BULLETIN OF THE
INSTITUTE OF MATHEMATICS
ACADEMIA SINICA
Volume 10, Number 3, September 1982

ON THE LOCAL EXISTENCE OF SOLUTIONS OF INITIAL VALUED PROBLEM

BY

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1. **Introduction.** Let E be a real Banach space with a norm $\| \ \|$, $\bar{B}_r(x_0) = \{x \in E : \|x - x_0\| \leq r\}$ for all $x_0 \in E$ and $J = [0, T] \subset R$. In this paper we consider the existence of a local solution for the initial value problem

$$(1.1) \qquad x' = f(t, x), \quad x(0) = x_0$$

where $f : J \times \bar{B}_r(x_0) \to E$ is uniformly continuous. It is well-known (for example, see Deimling [3]) that if f is either Lipschitzian or dissipative then (1.1) has a solution. Browder [1] shows the existence of (1.1) for the case that f is ω-Lipschitz and Cellina [2] shows the existence for the case that f is α-dissipative. More recently, Li [5] shows the existence for the case that f is α-Lip-dissipative. Here we show that if f is α-ω-dissipative then (1.1) has a solution.

2. **Notations and definitions.** Let E^* denote the dual space of E. The duality mapping F of E into the class of subsets of E^* is defined as follows

$$F(x) = \{x^* \in E^* : x^*(x) = \|x\|^2 = \|x^*\|^2\} \quad \text{for} \quad x \in E$$

and for each $(y, x) \in E \times E$, we denote $\inf\{x^*(y) : x^* \in F(x)\}$ and $\sup\{x^*(y) : x^* \in F(x)\}$ by $\langle y, x\rangle_-$ and $\langle y, x\rangle_+$ respectively. It is obvious that for $x, y, z \in E$ we have

$$(2.1) \qquad \langle y + z, x\rangle_+ \leq \langle y, x\rangle_+ + \|z\|\|x\| .$$

Let 2^E be the family of all bounded subsets of E. We define a

Received by the editors October 4, 1981.
* This research is partly supported by NSC, Republic of China.

real valued function α on 2^E as follows

$$\alpha(B) = \inf\{d > 0 : B \text{ can be covered by finitely many of sets of diameter} \leq d\},$$

where diameter of $A = \sup\{\|a_1 - a_2\|;\ a_1, a_2 \in A\}$ is denoted by diam A, such α is known to be a measure of noncompactness which is first defined by Kuratowski [4]. Some properties of α will be used in the sequel, we list them as the following

LEMMA 2.1. (i) $\alpha(A) = 0$ *if and only if* $\bar{A}$ *is compact*

(ii) $\alpha(\bar{B}_r(x_0)) \leq 2r$

(iii) $\alpha(\lambda A) = |\lambda|\alpha(A)$ *and* $\alpha(A_1 + A_2) \leq \alpha(A_1) + \alpha(A_2)$

(iv) $A_1 \subset A_2$ *implies* $\alpha(A_1) \leq \alpha(A_2)$

(v) *If* $\{a_n\}$, $\{b_n\}$ *are bounded sequence then*

$$\alpha(\{a_n\}) - \alpha(\{b_n\}) \leq \alpha(\{a_n - b_n\}).$$

One may see [3] and [5] for the proof.

DEFINITION 2.1. A function $\omega : (0, T] \times R_+ \to R$ is said to be of class U if for each $\varepsilon > 0$, there exist $\delta > 0$, a sequence $t_i \to 0^+$ and a sequence of continuous functions $\rho_i : [t_i, T] \to R$ with

$$\rho_i(t_i) \geq \delta t_i,\ D\bar{\rho}_i(t) > \omega(t, \rho_i(t)),\ 0 < \rho_i(t) \leq \varepsilon \text{ in } (t_i, T]; \tag{2.2}$$

ω is said to be of class U_c if ω is of class U and for each $t \in [0, T]$ the function $\omega(t, \cdot) : R_+ \to R$ is upper semicontinuous.

REMARK. It is clear that if ω satisfies either the Lipschitz condition $\omega(t, r) = Lr$ for some $L \geq 0$ or Nagumo condition $\omega(t, r) = r/t$ then ω is of the class U_c.

Let $D_b = [0, T] \times \bar{B}_b(x_0)$ and $f : D_b \to E$ continuous, where T is chosen such that $\|f\| \leq b/T = M$ on D_b (we may assume that $M \geq 1$).

DEFINITION 2.2. A mapping f of D_b into E is said to be α-ω-dissipative if for each $\varepsilon > 0$ there exist a finite covering (Ω_j) of D_b such that

$$\langle f(t, x_1) - f(t, x_2), x_1 - x_2 \rangle_- \le \omega(t, \|x_1 - x_2\|)\|x_1 - x_2\| + \varepsilon\|x_1 - x_2\|$$

for (t, x_1) and (t, x_2) are in same Ω_j.

3. **Existence theorems.** In this section we prove our main results in this paper, first we state our first theorem.

THEOREM 3.1. *If $f : D_b \to E$ is uniformly continuous, ω is of the class U and $g_h(t, x) = x - hf(t, x)$ for $h > 0$, if we assume further that for any subset $X \subseteq B_b(x_0)$, $h > 0$, $t \in [0, T]$, the following*

$$\alpha(g_h(t, X)) \ge \alpha(X) - h\omega(t, \alpha(X)) \tag{3.1}$$

holds, then equation (1.1) *has at least one solution on* $[0, T]$.

In order to show theorem 3.1 we need the following lemmas. The first lemma is well-known, for example, onn may fined in Deimling[3].

LEMMA 3.1. *Let $f : [0, T] \times \bar{B}_b(x_0) \to E$ is continuous and $\|f(t, x)\| \le b/T = M$ on D_b. Then for each $\varepsilon > 0$ there is a continuous differentiable function $x_\varepsilon : [0, T] \to \bar{B}_b(x_0)$ such that*

$$\begin{gathered} x'_\varepsilon(t) = f(t, x_\varepsilon(t)) + y_\varepsilon(t), \; x_\varepsilon(0) = x_0 \\ \textit{and} \quad \|y_\varepsilon(t)\| \le \varepsilon \quad \textit{on} \quad [0, T]. \end{gathered} \tag{3.2}$$

Such an x_ε is called an ε-approximation solution of (1.1). In the following we denote x_n as an $1/n$-approximation solution of f, then it follows from (3.2) that for any $n \in N$

$$\|x_n(t) - x_n(\bar{t})\| \le \left(M + \frac{1}{n}\right)|t - \bar{t}| \le 2M|t - \bar{t}|. \tag{3.3}$$

Hence we have

$$\{x_n\} \text{ is equicontinuous.} \tag{3.4}$$

LEMMA 3.2. *Let ω and f satisfy the hypothesis in Theorem 3.1 and let $\{x_n\}$ be defined as above. If $X(t) = \{x_n(t) : n \ge 1\}$ and $p(t) = \alpha(X(t))$ for each $t \in [0, T]$ then*

$$D^- p(t) \le \omega(t, p(t))$$

for all $t \in [0, T]$.

Proof. It is due to the uniform continuity of f on D_b, for each $\varepsilon > 0$ there exist an $\delta > 0$ such that $\|f(t, x) - f(\bar{t}, y)\| \leq \varepsilon/4$ whenever $|t - \bar{t}| < \delta$ and $\|x - y\| < \delta$.

The fact that $\{x_n\}$ is equicontinuous, we have that for $\delta > 0$ there is a real number h_0 with $0 < h_0 \leq \delta$ such that

$$\|x_n(t) - x_n(\bar{t})\| < \delta$$

for $|t - \bar{t}| < h_0$ and for all n.

It follows from (v) of Lemma 2.1 and (3.1) that for $t \in [0, T]$ and $0 < h \leq h_0$, we have

$$\begin{aligned} h\omega(t, p(t)) &\geq \alpha(X(t)) - \alpha(g_h(t, X(t))) \\ &\geq \alpha(X(t)) - \alpha(X(t-h)) \\ &\qquad - [\alpha(g_h(t, X(t))) - \alpha(X(t-h))] \\ &\geq p(t) - p(t-h) \\ &\qquad - \alpha(\{x_n(t) - x_n(t-h) - hf(t, x_n)t)\};\ n \geq 1\}). \end{aligned} \tag{3.5}$$

By the choice of δ and h_0, we have

$$\begin{aligned} &\|x_n(t) - x_n(t-h) - hf(t, x_n(t))\| \\ &\quad = \left\| \int_{t-h}^{t} f(s, x_n(s)) - f(t, x_n(t)) - y_n(s)\, ds \right\| \\ &\quad \leq \int_{t-h}^{t} \|f(s, x_n(s)) - f(t, x_n(t))\| ds + \int_{t-h}^{t} \|y_n(s)\|\, ds \\ &\quad \leq \frac{\varepsilon}{4} h + \frac{1}{n} h \\ &\quad \leq \frac{\varepsilon}{2} h . \end{aligned} \tag{3.6}$$

whenever $n > 4/\varepsilon$ and $0 < h \leq h_0$. The inequalities (3.5) and (3.6) imply that $h\omega(t, p(t)) \geq p(t) - p(t-h) - \varepsilon h$ for $0 < h \leq h_0$. Therefore

$$\omega(t, p(t)) \geq \limsup_{h \to 0^+} \frac{p(t) - p(t-h)}{h} - \varepsilon = D^- p(t) - \varepsilon$$

and

$$D^- p(t) \leq \omega(t, p(t)) .$$

LEMMA 3.3. *For each $\eta > 0$ there is $t_\eta > 0$ such that*

$$p(t) \leq \eta t \quad \textit{for all} \quad t \in [0, t_\eta] .$$

Proof. Since f is uniformly continuous on D, we have that for each $\eta > 0$ there is a $\delta > 0$ such that

$$\|f(t, x) - f(t, y)\| \leq \frac{\eta}{2}$$

whenever $0 < t < 2\delta$ and $\|x - y\| < 2\delta$.

For the equicontinuity of $\{x_n\}$ there is $t_\eta > 0$, and $t_\eta < \delta$ such that $\|x_n(t) - x_n(\bar{t})\| < \delta$ for $|t - \bar{t}| < t_\eta$ and for all n.

Take $t \in [0, t_\eta]$ we have

$$\|x_n(t) - x_m(t)\| \leq \|x_n - x_0\| + \|x_m - x_0\| < 2\delta$$

for all m and n. Choose k_0 such that $k_0 \geq 4/\eta$. Then for $m, n \geq k_0$

$$\begin{aligned}\|x_n(t) - x_m(t)\| &= \left\| \int_0^t f(s, x_n(s)) + y_n(s)\, ds \right. \\ &\qquad \left. - \int_0^t f(s, x_m(s)) + y_m(s)\, ds \right\| \\ &\leq \frac{\eta}{2} t + \left(\frac{1}{n} + \frac{1}{m}\right) t \\ &\leq \eta t .\end{aligned}$$

Then $p(t) = \alpha(\{x_n(t);\ n \geq 1\}) = \alpha(\{x_n(t);\ n \geq k_0\}) \leq \eta t$ for all t in $[0, t_\eta]$.

Now we are ready to prove Theorem 3.1.

Proof of Theorem 3.1. By the definition of ω being of class U, for given $\varepsilon > 0$, we may choose $\delta > 0$, a sequence $t_i \to 0^+$ and a sequence of continuous functions $\rho_i : [t_i, T] \to R$ satisfying (2.2). By Lemma 3.3, there is $t_0 > 0$ such that $p(t) \leq (\delta/2)t$ for all $t \in [0, t_0]$.

For sufficiently large i, we have $t_i < t_0$ and $p(t_i) < \rho_i(t_i)$. We assert that $p(t) < \rho_i(t)$ on $[t_i, T]$. For if not, then there is a first time $T \geq t^* > t_i$ with $\rho_i(t^*) = p(t^*)$ then $p(*t) > 0$ and thus, by Lemma 3.2 that

$$D^- p(t^*) \leq \omega(t^*, p(t^*)) = \omega(t^*, \rho_i(t^*)) < D^- \rho_i(t^*) .$$

However, it is impossible since $p(t) < \rho_i(t)$ for all $t \in [t_i, t^*)$. Hence $p(t) < \rho_i(t)$ for all $t \in [t_i, T]$ and then $p(t) \leq \varepsilon$ in $[t_i, T]$ for all i and arbitrary $\varepsilon > 0$. That is, $p(t) = 0$ or $\{x_n(t); n \geq 1\}$ is relatively compact for all $t \in [0, T]$.

Since $\{x_n\}$ is equicontinuous, Ascoli-Arzela theorem asserts that $\{x_n\}$ is relatively compact in $C([0, T], E)$, the space of continuous functions on $[0, T]$ into E with superior norm. Therefore there is a uniformly convergent subsequence $\{x_{n_k}\}$ of $\{x_n\}$ on $[0, T]$, say x as its limit then x is a solution of (1.1).

Now we shall make use of Theorem 3.1 to show the following theorem.

THEOREM 3.2. *Let $f : D_b \to E$ be α-ω-dissipative and uniformly continuous where ω is of class U_c. Then Equation* (1.1) *has a local solution on* $[0, T]$.

Proof. Due to Theorem 3.1, it suffices to show that (3.1) holds in our case.

Now let $h > 0$ be fixed. By the definition α, for any $X \subset B_b(x_0)$ and for given $\varepsilon > 0$ there is a covering $\{C_i;\ 1 \le i \le n\}$ of $g_h(t, X)$ with diam $C_i \le \alpha(g_h(t, X)) + \varepsilon/4$

$$A_i = \{x \in E : x - hf(t, x) \in C_i\}, \quad 1 \le i \le n.$$

We have

$$\bigcup_{i=1}^{n} A_i \supseteq X.$$

For f being α-ω-dissipative, there is a finite covering $\{O_j\}_{j=1}^{m}$ or $[0, T] \times B_b(x_0)$ such that

$$h\left\langle f(t', x_1) - f(t', x_2), \frac{x_1 - x_2}{\|x_1 - x_2\|}\right\rangle_- \le h\omega(t', \|x_1 - x_2\|) + \frac{\varepsilon}{4}$$

for (t', x_1) and (t', x_2) in same O_j, $1 \le j \le m$ and $x_1 \ne x_2$.

For $0 < t \le T$, define $A'_j = \{x \in E;\ (t, x) \in O_j\}$, $1 \le j \le m$. It follows from the definition of U_c that ω is upper semicontinuous and there is η with $0 < \eta < \varepsilon/2$ such that

$$\omega(t, r) \le \omega(t, \alpha(X)) + \frac{\varepsilon}{4} \quad \text{for} \quad \alpha(X) - \eta < r < \alpha(X) + \eta.$$

Let $\{A''_k\}$ be a finite covering of X with $\max\{\text{diam } A''_k : 1 \leq k \leq l\} < \alpha(X) + \eta$, and $A_{i,j,k} = A_i \cap A'_j \cap A''_k$ for $1 \leq i \leq n$, $1 \leq j \leq m$ and $1 \leq k \leq l$. Then $\bigcup\{A_{i,j,k};\ 1 \leq i \leq n, 1 \leq j \leq m, 1 \leq k \leq l\} \supseteq X$ and there are $0 \leq i_0 \leq n$, $1 \leq j_0 \leq m$ and $1 \leq k_0 \leq l$ with $\alpha(X) \leq \text{diam } A_{i_0, j_0, k_0} < \alpha(X) + \eta$. Hence there are x_1^0, x_2^0 in A_{i_0, j_0, k_0} with

$$\alpha(X) - \eta < \|x_1^0 - x_2^0\| < \alpha(X) + \eta$$

and then $\omega(t, \|x_1^0 - x_2^0\|) \leq \omega(t, \alpha(X)) + \varepsilon/4$. Since x_1^0, x_2^0 are in $A_{i_0, j_0, k_0} \subseteq A_{i_0} \cap A'_{j_0}$ we have

$$h\left\langle f(t, x_1^0) - f(t, x_2^0), \frac{x_0^1 - x_2^0}{\|x_1^0 - x_2^0\|}\right\rangle_- \leq hw(t, \|x_1^0 - x_2^0\|) + \frac{\varepsilon}{4}$$

$$\begin{aligned}
&\|x_1^0 - x_2^0\| - h\omega(t, \|x_1^0 - x_2^0\|) - \frac{\varepsilon}{4} \\
&\quad \leq \left\langle (x_1^0 - hf(t, x_1^0)) - (x_2^0 - hf(t, x_0)), \frac{x_0^1 - x_2^0}{\|x_1^0 - x_2^0\|}\right\rangle_+ \\
&\quad \leq \|g_h(t, x_1^0) - g_h(t, x_2^0)\| \leq \text{diam } C_{i_0} \\
&\quad \leq \alpha(g_h(t, X)) + \frac{\varepsilon}{4}.
\end{aligned}$$

Thus

$$\alpha(X) - h\omega(t, \alpha(X)) \leq \alpha(g_h(t, X)) + \varepsilon + \frac{\varepsilon}{4} h.$$

For ε being arbitrary, we get

$$\alpha(X) - h\omega(t, \alpha(X)) \leq \alpha(g_h(t, X)).$$

As a result of Theorem 3.2 and Remark in Section 2, we have the following Corollary:

COROLLARY 3.1. *Let $f : D_b \to E$ be uniformly continuous satisfying either of the following two conditions:*

(i) *There exists $L \geq 0$ such that any given $\varepsilon > 0$ there is a finite covering $\{O^s\}$ of D^b with*

$$\langle f(t, x_1) - f(t, x_2), x_1 - x_2 \rangle \leq L\|x_1 - x_2\|^2 + \varepsilon\|x_1 - x_2\|$$

whenever (t, x_1), (t, x_2) in same O^s.

(ii) *For any given* $\varepsilon > 0$ *there is a finite covering* $\{O^s\}$ *of* D_b *with*

$$\langle f(t,x_1) - f(t, x_2), x_1 - x_2\rangle \leq \frac{\|x_1 - x_2\|^2}{t} + \varepsilon\|x_1 - x_2\|$$

whenever (t, x_1), (t, x_2) *in same* O^s.

Then equation (1.1) *of* f *has a local solution on* $[0, T]$.

REMARK 1. The part (i) of Corollary 3.1 generalized the proposition (3.1) in Li [5].

REMARK 2. Theorem 3.2 is also true for the case that ω is a Kamke function in the sense of Li [5]. The proof will followed easily.

REMARK 3. Theorem 3.1 is proved for the case that ω is a Kamke function in the sense of Li.

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