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RESTRICTIONS OF FOURIER TRANSFORMS TO PLANE CURVES

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Abstract. A technique for obtaining necessary conditions on restricting the Fourier transform is introduced.

Let $I = [a, b]$ be a finite interval in $\boldsymbol{R}$ and let $\psi \in C^2(I)$ be real-valued and $\psi''(t) \geq 0$ for all t in I. Put $\gamma(t) = (t, \psi(t))$, $t \in I$. $\gamma(t)$ is a C^2 curve in $\boldsymbol{R}^2$. For a Lebesgue measurable function h on I, we shall also use $\|h\|_{L^p(\gamma)}$ to denote the L^p-norm of h. Let $\mathcal{S}$ be the set of all rapidly decreasing smooth functions on $\boldsymbol{R}^2$. For $g \in \mathcal{S}$, let Tg be the restriction to γ of the Fourier transform $\hat{g}$ of g. That is

$$Tg(t) = \hat{g}|_\gamma(t) = \hat{g}(\gamma(t)).$$

In [1], Per Sjölin proved, among many other things, that

THEOREM 1. *Assume* $1 \leq p < 4/3$ *and* $3(1-1/p) \leq 1/q \leq 1$. *Then*

$$\|(Tg)\psi''^{(1-1/p)}\|_{L^q(\gamma)} \leq C_p \|g\|_{L^p(\boldsymbol{R}^2)}, \qquad g \in \mathcal{S}. \tag{1}$$

The following estimate was then derived from Theorem 1 by applying Holder's inequality.

THEOREM 2. *Let* γ *be a* C^{n+1} *curve in* $\boldsymbol{R}^2$, *for some* $n \geq 3$, *which has non-vanishing curvature except at finitely many points. Assume*

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that the highest order of contact of the tangent at these points is $n-1$. *Then*

$$\|Tg\|_{L^q(\gamma)} \leq C_p \|g\|_{L^p(R^2)}, \qquad g \in \mathcal{S}, \tag{2}$$

if $1 \leq p,\ q \leq \infty$ *and* $(n+1)(1-1/p) < 1/q$. *The inequality need not hold if* $(n+1)(1-1/p) > 1/q$.

Theorem 2 does not contain the case $(n+1)(1-1/p) = 1/q$. This note is devoted to a more detailed study of this problem. It will be shown in Lemma 4 that the inequality (2) need not hold if $(n+1)(1-1/p) = 1/q$ and $1/q > 1/p$.

In the sequel, let $I = [0, \eta]$, $\eta > 0$. Assume that $\psi(t) = t^\theta(1+\phi(t))$, $\theta \geq 2$, $\phi \in C^2(I)$, $\phi(0) = 0$, and $\psi'' \geq 0$ on $(0, \eta]$. Let $\gamma(t) = (t, \psi(t))$. For any complex number $z = \alpha + i\beta$, $\alpha \geq 0$, and $g \in \mathcal{S}$, let

$$(T_z g)(t) = \hat{g}(\gamma(t))\, t^z.$$

The following is the main result of this note.

THEOREM 3. (i) *Assume that* $2 + 4\alpha \geq \theta$. *Then*

$$\|T_{\alpha+i\beta}\, g\|_{L^q(\gamma)} \leq C_p \|g\|_{L^p(R^2)}, \qquad g \in \mathcal{S}, \tag{3}$$

if $1 \leq p < 4/3$ *and* $3(1-1/p) \leq 1/q$. *The inequality* (3) *does not hold if* $p \geq 4/3$ *or* $3(1-1/p) > 1/q$.

(ii) *Assume that* $2 + 4\alpha < \theta$. *The inequality* (3) *holds if* $1 \leq p < 4/3$ *and* $\max\{3(1-1/p), (1+\theta)(1-1/p) - \alpha\} < 1/q$, *or if* $3(1-1/p) = 1/q$ *and* $1 \leq p \leq (\theta-2)/(\theta-2-\alpha)$. *The inequality* (3) *does not hold if* $p \geq 4/3$ *or* $\max\{3(1-1/p), (1+\theta)(1-1/p) - \alpha\} > 1/q$ *or* $(1+\theta)(1-1/p) - \alpha = 1/q > 1/p$.

Proof. We shall begin with the following lemma.

LEMMA 4. *The inequality* (3) *does not hold if*

$$(1+\theta)(1-1/p) - \alpha > 1/q$$

or

$$(1+\theta)(1-1/p) - \alpha = 1/q > 1/p.$$

Proof of lemma. Choose a large positive number M. For any positive integer k, set $\eta_k = M^{-k}$ and $\delta_k = M^{-k-1}$. Let Q_k be the

rectangle whose dimensions are δ_k, δ_k^θ along the tangent and normal directions of γ at η_k respectively. By choosing M very large, we may assume that the collection of rectangles $\{(1+1/M)Q_k : k=1,2,\cdots\}$ is pairwise disjoint and that there exists $c>0$ such that, for every $k\geq 1$, $\gamma(t)$ lies in Q_k if $|t-\eta_k|\leq c\delta_k$. Choose a smooth function ϕ such that $\phi(x, y)=1$ if $(x, y)\in Q$, the unit cube centered at the origin, and $\phi(x, y)=0$ if $(x, y)\notin(1+1/M)Q$. Let $h_k(x, y) = \phi(x/\delta_k, y/\delta_k^\theta)$. After performing a suitable rotation and a translation to h_k, we may assume that $h_k=1$ on Q_k and $h_k=0$ outside of $(1+1/M)\,Q_k$. Let g_k be the smooth function such that $\hat{g}_k=h_k$. Then

$$\|g_k\|_{L^p(\boldsymbol{R}^2)} = \delta_k^{(1+\theta)(1-1/p)}\|\hat{\phi}\|_{L^p(\boldsymbol{R}^2)}.$$

Put $v_k=M^{k(\alpha+1/q)}$. Let N be an arbitrary positive integer. We choose points $w_1,\cdots, w_N$ in $\boldsymbol{R}^2$ such that

$$\left\|\sum_{j=1}^N v_j\,\tau_{w_j}\,g_j\right\|^p_{L^p(\boldsymbol{R}^2)}\leq 2\sum_{j=1}^N \|v_j\,\tau_{w_j}\,g_j\|^p_{L^p(\boldsymbol{R}^2)},$$

where τ_w is the translation by w. Note that

$$\left|\left(\sum_{j=1}^N v_j\,\tau_{w_j}\,g_j\right)\hat{}\,(\gamma(t))\right| = v_k,$$

if $|t-\eta_k|\leq c\delta_k$, $k=1, 2,\cdots, N$. Thus

$$\left\|T_{\alpha+i\beta}\left(\sum_{j=1}^N v_j\,\tau_{w_j}\,g_j\right)\right\|^q_{L^q(\gamma)}\geq\sum_{j=1}^N\int_{\eta_j-c\delta_j}^{\eta_j+c\delta_j} v_j^q\,t^{\alpha q}\,dt\geq AN,$$

where A is a constant independent of N. If the inequality (3) is true, then we would have for each positive integer N,

$$C\sum_{j=1}^N M^{j(\alpha+1/q)p}\cdot\delta_j^{(1+\theta)(p-1)}\geq N^{p/q},$$

for some constant C independent of N. Therefore, since $\delta_j=M^{-j-1}$,

$$C_1\sum_{j=1}^N M^{jp(1/q+\alpha-(1+\theta)(1-1/p))}\geq N^{p/q},$$

for all positive integer N. This cannot hold if

$$(1+\theta)(1-1/p)-\alpha>1/q$$

or

$$(1+\theta)(1-1/p)-\alpha=1/q \quad\text{and}\quad p>q.$$

This completes the proof of Lemma 4.

REMARK. This technique applies to similar situations. It, with some modification, is also useful if the curvature is everywhere nonvanishing (see [2]).

Let us continue the proof of of Theorem 3. Note that on the subinterval $[\eta/2, \eta]$, γ has non-vanishing curvature, and $|(T_{\alpha+i\beta}g)(t)| \geq |\eta/2|^{\alpha}|\hat{g}(\gamma(t))|$. By the well known result, the inequality (3) can not hold if $p \geq 4/3$ or $3(1-1/p) > 1/q$. Now assume $1 \leq p < 4/3$, $3(1-1/p) \leq 1/q$ and $2+4\alpha \geq \theta$. Then

$$-\alpha + (\theta-2)(1-1/p) \leq -\alpha + (\theta-2)/4 \leq 0.$$

By Theorem 1, for $g \in \mathcal{S}$,

$$\begin{aligned} C\|g\|_{L^p(R^2)} &\geq \|(Tg)(\psi'')^{1-1/p}\|_{L^p(\gamma)} \\ &\geq C_1\|(T_{\alpha+i\beta}\, g)\cdot t^{-\alpha}\cdot t^{(\theta-2)(1-1/p)}\|_{L^p(\gamma)} \\ &\geq C_2\|T_{\alpha+i\beta}\, g\|_{L^q(\gamma)}. \end{aligned}$$

This finishes the proof of (i). Next assume $2+4\alpha < \theta$. In the $(1/p, 1/q)$ plane, the lines $3(1-1/p) = 1/q$ and $(1+\theta)(1-1/p) - \alpha = 1/q$ intersect at $(1/p_0, 1/q_0)$, where $1/p_0 = (\theta-2-\alpha)/(\theta-2)$ and $1/q_0 = 3\alpha/(\theta-2)$. Since $1 \leq p_0 < 4/3$, Theorem 1 implies, for all $g \in \mathcal{S}$,

$$\begin{aligned} C\|g\|_{L^{p_0}(R^2)} &\geq \|(Tg)(\psi'')^{1-1/p_0}\|_{L^{q_0}(\gamma)} \\ &\geq C_1\|(T_{\alpha+i\beta}\, g)\cdot t^{-\alpha}\cdot t^{(\theta-2)(1-1/p_0)}\|_{L^{q_0}(\gamma)} \\ &\geq C_1\|T_{\alpha+i\beta}\, g\|_{L^{q_0}(\gamma)}. \end{aligned}$$

By interpolation, the inequality (3) holds if $3(1-1/p) \leq 1/q$ and $1 \leq p \leq (\theta-2)/(\theta-2-\alpha)$. Next assume $p_0 < p < 4/3$. Then $3(1-1/p) < (1+\theta)(1-1/p) - \alpha$. Assume also $(1+\theta)(1-1/p) - \alpha < 1/q$. Define q_1 by $3(1-1/p) = 1/q_1$. Then $q_1 > q$ and

$$\begin{aligned} &[(\theta-2)(1-1/p) - \alpha]q\cdot q_1/(q_1-q) \\ &\quad = [(1+\theta)(1-1/p) - \alpha - 1/q_1]/(1/q - 1/q_1) \\ &\quad < 1. \end{aligned}$$

Therefore

$$\|t^{[\alpha-(\theta-2)(1-1/p)]q}\|_{L^{q_1/(q_1-q)}(\gamma)} < \infty.$$

Now

$$\begin{aligned}|T_{\alpha+i\beta}\, g(t)|^q &= |\, Tg(t) \cdot t^{\alpha}|^q \\ &\leq C|\, Tg(t) \cdot (\psi'')^{1-1/p}|^q \cdot t^{[\alpha-(\theta-2)(1-1/p)]q}.\end{aligned}$$

By Holder's inequality and Theorem 1,

$$\begin{aligned}\|T_{\alpha+i\beta}\, g\|^q_{L^q(\gamma)} &\leq C\|[(Tg)(\psi'')^{1-1/p}]^q\|_{L^{q_1/q}(\gamma)} \\ &\leq C\|Tg(\psi'')^{1-1/p}\|^q_{L^{q_1}(\gamma)} \\ &\leq C_1\|g\|^q_{L^p(\gamma)}.\end{aligned}$$

This completes the proof of Theorem 3.

Remark. When $2 + 4\alpha < \theta$, the problem remains unsettled if $(1 + \theta)(1 - 1/p) - \alpha = 1/q \leq 1/p < (\theta - 2 - \alpha)/(\theta - 2)$.

References

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