thus we have $(a, b, c, d) \equiv (14, 2, 10, 4) \pmod{(18, 3, 18, 6)}$ so that in fact $(a, b, c, d) \equiv (32, 2, 10, 4) \pmod{(36, 12, 36, 6)}$. Hence using mod 27, $b \equiv 2$ (mod) 9). Therefore using mod 37 we conclude that $2^{32} + 7^2 \equiv 2^{10} + 19^d \pmod{37}$, $d \equiv 27 \pmod{36}$, a contradiction.

Subcase 17.5.2. $b \equiv 0 \pmod 4$. Using mod 5, $a \equiv c \pmod 4$. Applying the above results and using mod 27 we have $(a, b, c, d) \equiv (3, 4, 11, 2)$, $(11, 1, 13, 4)$ or $(14, 2, 10, 4) \pmod{(18, 9, 18, 6)}$. Combining these results and using mod 37 we get $(a, b, c, d) \equiv (3, 4, 11, 2)$ or $(32, 2, 28, 4) \pmod{(36, 9, 36, 36)}$ so that in fact $(a, b, c, d) \equiv (3, 4, 11, 2)$ or $(32, 20, 28, 4) \pmod{36}$. Hence using mod 73 we have $b \equiv 4 \pmod{24}$ so that in fact $(a, b, c, d) \equiv (3, 4, 11, 2) \pmod{(36, 72, 36, 36)}$. Therefore using mod 16 we conclude that $a = 3$, $8 + 7^b = 2^c + 19^d$, $b > 4$. Thus using mod 343, $c \equiv 11 \pmod{147}$, so that using mod 43, $d \equiv 2 \pmod{42}$. Applying the above results and using mod 197 we have $b \equiv 4 \pmod{98}$ so that in fact $(b, c, d) \equiv (4, 11, 2) \pmod{(1764, 1764, 252)}$. Hence using mod 883, $d \equiv 2 \pmod{882}$, so that using mod 16807, $c \equiv 5156 \pmod{7203}$. Combining these results and using mod 1373 we conclude that $(b, d) \equiv (4, 2) \pmod{(343, 686)}$. Therefore, $(b, c, d) \equiv (4, 5156, 2) \pmod{(2058, 7203, 2058)}$. Finally, we consider our equation modulo the prime 14407. Since the order of 2 is 2401 and the order of 19 is 14406, we have the following possibilities: $7^b \equiv 2^{354} + 19^{2+2058k} - 8 \equiv 19^{2+2058k} - 2013 \pmod{14407}$, $k = 0, 1, 2, 3, 4, 5, 6$. Hence we get $(b, c, d) \equiv (3844, 354, 2) \pmod{(4802, 2401, 14406)}$, a contradiction.

THEOREM 18. *The only solution to* $2^a + 11^b = 2^c + 17^d$ *in positive integers is* $(a, b, c, d) = (3, 1, 1, 1)$.

Proof. Let (a, b, c, d) be another solution. We consider seven cases.

Case 18.1. $a = 1$. Then $2 + 11^b = 2^c + 17^d$. Using mod 4, b is odd, so that using mod 3, c is odd, d is odd. Thus we have a contradiction mod 5.

Case 18.2. $c = 1$. Then $2^a + 11^b = 2 + 17^d$. Using mod 4, b is odd, so that using mod 3, $a \equiv d \pmod 2$. Hence using mod 5 we have $(a, d) \equiv (3, 1) \pmod 4$. Thus using mod 17, $(a, b) \equiv (3, 1)$ or $(7, 5) \pmod{(8, 16)}$, Combining these results and using mod 64 we conclude that $a = 3$, $(b, d) \equiv (1, 1) \pmod{(16, 4)}$. Therefore $6 + 11^b = 17^d$, $d > 1$. Using mod 289, $b \equiv 97 \pmod{272}$. Hence using mod 137, $d \equiv 2 \pmod{68}$, a contradiction.

Case 18.3. $a = 2$. Then $4 + 11^b = 2^c + 17^d$. From Case 18.2, $c > 2$. We have a contradiction mod 8.

Case 18.4. $c = 2$. Then $2^a + 11^b = 4 + 17^d$. By Case 18.1, $a > 2$. We have a contradiction mod 8.

Case 18.5. $a = 3$. Then $8 + 11^b = 2^c + 17^d$. From Cases 18.2 and 18.4, $c > 3$. Using mod 16, $b \equiv 2 \pmod 4$, so that using mod 3, $c - d$ is odd. Thus using mod 5 we have $(c, d) \equiv (0, 3)$ or $(3, 0) \pmod 4$. Applying these results and using mod 17 we conclude that $(b, c, d) \equiv (6, 4, 3) \pmod{(16, 8, 4)}$. Hence using mod 9, $b \equiv c \pmod 6$. Therefore using mod 7, $d \equiv 0 \pmod 6$, a contradiction.

Case 18.6. $c = 3$. Then $2^a + 11^b = 8 + 17^d$. By Cases 18.1 and 18.3, $a > 3$. Using mod 16, $b \equiv 2 \pmod 4$, so that using mod 3, d is even, a is odd. Hence using mod 5, $(a, d) \equiv (3, 0) \pmod 4$. Thus using mod 17 we have the following possibilities: $11^b \equiv 8 - 2^{3+4k} \pmod{17}$, $k = 0, 1$. Therefore $(a, b) \equiv (7, 8) \pmod{(8, 16)}$, a contradiction.

Case 18.7. $a \geq 4$ and $c \geq 4$. Using mod 16, $b \equiv 0 \pmod 4$.

Suppose that d is odd. Hence using mod 3, c is even, a is odd, so that using mod 5, $(a, c, d) \equiv (1, 0, 1)$ or $(3, 0, 3) \pmod 4$. Combining the above results and using mod 17 we conclude that $(a, b, c, d) \equiv (5, 0, 4, 1)$ or $(1. 8, 0, 1) \pmod{(8, 16, 8, 4)}$. Thus using mod 32 we have $c = 4$, $(a, b, d) \equiv (5, 0, 1) \pmod{(8, 16, 4)}$. Therefore using mod 64 we conclude that $a = 5$, $16 + 11^b = 17^d$. Using mod 11, $d \equiv 6 \pmod{10}$, a contradiction.

Hence d is even. Using mod 3, $a \equiv c \pmod 2$, so that using mod 5 we have $(a, c, d) \equiv (t, t, 0)$ or $(2. 0, 2) \pmod 4$. Combining the above results and using mod 17 we conclude that $(a, b, c, d) \equiv (5, 4, 1, 0), (1, 12, 5, 0), (3, 0, 7, 0)$ or $(7, 8, 3, 0) \pmod{(8, 16, 8, 4)}$. Thus using mod 64 we get $(a, b, c, d) \equiv (3, 0, 7, 0) \pmod{(8, 16, 8, 4)}$. Finally, using mod 257 we have the following possibilities: $17^d \equiv 2^{3+8n} + 11^{16m} - 2^{7+8k} \pmod{257}$, $n, k = 0, 1$, $m = 0, 1, 2, 3$. Hence $(a, b, c, d) \equiv (3, 0, 15, 7), (11, 48, 7, 7), (3, 16, 15, 23), (11, 32, 7, 23), (3, 16, 7, 31), (11, 0, 15, 31), (3, 32, 7, 15)$ or $(11, 48, 15, 15) \pmod{(16, 64, 16, 32)}$, each of these congruences is a contradiction.

THEOREM 19. *The only solutions of* $2^a + 11^b = 2^c + 19^d$ *in positive integers are* $(a, b, c, d) = (4, 1, 3, 1)$ *and* $(8, 2, 4, 2)$.

Proof. Let (a, b, c, d) be another solution. We consider five cases.

Case 19.1. $a = 1$. Then $2 + 11^b = 2^c + 19^d$. Using mod 3, b is even, c is odd, so that using mod 8, d is odd. Hence using mod 5, $c \equiv 2 \pmod 4$, a contradiction.

Case 19.2. $c = 1$. Then $2^a + 11^b = 2 + 19^d$. Using mod 5, $(a, d) \equiv (1, 0) \pmod{(4, 2)}$, so that using mod 3, b is even. Thus we have a contradiction mod 8.

Case 19.3. $a = 2$. Then $4 + 11^b = 2^c + 19^d$. From Case 19.2, $c > 2$. We have a contradiction mod 8.

Case 19.4. $c = 2$. Then $2^a + 11^b = 4 + 19^d$. By Case 19.1, $a > 2$. We have a contradiction mod 8.

Case 19.5. $a \geq 3$ and $c \geq 3$. Using mod 8, $b \equiv d \pmod 2$. We consider the following subcases: $d \equiv 0$ or $1 \pmod 2$.

We will show that $c > 3$. Assume the contrary. Then $2^a + 11^b$

$= 8 + 19^d$. Using mod 19, $(a, b) \equiv (6, 0)$, $(4, 1)$ or $(0, 2)$ (mod $(18, 3)$), so that using mod 5, $(a, d) \equiv (0, 1)$ (mod $(4, 2)$). Thus using moduli 9, 7 successively we conclude that $(a, b, d) \equiv (4, 1, 1)$ (mod 6). Hence using mod 13, $(a, b, d) \equiv (4, 1, 1)$ (mod 12). Therefore using mod 17 we have $(a, b, d) \equiv (4, 1, 1)$ or $(4, 13, 5)$ (mod $(8, 16, 8)$). Thus using mod 32 we conclude that $a=4$, $8 + 11^b = 19^d$, $b > 1$. Using mod 121, $d \equiv 61$ (mod 110). Hence using mod 23, $b \equiv 4$ (mod 22), a contradiction. Thus $c > 3$.

Subcase 19.5.1, $d \equiv 0$ (mod 2). Using mod 5, $a \equiv c$ (mod 4). Hence using moduli 9, 7 successively we have $(a, b, c, d) \equiv (t, 0, t, 0)$, $(2, 2, 4, 2)$, $(3, 2, 1, 0)$, $(3, 4, 5, 0)$ or $(4, 4, 2, 2)$ (mod 6). Therefore using mod 19 we get $(a, b, c, d) \equiv (1, 0, 13, 0)$, $(4, 0, 10, 0)$, $(9, 4, 17, 0)$, $(10, 4, 8, 2)$ or $(8, 2, 4, 2)$ (mod $(18, 6, 18, 6)$). Combining the above results and using mod 37 we conclude that $(a, b\ d, c)$ $\equiv (8, 2, 4, 2)$ (mod $(36, 6, 36, 36)$). Thus using mod 73, $b \equiv 2$ (mod 72). Combining these results and using mod 17 we have (a, b, c, d) $\equiv (0, 2, 4, 2)$ or $(4, 10, 0, 6)$ (mod $(8, 16, 8, 8)$) so that in fact $(a, b, c, d) \equiv (8, 2, 4, 2)$ or $(20, 26, 16, 14)$) (mod $(24, 48, 24, 24)$). Therefore using mod 97 we get $(a, b, c, d) \equiv (8, 2, 4, 2)$ or $(20, 26, 16, 14)$ (mod $(48, 48, 48, 32)$). Combining these results and using mod 193 we conclude that $(a, b, c, d) \equiv (8, 2, 4, 2)$ or $(56, 34, 52, 98)$ (mod $(96, 64, 96, 192)$). Hence using mod 32 we have $c = 4$, $(a, b, d) \equiv (8, 2, 2)$ (mod $(96, 192, 192)$), $2^a + 11^b = 16 + 19^d$. Thus using mod 257, $d \equiv 2$ (mod 256) so that in fact $d \equiv 2$ (mod 768). Therefore using mod 769 we conclude that $(a, b, d) \equiv (8, 2, 2)$ (mod $(384, 768, 768)$). Hence using mod 512 we get $a = 8$, $240 + 11^b = 19^d$, $b > 2$. Using mod 1331, $d \equiv 992$ (mod 1210). This yields a contradiction mod 3631.

Subcase 19.5.2. $d \equiv 1$ (mod 2). Thus b is odd, so that using mod 3, c is odd, a is even, Hence using mod 5, $(a, c) \equiv (0, 3)$ (mod 4). Applying these results and using moduli 9, 7 successively we have $(a, b, c, d) \equiv (0, 1, 1, 5)$, $(0, 5, 5, 3)$, $(2, 3, 1, 5)$, $(2, 5, 3, 1)$, $(4, 1, 3, 1)$ or $(4, 3, 5, 3)$ (mod 6). Combining the above results and using mod 13 we conclude that $(a, b, c, d) \equiv (4, 1, 3, 1)$ or $(8, 11, 3, 7)$ (mod 12). In each case we have a contradiction mod 16.

THEOREM 20. *The only solutions to* $2^a + 13^b = 2^c + 17^d$ *in positive integers are* $(a, b, c, d) = (3, 1, 2, 1)$ *and* $(7, 2, 3, 2)$.

Proof. Let (a, b, c, d) be another solution. Using mod 4, $a \geq 2$, $c \geq 2$. We consider five cases.

Case 20.1. $a = 2$. Then $4 + 13^b = 2^c + 17^d$. Using mod 8, b is odd. Further, using mod 3, c is even, d is even. Thus using mod 16, $b \equiv 1 \pmod 4$. This yields a contradiction mod 17.

Case 20.2. $c = 2$. Then $2^a + 13^b = 4 + 17^d$. Using mod 17, $(a, b) \equiv (3, 1) \pmod{(8, 4)}$, so that using mod 5, $d \equiv 1 \pmod 4$. Hence using mod 16 we conclude that $a = 3$, $4 + 13^b = 17^d$, $d > 1$. Using mod 289, $b \equiv 29 \pmod{68}$. Thus using mod 137, $(b, d) \equiv (29, 52) \pmod{(136, 68)}$, a contradiction.

Case 20.3. $a = 3$. Then $8 + 13^b = 2^c + 17^d$. From Case 20.2, $c > 3$. Using mod 16, $b \equiv 2 \pmod 4$. Hence we have a contradiction mod 17.

Case 20.4. $c = 3$. Then $2^a + 13^b = 8 + 17^d$. By Case 20.1, $a > 3$. Using mod 16, $b \equiv 2 \pmod 4$, so that using mod 17, $a \equiv 7 \pmod 8$. Thus using mod 5, $d \equiv 2 \pmod 4$. Combining these results and using mod 13 we have $(a, d) \equiv (7, 2) \pmod{(12, 6)}$, so that using mod 9, $b \equiv 2 \pmod 3$. Hence $(a, b, d) \equiv (7, 2, 2) \pmod{(24, 12, 12)}$, so that using mod 64, $b \equiv 2 \pmod{16}$. Therefore using mod 97 we get $(a, b, d) \equiv (7, 2, 2) \pmod{(48, 96, 96)}$. Thus using mod 193 we have $(a, b, d) \equiv (7, 2, 2) \pmod{(96, 64, 192)}$. Hence using mod 256 we conclude that $a = 7$, $120 + 13^b = 17^d$, $b > 2$. Using mod 2197, $d \equiv 860 \pmod{1014}$ so that in fact $d \equiv 522 \pmod{676}$. Thus we have a cotradiction mod 677.

Case 20.5. $a \geq 4$ and $c \geq 4$. Using mod 16, $b \equiv 0 \pmod 4$.

Suppose that d is odd. Hence using mod 3, a is odd, c is even, so that using mod 5, $(a, b, c, d) \equiv (1, 0, 0, 1)$ or $(3, 0, 0, 3) \pmod 4$. Thus using mod 17 we conclude that $(a, b, c, d) \equiv (5, 0, 4, 1) \pmod{(8, 4, 8, 4)}$. Therefore using mod 13 we get $(a, b, c, d) \equiv (9, 0, 0, 1)$, $(1, 0, 4, 3)$ or $(5, 0, 8, 5) \pmod{(12, 4, 12, 6)}$. In each case we have a contradiction mod 7.

Hence d is even, Thus using mod 3, $a \equiv c \pmod 2$, so that using mod 5, $(a, b, c, d) \equiv (t, 0, t, 0)$ or $(2, 0, 0, 2) \pmod 4$. Therefore using mod 17 we conclude that $(a, b, c, d) \equiv (3, 0, 7, 0)$

(mod (8, 4, 8, 4)). Thus using mod 13 we get $(a, c, d) \equiv (3, 11, 0)$ $(7, 3, 2)$ or $(11, 7, 4)$ (mod (12, 12, 6)). Hence using mod 7, $(a, b, c, d) \equiv (7, 0, 3, 2)$ (mod (12, 4, 12, 6)), so that using mod 9, $b \equiv 2 \pmod 3$. Therefore using moduli 27, 19 successively we conclude that $(a, b, c, d) \equiv (7, 2, 3, 2)$ (mod (18, 18, 18, 9)). Thus $(a, b, c, d) \equiv (7, 20, 3, 20) \pmod{36}$. This yields a contradiction mod 37.

THEOREM 21. *The only solutions of* $2^a + 13^b = 2^c + 19^d$ *in positive integers are* $(a, b, c, d) = (3, 1, 1, 1)$ *and* $(8, 2, 6, 2)$.

Proof. Let (a, b, c, d) be another solution. Using mod 3, $a \equiv c \pmod 2$. We consider five cases.

Case 21.1. $a = 1$. Then $2 + 13^b = 2^c + 19^d$. Thus c is odd, so that using mod 8, d is odd, b is even. Hence using mod 5, $(b, c) \equiv (2, 1) \pmod 4$, so that using mod 17 we have the following possibilities: $19^d \equiv 2 + 13^2 - 2^{1+4n} \pmod{17}$, $n = 0, 1$. Thus $(c, d) \equiv (1, 4) \pmod 8$, a contradiction.

Case 21.2. $c = 1$. Then $2^a + 13^b = 2 + 19^d$. Hence a is odd, so that using mod 8, b is odd. d is odd. Thus using mod 5, $(a, b) \equiv (3, 1) \pmod 4$, so that using mod 17 we have $(a, b, d) \equiv (3, 1, 1)$ (mod (8, 4, 8)). Therefore using mod 16 we conclude that $a = 3$, $6 + 13^b = 19^d$, $b > 1$. Thus we have a contradiction mod 169.

Case 21.3. $a = 2$. Then $4 + 13^b = 2^c + 19^d$. Hence c is even,so that using mod 8, b is odd, d is even. Thus using mod 5, $(b, c) \equiv (1, 0) \pmod 4$. Applying these results and using moduli 13, 7 successively we conclude that $(b, c, d) \equiv (1, 4, 0)$ (mod (4, 12, 12)). Therefore using moduli 27, 37 successively we have $(b, c, d) \equiv (1, 4, 0) \pmod{36}$. This yields a contradiction mod 19.

Case 21.4. $c = 2$. Then $2^a + 13^b = 4 + 19^d$. Hence a is even, so that using mod 8, b is odd, d is even. Thus using mod 5 we have the following possibilities: $2^a \equiv 4 + 1 - 13^{1+2k} \pmod 5$, $k = 0, 1$. Therefore $a \equiv 1$ or $3 \pmod 4$, each of these congruences is a contradiction.

Case 21.5. $a \geq 3$ and $c \geq 3$. Using mod 8, b is even, d is even. We consider the following subcases: $b \equiv 2$ or $0 \pmod 4$.

Subcase 21.5.1. $b \equiv 2 \pmod 4$. Applying the above results and using mod 5, $(a, c) \equiv (0, 2) \pmod 4$, so that using mod 16, $d \equiv 2 \pmod 4$. Hence using moduli 13, 7 successively we have $(a, c, d) \equiv (8, 6, 2)$ or $(4, 6, 10) \pmod{12}$. Thus using mod 9 we get $(a, b, c, d) \equiv (8, 2, 6, 2)$ or $(4, 1, 6, 10)$ (mod (12, 3, 12, 12)) so that in fact $(a, b, c, d) \equiv (8, 2, 6, 2)$ or $(4, 10, 6, 10) \pmod{12}$. Therefore using moduli 27, 19 successively we conclude that $(a, b, c, d) \equiv (8, 2, 6, 2)$ or $(10, 16, 6, 10)$ (mod (18, 18, 18, 12)). Combining these results and using mod 37 we have $(a, b, c, d) \equiv (8, 2, 6, 2) \pmod{36}$. Hence using mod 73, $b \equiv 2 \pmod{72}$, so that using mod 32, $d \equiv 2 \pmod 8$. Combining these results and using mod 17, $(a, c) \equiv (0, 6) \pmod 8$ so that in fact $(a, b, c, d) \equiv (8, 2, 6, 2) \pmod{24}$. Thus using mod 97 we have $(a, b, c, d) \equiv (8, 2, 6, 2)$ or $(32, 50, 30, 18)$ (mod (48, 96, 48, 32)). Therefore using mod 128 we conclude that $c = 6$, $(a, b, d) \equiv (8, 2, 2)$ (mod (48, 96, 32)), $2^a + 13^b = 64 + 19^d$. Applying the above results and using mod 193 we get $(a, b, d) \equiv (8, 2, 2)$ (mod (96, 64, 192)), so that using mod 257, $(b, d) \equiv (2, 2)$ (mod (128, 256)). Hence using mod 512 we conclude that $a = 8$, $b > 2$, $192 + 13^b = 19^d$. Using mod 2197, $d \equiv 74 \pmod{156}$. Thus we have a contradiction mod 157.

Subcase 21.5.2. $b \equiv 0 \pmod 4$. Hence using mod 5, $a \equiv c \pmod 4$.

We will first show that $a > 3$. Assume the contrary. Then $8 + 13^b = 2^c + 19^d$. Thus using mod 16, $d \equiv 2 \pmod 4$, so that using mod 17 we have the following possibilities: $19^d \equiv 8 + 13^0 - 2^{3+4n} \pmod{17}$, $n = 0, 1$. Therefore $(b, c, d) \equiv (0, 3, 0)$ (mod (4, 8, 8)), a contradiction. Hence $a > 3$.

We will next show that $c > 3$. Assume the contrary. Then $2^a + 13^b = 8 + 19^d$. Thus using mod 16, $d \equiv 2 \pmod 4$, so that using mod 17 we have the following possibilities: $19^d \equiv 2^{3+4n} + 13^0 - 8 \pmod{17}$, $n = 0, 1$. Hence $d \equiv 0$ or $1 \pmod 8$, each of these congruences is a contradiction. Therefore $c > 3$.

Thus using mod 16, $d \equiv 0 \pmod 4$, Combining the above results and using mod 13 we conclude that $(a, b, c, d) \equiv (3, 0, 11, 0)$, $(5, 0, 9, 0)$, $(1, 0, 5, 4)$, $(11, 0, 7, 4)$, $(7, 0, 3, 8)$ or $(9, 0, 1, 8)$ (mod (12, 4, 12, 12)). In each case we have a contradiction mod 7.

THEOREM 22. ***The only solutions to*** $1+3^a=2^b+2^c3^d$ ***in integers are given in Table*** 5. ***In this table,*** t ***denotes an arbitrary integer.***

Table 5.

a	b	c	d
0	-1	-1	1
1	1	1	0
2	3	1	0
2	1	3	0
t	0	0	t
2	2	1	1
4	6	1	2
3	4	2	1
3	2	3	1

Proof. Let (a, b, c, d) be another solution. Clearly $b \neq 0$, $a \geq 0$.

We will first show that $a>0$. Assume the contrary. Then $2=2^b+2^c3^d$. Clearly $b<-1$. We get $2^{1-b}=1+2^{c-b}3^d$, so that using mod 2, $c=b$. Thus $2^{1-b}=1+3^d$, so that we have a contradiction mod 8. Therefore $a>0$.

We will next show that $b>0$. Assume the contrary. Then $b<0, 2^{-b}+2^{-b}3^a=1+2^{c-b}3^d$. Hence using mod 2, $c=b$, so that using mod 8 we conclude that $b \geq -2$. Suppose $b=-2$. Then $4+4 \cdot 3^a=1+3^d$, $d>1$. This yields a contradiction mod 9. Thus $b=-1$, $2+2 \cdot 3^a=1+3^d$. Clearly $d>0$. We have a contradiction mod 3, Therefore $b>0$.

We will third show that $d>0$. Assume the contrary. Then $1+3^a=2^b+2^c$. By the symmetry we may assume $c \geq b$. By Theorem 4, $b>1$. Hence using mod 8 we conclude that $b=2$, $3^a=3+2^c$. This yields a contradiction mod 3.

Clearly $c>0$, $a>1$. Using mod 8, $b \leq 2$ or $c \leq 2$. Further, using mod 3, b is even, Therefore we consider three cases.

Case 22.1. $c=1$. Then $1+3^a=2^b+2 \cdot 3^d$. Suppose $b=2$. Then $3^a=3+2 \cdot 3^d$, $d>1$. We have a contradiction mod 9. Thus $b>2$, so that using mod 16, $(a, d) \equiv (0, 0)$ or $(0, 2)$ (mod 4). Hence using mod 5 we get $(a, b, d) \equiv (0, 2, 2) \pmod 4$. Thus using mod 9, $b \equiv 0 \pmod 6$. Combining these results and using mod 13 we have $(a, b, d) \equiv (1, 6, 2)$ (mod (3, 12, 3)) so that in fact $(a, b, d) \equiv (4, 6, 2) \pmod{12}$. Therefore using mod 73, $b \equiv 6 \pmod 9$ so that in fact $(a, b, d) \equiv (4, 6, 2)$ (mod (12, 36, 12)). Hence using mod 27 we conclude that $d = 2$, $3^a = 2^b + 17$, $b > 6$. Using mod 64, $a \equiv 4 \pmod{16}$, so that using mod 193, $b \equiv 6 \pmod{96}$. Therefore using mod 257, $a \equiv 4 \pmod{256}$. This yields a contradiction mod 128.

Case 22.2. $b = 2$. Then $1 + 3^a = 4 + 2^c 3^d$. Using mod 9 we conclude that $d = 1$, $3^a = 3 + 3 \cdot 2^c$, $a > 2$. Using mod 27, $c \equiv 9 \pmod{18}$. Thus we have a contradiction mod 19.

Case 22.3. $c = 2$. Then $1 + 3^a = 2^b + 4 \cdot 3^d$. By Case 22.2, $b > 2$. Using mod 16, $(a, d) \equiv (1, 0), (1, 2), (3, 1)$ or $(3, 3) \pmod 4$. Hence using mod 5 we have $(a, b, d) \equiv (3, 0, 1) \pmod 4$. Suppose $d > 1$. Using mod 9, $b \equiv 0 \pmod 6$ so that in fact $b \equiv 0 \pmod{12}$. Thus using mod 13 we have the following possibilities: $3^a \equiv 4 \cdot 3^k \pmod{13}$, $k = 0, 1, 2$. Each of these congruences is a contradiction. Therefore $d = 1$, $3^a = 2^b + 11$, $b > 4$. Applying the above results and using mod 17 we conclude that $(a, b) \equiv (3, 4)$ (mod (16, 8)). This yields a contradiction mod 32.

THEOREM 23. *The only solutions to* $3^a + 7^b = 1 + 2^c + 5^d$ *in nonnegative integers are given in Table* 6.

Table 6.

a	b	c	d
0	1	1	1
2	0	3	0
2	0	2	1
1	0	1	0
3	0	1	2
3	1	5	0
4	2	7	0
1	1	3	0
1	1	2	1
4	2	2	3
6	4	2	5
3	1	3	2
2	2	5	2

Proof. Let (a, b, c, d) be another solution. Clearly $c > 0$.

We will show that $a > 0$. Assume the contrary. Then $7^b = 2^c + 5^d$. Using mod 3, c is odd, d is odd, so that using mod 5, $(b, c) \equiv (1, 1)$ or $(3, 3) \pmod 4$. Thus using mod 4 we conclude that $c = 1$, $7^b = 2 + 5^d$, $d > 1$. We have a contradiction mod 25. Hence $a > 0$.

LEMMA 23.1. $b > 0$.

Proof. Assume the contrary. Then $3^a = 2^c + 5^d$. By Theorem 4, $c > 1$, $d > 0$. We consider three cases.

Case 23.1.1. $c = 2$. Then $3^a = 4 + 5^d$. Clearly $d > 1$. Using mod 25, $a \equiv 6 \pmod{20}$, so that we have a contradiction mod 11.

Case 23.1.2. $c = 3$. Then $3^a = 8 + 5^d$. Using mod 5, $a \equiv 1 \pmod 4$, so that we have a contradiction mod 16.

Case 23.1.3. $c \geq 4$. Using moduli 16, 5 successively we have $(a, c, d) \equiv (0, 0, 0)$ or $(2, 2, 2) \pmod 4$. In each case we have a contradiction mod 3.

LEMMA 23.2 $d > 0$.

Proof. Assume the contrary. Then $3^a + 7^b = 2 + 2^c$. Clearly $c > 5$. By Theorems 4 and 1, $b > 0$, $a > 1$. Using mod 3, c is odd. Further, using mod 7, $(a, c) \equiv (1, 0)$, $(4, 1)$ or $(3, 2)$ (mod $(6, 3)$). Combining these results and using mod 9 we have $(a, b, c) \equiv (1, 0, 3)$, $(4, 2, 1)$ or $(3, 1, 5)$ (mod $(6, 3, 6)$). Hence using mod 13 we get $(a, b, c) \equiv (4, 2, 7)$ or $(3, 1, 5)$ (mod $(6, 12, 12)$), so that using mod 5, $(a, b, c) \equiv (0, 2, 7)$ or $(3, 1, 5)$ (mod $(4, 12, 12)$). Thus using mod 73 we get $(a, b, c) \equiv (4, 2, 7)$ or $(3, 1, 5)$ (mod $(12, 24, 9)$). Combining these results and using mod 32, $(a, b, c) \equiv (3, 1, 5)$ or $(4, 2, 7)$ (mod $(8, 24, 36)$) so that in fact $(a, b, c) \equiv (3, 1, 5)$ or $(4, 2, 7)$ (mod $(24, 24, 36)$). Hence using mod 193 we conclude that $(a, b, c) \equiv (3, 1, 5)$ or $(4, 2, 7)$ (mod $(16, 24, 96)$). Therefore using mod 64 we have $(a, b, c) \equiv (4, 2, 7)$ (mod $(16, 24, 96)$) so that in fact $(a, b, c) \equiv (4, 2, 7)$ (mod $(48, 24, 96)$). Thus using mod 97, $b \equiv 2 \pmod{96}$ so that in fact $(a, b, c) \equiv (4, 2, 7)$ (mod $(48, 96, 96)$). Hence using mod 769 we get $(a, b, c) \equiv (4, 2, 7)$ (mod $(48, 256, 384)$), so that using mod 257, $a \equiv 4 \pmod{256}$.

Therefore using mod 256 we conclude that $c=7$. Thus $a=4$, $b=2$, a contradiction.

Proof of Theorem 23. By [2], $c>3$. Using mod 3, $c-d$ is odd. Thus using moduli 16, 5 successively we conclude that $(a, b, c, d) \equiv (2, 2, 1, 2)$ or $(3, 1, 3, 0) \pmod 4$.

Suppose that $(a, b, c, d) \equiv (3, 1, 3, 0) \pmod 4$. Hence using moduli 9, 7 successively we get $(a, b, c, d) \equiv (1, 0, 3, 0)$, $(5, 1, 1, 4)$, $(3, 1, 3, 2)$ or $(3, 1, 5, 0)$ $(\bmod\ (6, 3, 6, 6))$ so that in fact $(a, b, c, d) \equiv (7, 9, 3, 0)$, $(11, 1, 7, 4)$, $(3, 1, 3, 8)$ or $(3, 1, 11, 0) \pmod{12}$. In each case we have a contradiction mod 13.

Therefore $(a, b, c, d) \equiv (2, 2, 1, 2) \pmod 4$. Using moduli 32, 17 successively we have $(a, b, c, d) \equiv (2, 10, 1, 2)$ or $(2, 2, 5, 2)$ $(\bmod\ (16, 16, 8, 16))$. Thus using mod 64 we conclude that $c=5$, $(a, b, d) \equiv (2, 2, 2) \pmod{16}$, $3^a + 7^b = 33 + 5^d$. Hence using moduli 9, 7 successively we have $(a, b, d) \equiv (2, 2, 2)$ $(\bmod\ (6, 3, 6))$ so that in fact $(a, b, d) \equiv (2, 2, 2) \pmod{48}$. Thus using mod 19, $(a, d) \equiv (2, 2)$ $(\bmod\ (18, 9))$, so that using mod 37, $(a, b, d) \equiv (2, 2, 2)$ $(\bmod\ (18, 9, 36))$. Therefore $(a, b, d) \equiv (2, 2, 2) \pmod{36}$, so that using mod 27 we conclude that $a=2$, $7^b = 24 + 5^d$, $d>2$. Using mod 125, $b \equiv 6 \pmod{20}$. This yields a contradiction mod 11.

THEOREM 24. *The only solutions of* $5^a = 1 + 2^b + 2^c$ *in integers are* $(a, b, c) = (1, 1, 1)$, $(2, 3, 4)$ *and* $(2, 4, 3)$.

Proof. Let (a, b, c) be another solution. Clearly $a>0$, $b>0$, $c>0$. By the symmetry we may assume $c \geq b$. We consider four cases.

Case 24.1. $b=1$. Then $5^a = 3 + 2^c$. Clearly $c>1$. We have a contradiction mod 4.

Case 24.2. $b=2$. Then $5^a = 5 + 2^c$. We have a contradiction mod 5.

Case 24.3. $b=3$. Then $5^a = 9 + 2^c$. Clearly $a>2$. Using mod 125, $c \equiv 64 \pmod{100}$. Hence we have a contradiction mod 601.

Case 24.4. $b \geq 4$. Using mod 16, $a \equiv 0 \pmod 4$, so that using mod 3, $c-b$ is odd. Therefore using mod 13 we get $5^0 \equiv 1 + 2^b + 2^c \equiv 1 + 2^b(2^{c-b}+1) \pmod{13}$. Thus $c - b \equiv 6 \pmod{12}$, a contradiction.

THEOREM 25. *The only solutions to* $1+3^a=2^b5^c+2\cdot 3^d5^e$ *in nonnegative integers are given in Table 7.*

Table 7.

a	b	c	d	e
1	1	0	0	0
2	3	0	0	0
2	2	0	1	0
4	6	0	2	0
4	5	0	0	2
3	1	1	2	0
4	4	1	0	0
8	8	2	4	0
6	7	1	2	1

Proof. Let (a, b, c, d, e) be another solution. By Theorem 22, $c>0$ or $e>0$. Clearly $a>1$, $b>0$.

LEMMA 25.1. $c>0$.

Proof. Assume the contrary. Then $1+3^a=2^b+2\cdot 3^d5^e$. Suppose $d>0$. Using mod 3, b is even, so that using mod 5, $(a, b)\equiv(1, 2) \pmod 4$. This yields a contradiction mod 4. Thus $d=0$, $1+3^a=2^b+2\cdot 5^e$. We consider two cases.

Case 25.1.1. $b=1$. Then $3^a=1+2\cdot 5^e$. Using mod 5, $a\equiv 0 \pmod 4$. We have a contradiction mod 16.

Case 25.1.2. $b\geq 2$. Using mod 4, a is even, so that using mod 5, $(a, b)\equiv(0, 1) \pmod 4$. Hence using moduli 9, 7 successively we have $(a, b, e)\equiv(2, 3, 0)$ or $(4, 5, 2) \pmod 6$. Combining these results and using mod 13 we get $(a, b, e)\equiv(4, 5, 2) \pmod{(6, 12, 4)}$ so that in fact $(a, b, e)\equiv(4, 5, 2) \pmod{12}$. Thus using mod 73, $(b, e)\equiv(5, 2) \pmod{(9, 72)}$, so that using mod 17 we conclude that $(a, b, e)\equiv(4, 5, 2) \pmod{(16, 8, 16)}$. Therefore using mod 64 we have $b=5$, $3^a=31+2\cdot 5^e$, $e>2$. Using mod 125, $a\equiv 64 \pmod{100}$ so that in fact $a\equiv 64 \pmod{300}$. Hence we have a contradiction mod 601.

LEMMA 25.2. $e>0$.

Proof. Assume the contrary. Then $1+3^a=2^b5^c+2\cdot 3^d$. We consider two cases.

Case 25.2.1. $b=1$. Then $1+3^a=2\cdot 5^c+2\cdot 3^d$. Using mod 4, a is odd, so that using mod 5, $(a, d)\equiv(1, 3)$ or $(3, 2)$ (mod 4). Hence using mod 16 we have $(a, c, d)\equiv(3, 1, 2)$ (mod $(4, 2, 4)$), so that using mod 9, $c\equiv 1$ (mod 6). Thus using mod 7, $(a, c, d)\equiv(1, 1, 4)$ or $(3, 1, 2)$ (mod 6). Combining these results and using mod 13 we get $(a, c, d)\equiv(3, 1, 2)$ (mod $(6, 4, 6)$) so that in fact $(a, c, d)\equiv(3, 1, 2)$ (mod 12). Hence using mod 73, $c\equiv 1$ (mod 72). Therefore using mod 27 we conclude that $d=2$, $3^a=2\cdot 5^c+17$, $c>1$. Using mod 25, $a\equiv 19$ (mod 20). This yields a contradiction mod 11.

Case 25.2.2. $b\geq 2$. Using mod 4, a is even, so that using mod 5, $(a, d)\equiv(0, 0)$ (mod 4).

We will show that $d>0$. Assume the contrary. Then $3^a=2^b5^c+1$. Suppose $c>1$. Using mod 25, $a\equiv 0$ (mod 20), so that we have a contradiction mod 11. Hence $c=1$, $3^a=5\cdot 2^b+1$. Clearly $b>4$. Using mod 32, $a\equiv 0$ (mod 8), Thus using mod 17 we have the following possibilities: $5\cdot 2^b\equiv 3^{8n}-1$ (mod 17), $n=0, 1$. Each of these congruences is a contradiction. Therefore $d>0$.

Using mod 9, $b\equiv c$ (mod 6), so that using mod 7 we get $(a, b, c, d)\equiv(0, 0, 0, 4)$, $(2, 0, 0, 0)$, $(4, 0, 0, 2)$, $(2, 2, 2, 4)$ $(2, 3, 3, 2)$, $(4, 4, 4, 4)$, $(0, 5, 5, 2)$ or $(4, 1, 1, 0)$ (mod 6). Combining the above results and using mod 13 we have $(a, b, c, d)\equiv(4, 6, 0, 2)$, $(4, 0, 2, 2)$, $(2, 2, 0, 4)$ or $(2, 8, 2, 4)$ (mod $(6, 12, 4, 6)$) so that in fact $(a, b, c, d)\equiv(4, 6, 0, 8)$, $(4, 0, 6, 8)$, $(8, 2, 8, 4)$ or $(8, 8, 2, 4)$ (mod 12). Thus using mod 73 we get $(a, b, c, d)\equiv(4, 0, 18, 8)$, $(4, 3, 66, 8)$, $(4, 6, 42, 8)$, $(8, 2, 50, 4)$, $(8, 5, 26, 4)$ or $(8, 8, 2, 4)$ (mod $(12, 9, 72, 12)$). Applying these results and using mod 37 we have $(a, b, c, d)\equiv(8, 20, 50, 4)$, $(8, 32, 26, 4)$ or $(8, 8, 2, 4)$ (mod $(18, 36, 72, 18)$), so that using mod 19, $(a, b, c, d)\equiv(8, 8, 2, 4)$

(mod (18, 36, 72, 18)). Therefore $(a, b, c, d) \equiv (8, 8, 2, 4)$ (mod (36, 36, 72, 36)), so that using mod 32, $(a, d) \equiv (0, 0)$ (mod (8, 4)). Thus using mod 17 we get $(a, b, c, d) \equiv (8, 0, 2, 4)$, $(8, 4, 10, 4)$, $(8, 0, 10, 12)$ or $(8, 4, 2, 12)$ (mod (16, 8, 16, 16)) so that in fact $(a, b, c, d) \equiv (8, 8, 2, 4)$, $(8, 20, 26, 4)$, $(8, 8, 26, 28)$ or $(8, 20, 2, 28)$ (mod (48, 24, 48, 48)). Hence using mod 97 we have $(a, b, c, d) \equiv (8, 8, 2, 4)$, $(8, 32, 50, 4)$ $(8, 20, 74, 4)$ or $(8, 44, 26, 4)$ (mod (48, 48, 96, 48)). Thus using mod 193, $(a, b, c, d) \equiv (8, 8, 2, 4)$, $(8, 56, 98, 4)$, $(8, 20, 170, 4)$, $(8, 68, 74, 4)$, $(8, 44, 122, 4)$, $(8, 92, 26, 4)$, $(8, 32, 146, 4)$ or $(8, 80, 50, 4)$ (mod (48, 96, 192, 48)). Therefore using mod 769 we conclude that $(a, b, c, d) \equiv (8, 8, 2, 4)$ or $(8, 200, 66, 4)$ (mod (48, 384, 128, 48)). Combining the above results and using mod 128, $a \equiv 8$ (mod 32) so that in fact $(a, b, c, d) \equiv (8, 8, 2, 4)$ or $(8, 8, 578, 4)$ (mod (288, 288, 1152, 144)). Hence using mod 1153 we have $(a, b, c, d) \equiv (8, 8, 2, 4)$ (mod (576, 288, 1152, 576)), so that using mod 257, $(a, b, c, d) \equiv (8, 8, 2, 4)$ (mod (256, 16, 256, 256)). Therefore using mod 512 we conclude that $b = 8$, $1 + 3^a = 256 \cdot 5^c + 2 \cdot 3^d$. Using mod 81, $c \equiv 2$ (mod 54). Combining the above results and using mod 109 we have $(a, c, d) \equiv (8, 2, 4)$ (mod 27) so that in fact $(a, c, d) \equiv (8, 2, 4)$ (mod 108). Hence using mod 163 we get $(a, c, d) \equiv (8, 2, 4)$ (mod (162, 108, 162)). Thus using mod 3889 we have $(a, c, d) \equiv (8, 2, 4)$ (mod (81, 972, 81)). Therefore using mod 243 we conclude that $d = 4$, $3^a = 256 \cdot 5^c + 161$, $c > 2$. Using mod 125, $a \equiv 88$ (mod 100). This yields a contradiction mod 101.

Proof of Theorem 25. Using mod 5, $a \equiv 2$ (mod 4). We consider three cases.

Case 25.1. $d = 0$. Then $1 + 3^a = 2^b 5^c + 2 \cdot 5^e$. Using mod 8, $b \geq 3$. Suppose $b > 3$. Using mod 16, e is odd, so that we have a contradiction mod 3. Thus $b = 3$, $1 + 3^a = 8 \cdot 5^c + 2 \cdot 5^e$. Using moduli 9, 7 successively we get $(a, c, e) \equiv (2, 0, 0)$ or $(2, 4, 2)$ (mod 6). Hence using mod 13, $(a, c, e) \equiv (2, 0, 0)$ (mod (6, 4, 4)) so that in fact $(a, c, e) \equiv (2, 0, 0)$ or $(2, 4, 8)$ (mod 12). Thus using mod 25, $a \equiv 10$ (mod 20). Therefore using mod 601 we conclude that $(a, c, e) \equiv (2, 0, 0)$ or $(29, 4, 8) \pmod{(75, 12, 12)}$, Each of these congruences is a contradiction.

Case 25.2. $d=1$. Then $1+3^a = 2^b 5^c + 6 \cdot 5^e$. Using mod 8 we conclude that $b=2$, $1+3^a = 4 \cdot 5^c + 6 \cdot 5^e$. Using mod 16, $(a, e) \equiv (2, 0) \pmod{(4, 2)}$. Hence using moduli 9, 7 successively we have $(a, c, e) \equiv (2, 0, 0)$ or $(0, 0, 4) \pmod{6}$, so that using mod 25, $a \equiv 10 \pmod{20}$. Therefore using mod 13 we get $(a, c, e) \equiv (2, 0, 0)$ or $(0, 2, 0) \pmod{(6, 4, 4)}$ so that in fact $(a, c, e) \equiv (2, 0, 0)$ or $(6, 6, 4) \pmod{12}$. Thus using mod 601 we conclude that $(a, c, e) \equiv (2, 0, 0) \pmod{(75, 12, 12)}$, a contradiction.

Case 25.3. $d \geq 2$. Using mod 9, $b \equiv c \pmod{6}$. Further, using mod 4, $b \geq 2$. Clearly $a > 3$. We consider three subcases.

Subcase 25.3.1. $b=2$. Then $1+3^a = 4 \cdot 5^c + 2 \cdot 3^d 5^e$. Using mod 16, $(a, d, e) \equiv (2, 1, 0) \pmod{(4, 2, 2)}$. Applying the above results and using mod 7 we conclude that $(a, c, d, e) \equiv (4, 2, 1, 2)$, $(4, 2, 3, 4)$ or $(4, 2, 5, 0) \pmod{6}$. Combining these results and using mod 13 we have $(a, c, d, e) \equiv (4, 2, 5, 2) \pmod{(6, 4, 6, 4)}$ so that in fact $(a, c, d, e) \equiv (10, 2, 5, 6) \pmod{(12, 12, 6, 12)}$. Hence using mod 27, $c \equiv 14 \pmod{18}$ so that in fact $c \equiv 14 \pmod{36}$. Consideration of our equation mod 37 we have the following possibilities: $3^a \equiv 2 \cdot 3^{5+6n} 5^{6+12m} \pmod{37}$, $n, m = 0, 1, 2$. Each of these congruences is a contradiction.

Subcase 25.3.2. $b=3$. Then $1+3^a = 8 \cdot 5^c + 2 \cdot 3^d 5^e$. Using mod 16, $(a, d, e) \equiv (2, 0, 0) \pmod{(4, 2, 2)}$. Applying the above results and using mod 7 we conclude that $(a, c, d, e) \equiv (2, 3, 0, 4)$, $(2, 3, 2, 0)$ or $(2, 3, 4, 2) \pmod{6}$. Combining these results and using mod 13 we have $(a, c, d, e) \equiv (2, 3, 0, 2) \pmod{(6, 4, 6, 4)}$ so that in fact $(a, c, d, e) \equiv (2, 3, 0, 10) \pmod{(12, 12, 6, 12)}$. Hence using mod 27, $c \equiv 3 \pmod{18}$ so that in fact $c \equiv 3 \pmod{36}$. Consideration of our equation mod 37 we have the following possibilities: $3^a \equiv 2 \cdot 3^{6n} 5^{10+12m} \pmod{37}$, $n, m = 0, 1, 2$. Each of these congruences is a contradiction.

Subcase 25.3.3. $b \geq 4$. Using mod 16, $(a, d, e) \equiv (2, 0, 1) \pmod{(4, 2, 2)}$. Therefore applying the above results and using mod 7 we have the following table of possibilities mod 6:

Table C. (a, b, c, d, e) mod 6.

a	b	c	d	e
0	1	1	0	5
0	1	1	2	1
0	1	1	4	3
4	2	2	0	1
4	2	2	2	3
4	2	2	4	5
0	4	4	0	3
0	4	4	2	5
0	4	4	4	1
2	4	4	0	5
2	4	4	2	1
2	4	4	4	3
2	5	5	0	3
2	5	5	2	5
2	5	5	4	1

Thus using mod 13 we get $(a, b, c, d, e) \equiv (0, 7, 1, 2, 1)$, $(0, 1, 3, 2, 1)$, $(0, 4, 0, 2, 1)$, $(0, 10, 2, 2, 1)$, $(2, 5, 3, 2, 3)$ or $(2, 11, 1, 2, 3) \pmod{(6, 12, 4, 6, 4)}$ so that in fact $(a, b, c, d, e) \equiv (6, 7, 1, 2, 1)$, $(6, 1, 7, 2, 1)$, $(6, 4, 4, 2, 5)$, $(6, 10, 10, 2, 5)$, $(2, 5, 11, 2, 11)$ or $(2, 11, 5, 2, 11) \pmod{(12, 12, 12, 6, 12)}$.

Suppose that $(a, b, c, d, e) \equiv (2, 5, 11, 2, 11)$ or $(2, 11, 5, 2, 11) \pmod{(12, 12, 12, 6, 12)}$. Using mod 25, $a \equiv 10 \pmod{20}$ so that in fact $a \equiv 20 \pmod{30}$. Hence using mod 31 we have the following possibilities: $1 + 5 \equiv 2^b 5^2 + 2 \cdot 3^{2+6n} 5^2 \pmod{31}$, $n = 0, 1, 2, 3, 4$. Each of these congruences is a contradiction.

Suppose that $(a, b, c, d, e) \equiv (6, 4, 4, 2, 5)$ or $(6, 10, 10, 2, 5) \pmod{(12, 12, 12, 6, 12)}$. Using mod 25, $a \equiv 10 \pmod{20}$ so that in fact $a \equiv 0 \pmod{30}$. Hence using mod 31 we have the following possibilities: $2 \equiv 5 \cdot 2^b + 2 \cdot 3^{2+6n} 5^2 \pmod{31}$, $n=0, 1, 2, 3, 4$. Each of these congruences is a contradiction.

Therefore $(a, b, c, d, e) \equiv (6, 1, 7, 2, 1)$ or $(6, 7, 1, 2, 1) \pmod{(12, 12, 12, 6, 12)}$. Suppose $c > 1$ and $e > 1$. Using mod 25, $a \equiv 10 \pmod{20}$ so that in fact $a \equiv 0 \pmod{30}$. Hence using mod 31 we have $(b, d) \equiv (0, 2) \pmod{(5, 30)}$, so that using mod 11, $(a, b, c, d, e) \equiv (0, 5, 1, 2, 0)$ or $(0, 5, 3, 2, 3) \pmod{(30, 10, 5, 30, 5)}$.

Combining the above results and using mod 41 we have the following possibilities: $1 + 3^{2+4n} \equiv 2^5 5^{11} + 2 \cdot 3^d 5^5$, $2^5 5^3 + 2 \cdot 3^d 5^{13}$, $5 \cdot 2^{15} + 2 \cdot 3^d 5^5$ or $2^{15} 5^{13} + 2 \cdot 3^d 5^{13}$ (mod 41), $n = 0, 1$. Thus $(a, b, c, d, e) \equiv (2, 5, 11, 7, 5)$ or $(2, 15, 1, 7, 5)$ (mod (8, 20, 20, 8, 20)), each of these congruences is a contradiction. Hence $c = 1$ or $e = 1$.

We will show that $c > 1$. Assume the contrary, Then $1 + 3^a = 5 \cdot 2^b + 2 \cdot 3^d 5^e$. Suppose $e > 1$. Using mod 25, $(a, b) \equiv (2, 1)$ or $(18, 3)$ (mod (20, 4)) so that in fact $(a, b) \equiv (12, 1)$ or $(18, 3)$ (mod (30, 4)). Combining the above results and using mod 31 we have the following possibilities: $5 \cdot 2^b + 10 \cdot 3^{2+6n} \equiv 5$ or 9 (mod 31), $n = 0, 1, 2, 3, 4$. Each of these congruences is a contradiction. Hence $e = 1$, $1 + 3^a = 5 \cdot 2^b + 10 \cdot 3^d$. Suppose $d > 2$. Using mod 27, $b \equiv 13$ (mod 18). Thus using mod 19 we have the following possibilities: $3^a \equiv 5 \cdot 2^{13} + 10 \cdot 3^{2+6n} - 1$ (mod 19), $n = 0, 1, 2$. Hence $a \equiv 2$, 16 or 10 (mod 18), each of these congruences is a contradiction. Therefore $d = 2$, $3^a = 5 \cdot 2^b + 89$, $b > 7$. Using mod 128, $a \equiv 6$ (mod 32) so that in fact $a \equiv 6$ (mod 96), Thus using mod 97, $b \equiv 7$ (mod 48), so that using mod 257, $a \equiv 6$ (mod 256). This yields a contradiction mod 256. Hence $c > 1$.

Therefore $e = 1$, $1 + 3^a = 2^b 5^c + 10 \cdot 3^d$. Using mod 25, $(a, d) \equiv (2, 0)$ or $(18, 2)$ (mod (20, 4)) so that in fact $(a, d) \equiv (12, 0)$ or $(18, 2)$ (mod (30, 4)). Hence using mod 31 we have the following possibilities $5 \cdot 2^b + 10 \cdot 3^{2+6n} \equiv 5$ or 9 (mod 31), $n = 0, 1, 2, 3, 4$. Each of these congruences is a contradiction.

The exponential Diophantine equation $1 + 2^a = 3^b 5^c + 2^d 3^e 5^f$ is solved, with computer assistance, by L. J. Alex [1]. The similar exponential Diophantine equation $1 + 3^a = 2^b 5^c + 2^d 3^e 5^f$ is not easily solved by hand, Theorem 22 is the case $c = f = 0$, Theorem 25 is the case $d = 1$.

REFERENCES

1. L. J. Alex, *Simple groups and a Diophantine equation*, Pacific J. Math., **104** (1983), 257–262.

2. J. L. Brenner and L. L. Foster, *Exponential Diophantine equations*, Pacific J. Math., **101** (1982), 263–301.

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