

A MEASURE-THEORETICAL MAX-FLOW PROBLEM

Part I: General Theory; Sup-inf Problems, Cuts

BY

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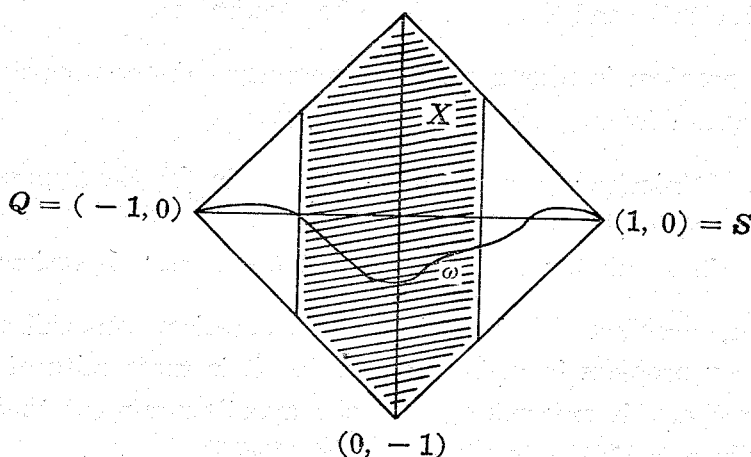
0. Introduction. Let

$$X = \{(x, y) \mid x, y \in \mathbf{R}, -\frac{1}{2} \leq x \leq \frac{1}{2}, |x| + |y| \leq 1\}$$

and

$$\mathcal{Q} = \{\omega \mid \omega \in [-1, 1]^{[-1, 1]}, |\omega(s) - \omega(t)| \leq |s - t| (|s|, |t| \leq 1)\}.$$

(0, 1)



Both X and \mathcal{Q} are compact metric spaces—with uniform approximation in \mathcal{Q} —and thus bear natural Baire-Borel structures. All measures on these structures will be tacitly understood as non-negative measures with finite total mass. Every $\omega \in \mathcal{Q}$ has a rectifiable plane curve as its graph. Let $P(\omega, \cdot)$ denote the arc

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* Footnote: Part of the work in this paper was done while the first author was visiting the Mathematical Institute of the Academia Sinica at Nankang, Taiwan in Spring 1979.

length measure living on the intersection of X and that graph. P is thus a kernel form \mathcal{Q} to X . It transports measures μ from \mathcal{Q} to X , and bounded measurable functions from X to \mathcal{Q} in the usual fashion. Let $c \geq 0$ be any measure on X . The main problem with which we will deal in this paper is the following

Among all measures μ in \mathcal{Q} fulfilling

$$(*) \quad \mu P \leq c$$

find one with maximal total mass $\mu 1$.

To this problem, the following interpretation may be given. Let $Q = (-1, 0)$, $S = (1, 0)$; animals are wandering from Q to S along paths $\omega \in \mathcal{Q}$; a measure μ in \mathcal{Q} is a flow of such animals; the animal which takes path ω feeds along the graph of ω according to arc length; the overall result of this feeding is the measure μP in X ; c represents the distribution of food available in X ; the condition $\mu P \leq c$ is to be obeyed; what is the maximal possible flow under this condition? $\mu 1$ is to be maximized.

Our problem is thus a measure-theoretical linear program. The dual program is:

$$(**) \quad \begin{array}{l} \text{Minimize, for a given measure } c \text{ in } X, \text{ the expression } cf \\ \text{while obeying the condition } Pf \geq 1, \text{ with } f \geq 0 \text{ varying} \\ \text{in a suitable class of bounded measurable functions on } X. \end{array}$$

Moreover, problem $(*)$ is a measure-theoretical generalization of a max-flow problem in a finite network. It is most natural to ask whether there is any analogon to the max-flow-min-cut theorem of Ford-Fulkerson [1962] (see also Jacobs [1969]).

This paper is subdivided into three sections. In §1 we list continuity and compactness properties of the framework for problem $(*)$.—In §2 we investigate the question to which extent classical optimality theorems (Holmes [1972], Rockafellar [1966], [1970], [1966a], [1968], Ky Fan [1953], [1964], [1972], Sion [1958], Nikaidô [1953], [1954], Moreau [1964], König [1968]) can be applied to our problem $(*)$; the answer is generally negative, because our problem $(*)$ doesn't have sufficient continuity and compactness properties;

a lot of examples are demonstrative for this.—In §3 we try to imitate the concept of a cut, as familiar in finite network theory (Ford-Fulkerson [1962], Jacobs [1969]), in our present context. Our imitations are not overall successful but help to determine solutions of (*) in various special cases.

We plan to continue this paper with another part on approximation. The present paper is based to a large extent on Seiffert [1981]. Partial results have been announced in Jacobs [1979].

1. Continuity and compactness. Since \mathcal{Q} and X are compact metric spaces, every norm bounded set of measures in one of these spaces is weakly conditionally compact, even sequentially.

THEOREM 1.1. *For every continuous $f \geq 0$ on X , the function Pf is lower semicontinuous on \mathcal{Q} .*

Proof. If $\omega_n \rightarrow \omega$ in \mathcal{Q} , then for any interval $[a, b] \subseteq [-\frac{1}{2}, \frac{1}{2}]$, the arc length of ω_n over $[a, b]$ is $> (\text{arc length of } \omega \text{ over } [a, b]) - \varepsilon$ if n is sufficiently large. Approximating $(Pf)(\omega) = \int_X P(\omega, dx) f(x)$ and $(Pf)(\omega_n)$ by Riemann sums, the desired result follows easily.

As a matter of fact, the only continuity points of $P1$ are those ω for which $(P1)(\omega) = \sqrt{2}$, i.e. ω , as a function of $t \in [-\frac{1}{2}, \frac{1}{2}]$, has slope ± 1 almost everywhere. We leave the proof as an exercise to the reader.

THEOREM 1.2. *If μ_n, μ are measures on \mathcal{Q} , and c_n, c are measures on X such that*

$$\mu_n \longrightarrow \mu \quad (\text{weakly})$$

$$c_n \longrightarrow c \quad (\text{weakly})$$

$$\mu_n P \leq c_n \quad (n = 1, 2, \dots)$$

then

$$\mu P \leq c.$$

Proof. Let $0 \leq f \in C(X, \mathbf{R})$. We have to show $Pf \leq cf$. Choose any $\varepsilon > 0$ and find a continuous $0 \leq \varphi \in C(\mathcal{Q}, \mathbf{R})$ such that

$\varphi \leq Pf$, $\mu\varphi \geq \mu Pf - \varepsilon$. Choose n such that $\mu_n \varphi \geq \mu\varphi - \varepsilon$. Then

$$\begin{aligned}\mu Pf &\leq \mu\varphi + \varepsilon \leq \mu_n \varphi + 2\varepsilon \leq \mu_n Pf + 2\varepsilon \\ &\leq c_n f + 2\varepsilon < cf + 3\varepsilon,\end{aligned}$$

if we choose, in addition, n large enough to ensure $c_n f < cf + \varepsilon$. As $\varepsilon > 0$ was arbitrary, the desired inequality $Pf \leq cf$ follows.

COROLLARY 1.3. *For every measure c on X , the set*

$$\{\mu \mid \mu \text{ is a measure on } \Omega \text{ and } \mu P \leq c\}$$

is convex and weakly compact.

COROLLARY 1.4. *For every measure c on X , problem (*) has at least one solution.*

The mapping $\mu \rightarrow \mu P$ is not weakly continuous, as is shown by

EXAMPLE 1.5. Let $\omega \equiv 0$ and $\omega_n \rightarrow \omega$ uniformly, $P(\omega_n, 1) = \sqrt{2}$. Let generally ε_η denote point mass 1 at $\eta \in \Omega$. Then

$$\varepsilon_{\omega_n} \rightarrow \varepsilon_\omega \quad (\text{weakly})$$

but

$$\lim_n \varepsilon_{\omega_n} P1 = \lim_n P(\omega_n, X) = \sqrt{2} > 1 = P(\omega, X) = \varepsilon_\omega P1,$$

hence $\varepsilon_{\omega_n} P \rightarrow \varepsilon_\omega P$ is false. But we have the

THEOREM 1.6. *Let $0 \leq f \in C(X, \mathbf{R})$ and μ_n, μ measures on Ω such that $\mu_n \rightarrow \mu$ (weakly). Then*

$$\liminf_{n \rightarrow \infty} \mu_n Pf \geq \mu Pf.$$

Proof. Pf is lower semicontinuous. For any $\varepsilon > 0$ find $0 \leq \varphi \in C(\Omega, \mathbf{R})$ such that $\varphi \leq Pf$, $\mu\varphi \geq \mu Pf - \varepsilon$. Then for n sufficiently large we have

$$\mu Pf \leq \mu\varphi + \varepsilon < \mu_n \varphi + 2\varepsilon \leq \mu_n Pf + 2\varepsilon$$

which proves the desired result.

THEOREM 1.7. *For any measure c on X let*

$$m(c) = \max \{\mu 1 \mid \mu P \leq c\}.$$

Then the mapping $c \rightarrow m(c)$ is upper semicontinuous, i. e. $c_n \rightarrow c$ (weakly) implies

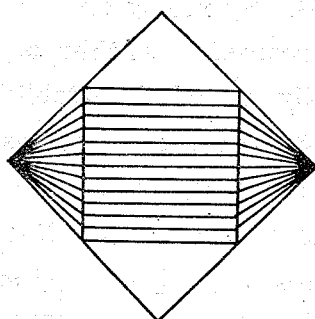
$$\limsup_{n \rightarrow \infty} m(c_n) \leq m(c).$$

Proof. Let $\mu_n P \leq c_n$, $\mu P \leq c$, $\mu_n 1 = m(c_n)$, $\mu 1 = m(c)$. We may assume $\mu_n \rightarrow \mu_0$ (weakly), for some μ_0 . By Theorem 1.2 we have $\mu_0 P \leq c$, hence $\mu_0 1 \leq m(c)$. Clearly $\lim_n m(c_n) = \lim_n \mu_n 1 = \mu_0 1 \leq m(c)$.

We now turn to questions of bijectivity. The mapping $\mu \rightarrow \mu P$ is not one-to-one, as is shown by

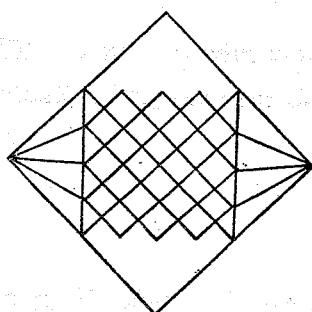
EXAMPLE 1.8. Equidistribute

1) mass 1 over all paths



(continuously)

2) mass $1/\sqrt{2}$ over all paths



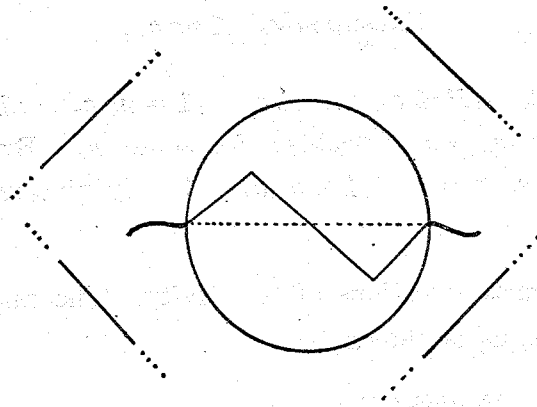
(continuously).

This yields two different measures μ, ν in \mathcal{Q} such that $\mu P = \nu P = 2$ -dimensional Lebesgue measure in $X \cap \{R \times H[-\frac{1}{2}, \frac{1}{2}]\}$.

But we have

THEOREM 1.9. The mapping $P : C(X, R) \rightarrow R_+^{\mathcal{Q}}$ is one-to-one.

Proof. Let $f \in C(X, \mathbf{R})$ be constant in the disc $D \subseteq X$. Parametrize the paths of shape



by their arc length σ inside $D: \sigma \rightarrow \omega^\sigma$. Then $(d/d\sigma)P(\omega^\sigma, f) =$ the constant value of f in D . By nearly obvious approximation arguments we see how we can recover a continuous f on X from the function Pf on \mathcal{Q} . Finally we shall establish a necessary condition for a measure in X to be representable as μP for some measure in P . In order to set the stage, let $\pi: \mathbf{R}^2 \rightarrow \mathbf{R}$ denote the first-component projection: $\pi(x, y) = x$ ($x, y \in \mathbf{R}$); moreover let M denote the set of all measures ρ in $[-\frac{1}{2}, \frac{1}{2}]$ such that $\rho \ll \lambda =$ linear Lebesgue measure, with a density σ fulfilling

$$\text{ess sup } \sigma \leq \sqrt{2} \text{ ess inf } \sigma.$$

THEOREM 1.10. 1) Let μ be a measure on \mathcal{Q} . Then $\pi(\mu P) \in M$.

2) Let $\rho \in M$. Then there is a nonnegative finite measure on μ such that

- a) μ lives on a single $\omega \in \mathcal{Q}$.
- b) $\pi(\mu P) = \rho$.

Proof. 1) The statement surely holds if μ sits on a single $\omega \in \mathcal{Q}$. If μ has mass one, then the statement holds again since averaging (the measurability problems involved here are easy to solve) lifts ess inf and lowers ess sup at most. The general case follows from this by multiplication by a constant.

2) Let $\rho \in M$. Put

$$\omega(t) = \begin{cases} 0 & \text{for } -1 \leq t \leq -\frac{1}{2} \\ \frac{1}{2\rho([- \frac{1}{2}, \frac{1}{2}])} \rho\left([- \frac{1}{2}, t]\right) & \text{for } -\frac{1}{2} \leq t \leq \frac{1}{2} \\ \frac{1}{2} - \left(t - \frac{1}{2}\right) & \text{for } \frac{1}{2} \leq t \leq 1. \end{cases}$$

Then $\mu = (1/2\rho([- \frac{1}{2}, \frac{1}{2}])) \varepsilon_\omega$ fulfils $\rho = \pi(\mu P)$.

2. **The non-applicability of some standard theorems from optimization theory.** In dealing with problems (*), (**) we easily obtain the inequality

$$\sup_{\substack{\mu \geq 0 \\ \mu P \leq c}} \mu 1 \leq \inf_{\substack{f \geq 0 \\ Pf \geq 1}} cf$$

from $\mu 1 \leq \mu Pf \leq cf(\mu, f \geq 0, \mu P \leq c, Pf \geq 1)$. It would be desirable to obtain equality here.

In this section we investigate the possibility of applying standard theorems from linear and convex optimization theory to that effect. The overall result is negative, and the reason lies, roughly speaking, in the fact that our problem has semicontinuity only where continuity is required.

1. *The Duality Theorem.*

Following Holmes [1972], we consider a Banach space E and its dual space E^* . For any convex function $F : E \rightarrow \mathbf{R} \cup \{\infty\}$ let

$$\text{dom } F = \{x \in E \mid F(x) < \infty\}$$

and let the *convex conjugate* $F^* : E^* \rightarrow \mathbf{R} \cup \{\infty\}$ be defined by

$$F^*(x^*) = \sup \{\langle x, x^* \rangle - F(x) \mid x \in \text{dom } F\}$$

where $\langle x, x^* \rangle$ denotes the value of $x^* \in E^*$ at $x \in E$. For concave functions $G : E \rightarrow \{-\infty\} \cup \mathbf{R}$ (i.e. functions G such that $-G$ is convex) the *concave conjugate* $G^+ : E^* \rightarrow \{-\infty\} \cup \mathbf{R}$ is defined by

$$G^+(x^*) = \inf \{\langle x, x^* \rangle - G(x) \mid x \in \text{dom } G\}.$$

Then we have the

THEOREM 2.1. (Duality Theorem of Fenchel-Rockafellar). *Let E, E^*, F, G, F^*, G^* be as above and assume that one of the functions*

F, G is continuous at some $x \in (\text{dom } F) \cap (\text{dom } G)$. Then

$$(1) \quad \inf_{x \in R} (F - G)(x) = \sup_{x^* \in R^*} (G^+ - F^*)(x^*).$$

For a proof see Holmes [1972].

Let us now specialize to our problems (*), (**). Thus let

$$E = C(X, \mathbf{R}).$$

Then E^* is the set of all finite signed measures on X . Let us denote the elements of E by f rather than by x . For any measure c on X define

$$F(f) = \begin{cases} cf & \text{if } f \geq 0 \\ +\infty & \text{else.} \end{cases}$$

Then F is convex, and $\text{dom } F = C_+(X, \mathbf{R})$. Define

$$G(f) = \begin{cases} 0 & \text{if } Pf \geq 1 \\ -\infty & \text{else.} \end{cases}$$

Then G is concave and $\text{dom } G = \{f \mid f \in C(X, \mathbf{R}), Pf \geq 1\}$. We now find

$$\inf_{f \in C(X, \mathbf{R})} (F - G)(f) = \inf_{\substack{f \geq 0 \\ Pf \geq 1}} cf,$$

which means precisely our problem (**). Now what does the right member of (1) look like in our special case? We find (exercise!)

$$\max_{m \in C(X, \mathbf{R})} (G^+ - F^*)(m) = \max_{\substack{m \in \text{dom } G^+ \\ m \leq c}} \left[\inf_{Pf \geq 1} mf \right]$$

$$(\text{dom } F) \cap (\text{dom } G) = \{f \mid 0 \leq f \in C(X, \mathbf{R}), Pf \geq 1\}$$

and G is continuous e.g. at $f \equiv 2$. Thus Theorem 2.1 applies and yields

$$\inf_{\substack{f \geq 0 \\ Pf \geq 1}} cf = \max_{\substack{m \in \text{dom } G^+ \\ m \leq c}} [\inf mf].$$

The question is e.g. what $\text{dom } G^+$ looks like. It is a set of measures containing all measures of the form μP ; the question whether $\text{dom } G^+$ equals the set of all μP is open; a modification of our procedure which makes the right member of (1) correspond to our original problem (*) leads into still greater difficulties (see Seiffert [1981] for details).

2. The Minimax Theorem.

For (*) and (**) the following inequalities are easy to prove

$$\max_{\substack{\mu \geq 0 \\ \mu P \leq c}} \mu 1 \leq \sup_{\substack{\mu \geq 0 \\ \mu P \leq c}} \inf_{\substack{f \geq 0 \\ Pf \geq 1}} \mu Pf \leq \inf_{\substack{f \geq 0 \\ Pf \geq 1}} \sup_{\substack{\mu \geq 0 \\ \mu P \leq c}} \mu Pf \leq \inf_{\substack{f \geq 0 \\ Pf \geq 1}} cf.$$

The question is whether a known minimax theorem may be applied to μPf as a function of μ and f in order to obtain equality instead of the middle inequality. In Seiffert [1981] the minimax theorems from the following papers are checked for this: Nikaidô [1953], [1954], Ky Fan [1953], [1964], [1972], Sion [1958], Moreau [1964], König [1968]. All these theorems don't apply in our situation because we have semicontinuity only where the theorems require continuity, and semicontinuity in the false direction (not corrigible by changing signs) where semicontinuity is required.

3. Lagrange Measures.

In Rockafellar [1968] the well-known Kuhn-Tucker theory of Lagrange parameters is carried over to an infinite dimensional situation. As is shown in Seiffert [1981], Rockafellar's theorem doesn't apply in the situation of (*) and (**) since Pf is semicontinuous and not continuous for continuous f on X , except in very special cases.

3. Cut-type estimates. In the theory of Ford-Fulkerson [1962] of flows in finite networks the notion of a cut is fundamental, and the basic theorem is the max-flow-min-cut theorem, proved with the help of the marking algorithm.

In this section we try, following Seiffert [1981], various possibilities of defining the notion of a cut for problem (*). We don't achieve a max-flow-min-cut theorem but find cut-type estimates helpful in proving the maximality of special flows for special measures c in X .

1. Cuts as subsets of X .

DEFINITION 3.1. A closed subset C of X is called a cut if for every $\omega \in \mathcal{Q}$ the intersection of C with the graph of ω is non-empty.

For any $\varepsilon > 0$ and any subset C of X we denote by C_ε the ε -neighborhood of the set C :

$$C_\varepsilon = \{(x, y) \mid \text{there is } (x_0, y_0) \in C \text{ such that} \\ \sqrt{(x - x_0)^2 + (y - y_0)^2} \leq \varepsilon\}.$$

Clearly, if C is closed, hence compact, every C_ε is compact as well. It is easy to prove

LEMMA 3.2. *A closed set $C \subseteq X$ is a cut if and only if $\liminf_{\varepsilon \rightarrow 0+0} (1/2\varepsilon) P(\omega, C_\varepsilon) \geq 1$ for all $\omega \in \mathcal{Q}$ such that the graph of ω hits C in an inner point of X , $(1/\varepsilon) P(\omega, C_\varepsilon) \geq 1$ for all other $\omega \in \mathcal{Q}$.*

Let ∂X denote the boundary of X and $\text{int } X = X \setminus \partial X$ the interior of X .

DEFINITION 3.3. Let $C \subseteq X$ be a cut and c a measure on X , then

$$c(C) \approx \liminf_{\varepsilon \rightarrow 0+0} \left[\frac{1}{2\varepsilon} c(((\text{int } X) \cap C)_\varepsilon) + \frac{1}{\varepsilon} c((\partial X \cap C)_\varepsilon) \right]$$

is called the c -capacity of the cut C .

Our Lemma 3.2 and our definition 3.3 are split into two cases only due to problems occurring at the boundary of X . In practical cases these problems are mostly easily settled, as we shall see.

THEOREM 3.4. *Let μ be a measure in \mathcal{Q} , c a measure in X and $C \subseteq X$ a cut. Then $\mu P \leq c$ implies*

$$\mu 1 \leq c(C).$$

Proof. Write

$$D = C \cap (\text{int } X) \quad \mathcal{Q}_D = \{\omega \mid \text{the graph of } \omega \text{ hits } D\}$$

$$E = C \cap (\partial X) \quad \mathcal{Q}_E = \mathcal{Q} \setminus \mathcal{Q}_D$$

Choose any $1 > \delta > 0$ and find $\varepsilon_0 > 0$ such that $0 < \varepsilon \leq \varepsilon_0$ implies

$$\mu \left(\left\{ \omega \mid \frac{1}{2\varepsilon} P(\omega, D_\varepsilon) \geq 1 - \delta \right\} \right) \geq \mu(\mathcal{Q}_D) - \delta.$$

Then $0 < \varepsilon \leq \varepsilon_0$ implies also

$$\begin{aligned}
\mu 1 &= \mu(\mathcal{Q}) = \mu(\mathcal{Q}_D) + \mu(\mathcal{Q}_E) \\
&\leq \mu\left(\left\{\omega \mid \frac{1}{2\varepsilon} P(\omega, D_\varepsilon) \geq 1 - \delta\right\}\right) + \delta \\
&\quad + \mu\left(\left\{\omega \mid \frac{1}{\varepsilon} P(\omega, E_\varepsilon) \geq 1\right\}\right) \\
&\leq \frac{1}{1 - \delta} \cdot \frac{1}{2\varepsilon} \int P(\omega, D_\varepsilon) \mu(d\omega) + \frac{1}{\varepsilon} \int P(\omega, E_\varepsilon) \mu(d\omega) \\
&\leq \frac{1}{1 - \delta} \left[\frac{1}{2\varepsilon} (\mu P)(D_\varepsilon) + \frac{1}{\varepsilon} (\mu P)(E_\varepsilon) \right] \\
&\leq \frac{1}{1 - \delta} \left[\frac{1}{2\varepsilon} c(D_\varepsilon) + \frac{1}{\varepsilon} c(E_\varepsilon) \right].
\end{aligned}$$

Varying ε and δ suitably, our theorem follows.

EXAMPLE 3.5. Let a finite network in the sense of the Ford-Fulkerson theory be given such that it can be represented by a planar drawing with $(-1, 0) = Q$ as the source and $S = (1, 0)$ as the sink. Modifying the drawing, we can assume that the drawing consists of finitely many sections of graphs of points $\omega \in \mathcal{Q}$, and the capacity distribution over the network is represented by some measure c in X living on these sections and being nonnegative multiples of arc length measure on them. A cut in the usual sense of the Ford-Fulkerson theory can then be represented by a cut $C \subseteq X$ in the sense of Definition 3.1, and $c(C)$ equals the Ford-Fulkerson capacity of the represented Ford-Fulkerson cut. Thus the cut theory of Ford-Fulkerson appears as a special case of our present approach, up to problems of planar representability.

EXAMPLE 3.6. Let c be a measure on X which is absolutely continuous with respect to planar Lebesgue measure λ^2 such that the density $(dc/d\lambda^2)(x, y)$ depends on y only, and may thus be written $\rho(y)$. Assume $\rho(y) = 0$ ($|y| \geq \frac{1}{2}$). Then e.g.

$$C = \left\{ (x, y) \mid (x, y) \in X, x = -\frac{1}{2} \right\}$$

is a cut with capacity

$$c(C) = \int_{-1/2}^{1/2} \rho(y) dy$$

and this is μ for the μ distributed over all paths of the form

$$\omega_y(t) = \begin{cases} 2y(t+1) & \text{for } -1 \leq t \leq -\frac{1}{2} \\ y & \text{for } -\frac{1}{2} \leq t \leq \frac{1}{2} \\ 2y(1-t) & \text{for } \frac{1}{2} \leq t \leq 1 \end{cases}$$

according to ρ . Thus μ is a solution of (*) in this case.

EXAMPLE 3.7. Let c be as in example 3.6, but ρ depend on x alone instead of y alone. Then every

$$C_x = \{(x, y) \mid (x, y) \in X\}$$

is a cut, and

$$\inf_{-1/2 \leq x \leq 1/2} c(C_x) = \text{ess inf}_{-1/2 \leq x \leq 1/2} \rho(x).$$

Equidistributing exactly this amount of mass over the paths ω_y considered in example 3.6 we obtain a solution μ of (*).

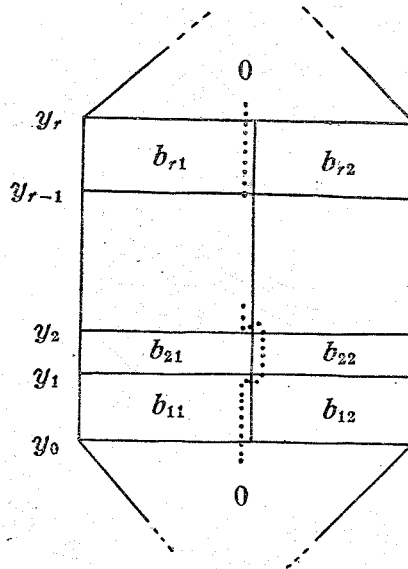
EXAMPLE 3.8. Let

$$B = \begin{bmatrix} b_{11} & b_{12} \\ \dots & \dots \\ b_{r1} & b_{r2} \end{bmatrix}$$

be a $r \times 2$ -matrix with nonnegative entries and let $-\frac{1}{2} = y_0 < y_1 < \dots < y_r = \frac{1}{2}$. Define

$$\rho(x, y) = \begin{cases} b_{k1} & \text{for } -\frac{1}{2} \leq x < 0, \quad y_{k-1} < y < y_k \\ b_{k2} & \text{for } 0 \leq x \leq \frac{1}{2}, \quad y_{k-1} < y < y_k \\ 0 & \text{else in } X \end{cases} \quad (k = 1, \dots, r)$$

and let c be the measure in X with λ^2 -density ρ .

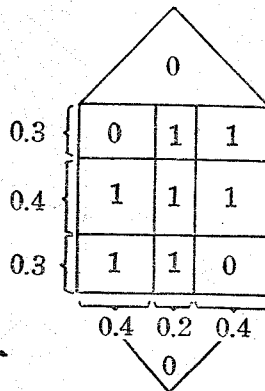


The picture shows how to zigzag a cut curve (closer and closer to the y -axis), the zigzag being dependent on which of b_{k1} , b_{k2} is smaller: the curve's vertical parts always choose the lower levels of ρ . We may thus bring $c(C)$ down to

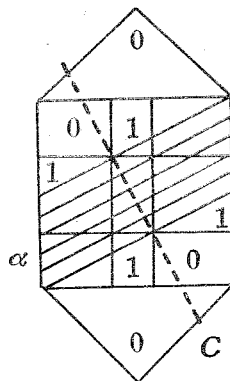
$$(2) \quad \sum_{k=1}^r (y_k - y_{k-1}) \min \{b_{k1}, b_{k2}\}$$

as close as we want. Equidistributing mass $(y_k - y_{k-1}) \min \{b_{k1}, b_{k2}\}$ over those ω_y (see example 3.6) which have $y_{k-1} < y < y_k$, and summing up the results, we obtain a μ with $\mu P \leq c$ and $\mu 1 = (2)$, thus solving (*).—In a similar way some more cases of “chessboard-like” c 's can be settled.

EXAMPLE 3.9. Let the λ^2 -density ρ of c be given by



Now draw a cut curve C according to



We find

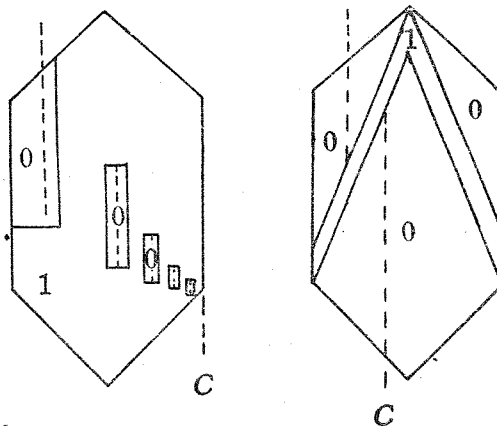
$$c(C) = \sqrt{(0, 2)^2 + (0, 4)^2} = \frac{\sqrt{4 + 16}}{10} = \frac{1}{\sqrt{5}}.$$

For any $-\frac{1}{2} \leq \alpha \leq 0$ define

$$\eta_\alpha(t) = \begin{cases} 2\alpha(t+1) & \text{for } -1 \leq t \leq -\frac{1}{2} \\ \frac{1}{2}\left(t + \frac{1}{2}\right) + \alpha & \text{for } -\frac{1}{2} \leq t \leq \frac{1}{2} \\ 2\left(\alpha + \frac{1}{2}\right)(1-t) & \text{for } \frac{1}{2} \leq t \leq 1 \end{cases}$$

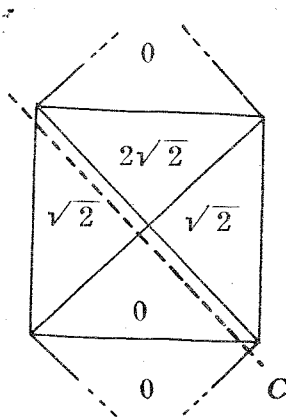
and equidistribute mass $1/\sqrt{5}$ over these $\eta_\alpha \in \mathcal{Q}$. This yields a μ in \mathcal{Q} which is easily seen to fulfil $\mu P \leq c$. It follows that μ solves (*).

EXAMPLE 3.10. Let $dc/d\lambda^2$ be given in X according to one of the following two pictures

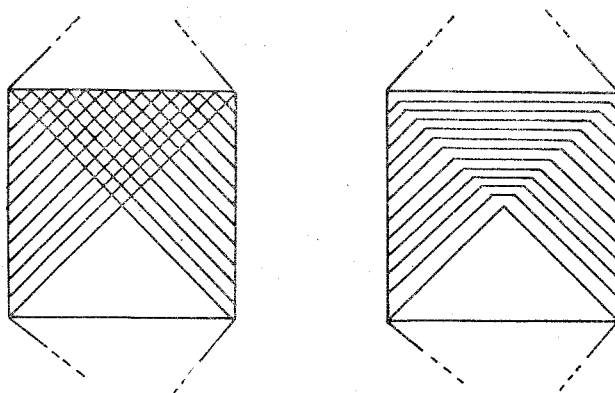


Here no $\omega \in \mathcal{Q}$ can—we content ourselves with a visual argument here—avoid passing through some, however short, interval where there is “no food”. Thus $\mu = 0$ is the solution to (*) in these cases.

EXAMPLE 3.11. Let $dc/d\lambda^2$ be given according to the following picture



The picture contains a cut curve C . If we let it approach the diagonal, its capacity $c(C)$ goes down to 1. Obvious equidistribution of mass 1 over the family of paths from in each of the following pictures

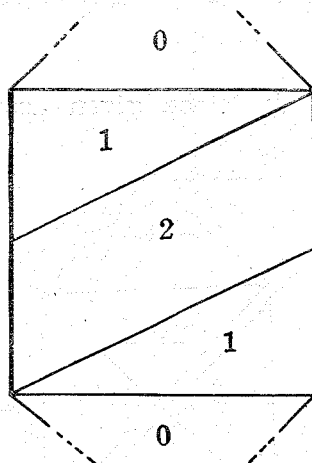


yields a μ in \mathcal{Q} with $\mu 1 = 1$ and $\mu P \leq c$, and thus two solutions to (*).

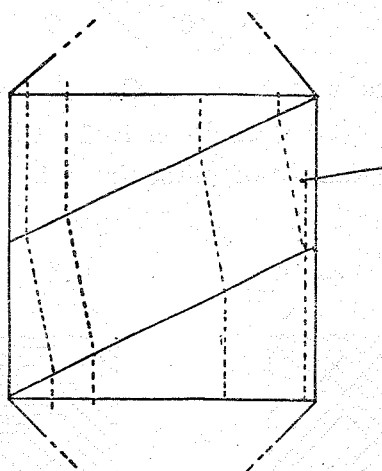
EXAMPLE 3.12. Let μ_1 be equidistribution of mass 1 on the paths ω_y (example 3.6) and μ_2 be equidistribution of mass $1/\sqrt{5}$ on the paths η_α (example 3.9), and

$$c = \mu_1 P + \mu_2 P.$$

The density dc/dl^2 is to be seen in the following picture



Cut curves can be drawn according to the following picture



depending upon an angle β . If we put $s = \tan \beta$, we get a capacity

$$\frac{s + 1 + 2\sqrt{1 + s^2}}{s + 2} \quad (0 \leq t < \infty).$$

Differentiation yields a minimum at

$$s_0 = \frac{8 - \sqrt{19}}{15}$$

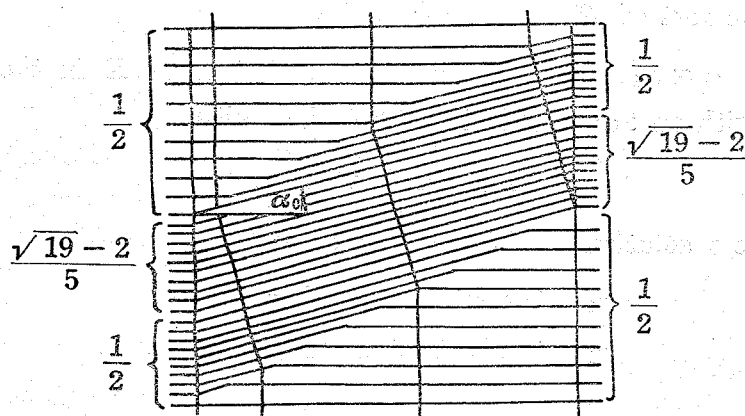
which corresponds to

$$\beta_0 = 13.7^\circ \quad (\text{approximately})$$

and yields a cut capacity

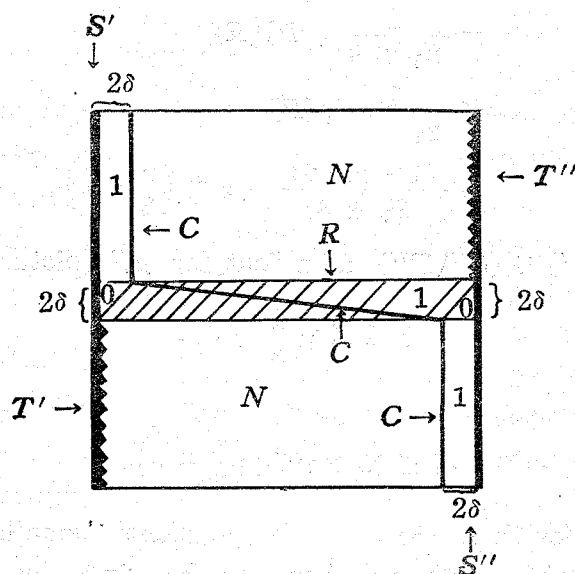
$$\frac{3 + \sqrt{19}}{5} > 1 + \frac{1}{\sqrt{5}}$$

The following picture displays a μ with $\mu 1 = (3 + \sqrt{19})/5$



The details^o are left to the reader (see Seiffert [1981]).

EXAMPLE 3.13. Let dc/dl^2 be given in X according to the following picture



with variables $\delta > 0$ and $N > 0$ such that the C in the picture yields a cut with minimal

$$(3) \quad c(C) = 1 - 2\delta + \sqrt{1 - 8\delta + 20\delta^2}$$

In view of the fact that any μ has to live on the Lip_1 paths $\omega \in \mathcal{Q}$, no μ with $\mu P \leq c$ and $\mu 1 = c(C)$ can exist. We get a lower bound than $c(C)$ for $\mu 1$ if we split \mathcal{Q} into

$\mathcal{Q}_1 =$ all ω passing through T' and T'' in the above picture,
 $\mathcal{Q}_2 =$ the rest of \mathcal{Q} .

Now the $\omega \in \mathcal{Q}_1$ pass through the parallelogram R in the above picture with an arc length $\geq 2\sqrt{2} \delta$, i. e. with

$$P(\omega, R) \geq 2\sqrt{2} \delta.$$

Let μ be a solution to (*) for our c , and

$\mu_1 =$ the restriction of μ to \mathcal{Q}_1 ,

$\mu_2 =$ the restriction of μ to \mathcal{Q}_2 .

Then

$$\begin{aligned} \mu_1 1 &= \frac{1}{2\sqrt{2} \delta} \int_{\mathcal{Q}_1} 2\sqrt{2} \delta \mu(d\omega) \\ &\leq \frac{1}{2\sqrt{2} \delta} \int \mu(d\omega) P(\omega, R) \\ &= \frac{1}{2\sqrt{2} \delta} (\mu P)(R) \\ &\leq \frac{1}{2\sqrt{2} \delta} c(R) \\ &= \frac{(1-2\delta) 2\delta}{2\sqrt{2} \delta} = \sqrt{2} \left(\frac{1}{2} - \delta \right). \end{aligned}$$

Taking $(S' \setminus T') \cup (S'' \setminus T'')$ as a "cut for μ_2 " (picture!) we find

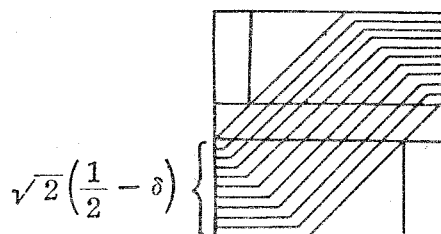
$$\mu_2 1 \leq 2 \left(\frac{1}{2} - \delta \right),$$

and thus

$$\mu 1 \leq (2 + \sqrt{2}) \left(\frac{1}{2} - \delta \right),$$

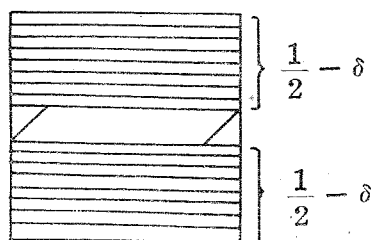
which is smaller than (3) if δ is sufficiently small. A μ_0 with $\mu_0 1 = (2 + \sqrt{2}) \left(\frac{1}{2} - \delta \right)$ can, however, be given by $\mu_0 = \mu_{01} + \mu_{02}$, where

$\mu_{01} =$ equidistribution of mass $\sqrt{2} \left(\frac{1}{2} - \delta \right)$ over the paths in the following picture



and

μ_{02} = equidistribution of mass $2(\frac{1}{2} - \delta)$ over the paths in the following picture



It is easy to verify (see Seiffert [1981]) that $\mu_0 P \leq c$. Thus μ_0 solves (*).

2. Cuts as continuous functions.

Given a measure c on X , we know that $\mu P \leq c$, $Pf \leq 1 \Rightarrow \mu 1 \leq cf$. Hence in trying to solve (*), i.e. to lift $\mu 1$ as much as possible (under $\mu P \leq c$) we may try to bring cf down as far as possible by proper choice of $f \geq 0$ with $Pf \leq 1$, i.e. to solve (**). If $\mu 1 = cf$ occurs with some $f \geq 0$, $Pf \leq 1$, we know that μ solves (*) (and f solves (**)). Now what does (**) mean? Visually, we have to try to find a "canyon" or "wadi" in c which brings cf down as far as possible. Sure such a "canyon" should be interpreted as sort of a cut. The difficulty in solving (**) results from the fact that there is not enough compactness in $C(X, \mathbb{R})$.

The examples in subsection 1 contain cuts as subsets which can be approximated by continuous functions $f \geq 0$ on X with $Pf \leq 1$ quite easily such that the solutions μ of (*) given there fulfil $\mu 1 = \inf_{\substack{f \geq 0 \\ Pf \leq 1}} cf$, except in example 3.13. For details, see Seiffert [1981].

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